ACHIEVING SUPERCONVERGENCE BY ONE DIMENSIONAL DISCONTINUOUS FINITE ELEMENTS. PART 1. THE WG METHOD

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Abstract. A simple stabilizer free weak Galerkin (SFWG) finite element method is introduced for solving second order elliptic problem in one dimension. In this SFWG method, weak function is formed by discontinuous $k$th order polynomial with additional unknowns defined on the vertex points and its weak derivative is approximated by polynomial of degree $k + 1$. The superconvergence of order two for the SFWG finite element solution is obtained. An elementwise lifted $P_{k+2}$ solution from the $P_k$ SFWG solution converges at the optimal order. The numerical results confirm the theory.

1. Introduction

The weak Galerkin (WG) finite element methods, first proposed and analyzed in [2, 3] provide a general finite element technique for solving partial differential equations. The novelty of the WG method is the introduction of weak function and its weakly defined derivative. The weak functions possess the form of $v = \{v_0, v_b\}$ with $v = v_0$ representing the value of $v$ in the interior of each element and $v = v_b$ on the boundary of the element. $v_0$ and $v_b$ are approximated by polynomials of $P_k(T)$ and $P_s(e)$ respectively, where $e$ stands for the edge or face of $T$. Weak derivative is specially designed for weak function approximated by $P_\ell(T)$ polynomials. Each combination of the WG element ($P_k(T), P_s(e), [P_\ell(T)]^d$) leads to a weak Galerkin finite element method.

For some special combinations of the WG element ($P_k(T), P_s(e), [P_\ell(T)]^d$) for $d \geq 2$, stabilizer is no longer needed in the corresponding WG method, which leads to stabilizer free weak Galerkin (SFWG) method. A stabilizer free weak Galerkin method was first introduced in [4] on polygonal and polyhedral meshes. It demonstrated that stabilizer term in the WG method can be eliminated if the WG element ($P_k(T), P_k(e), [P_{k+n-1}(T)]^d$) is used, where $n$ is the number of the sides in a polygon. This result has been improved in [1] by reducing the degree of polynomial for weak gradient on triangular mesh. Furthermore superconvergence is observed for some special WG element. An order one superconvergence of a SFWG method is derived in [5] for the WG element ($P_k(T), P_k(e), MRT_k(T)$) where $MRT_k(T)$ is a macro-Raviart-Thomas element on a polygon/polyhedron $T$. An order two superconvergence is obtained for the WG element ($P_k(T), P_{k+1}(e), [MP_{k+1}(T)]^d$) in [6], where $[MP_{k+1}(T)]^d$ is a macro-BDM element on a polygon/polyhedron $T$.

This paper have two purposes: 1. Investigating the performance of SFWG element ($P_k(T), P_s(e), [P_\ell(T)]^d$) in one dimension; 2. Providing necessary theory for
the paper Part 2 of the sequence. In one dimensional case, polynomial $P_k(v)$ degenerates to a single value at end points of an interval. This work answers the question: which WG element $(P_k(I), P_0(x_i), P_k(I))$ will maximize the order of convergence in one dimension, where $x_i$ is an end point of the interval $I$. Two orders higher convergence for the solution of the SFWG method is obtained in one dimension for the WG element $(P_k(T), P_0(x_i), |P_{k+1}(T)|^d)$. That is, the $P_k$ SFWG solution converges at order $k + 3$ in $L^2$ norm, and at order $k + 2$ in $H^1$ norm. Further, the $P_k$ SFWG solution is lifted to a $P_{k+2}$ solution elementwise, which converges at the optimal order. Numerical results are presented verifying the theorem.

2. SFWG Finite Element Schemes

The considered problem is seeking an unknown function $u$ satisfying

$$\begin{align*}
-u'' &= f \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega,
\end{align*}$$

where $\Omega = [a, b]$.

Let $\Omega = [a, b] = \bigcup_{i=1}^{N} I_i$ with $I_i = [x_{i-1}, x_i]$ and $T_h = \{ I_i \mid i = 1, \cdots, N \}$, where $h = \max |I_i|$. Let $Dv = \frac{dv}{dx}$.

For a given integer $k \geq 1$, let $V_h$ be the weak Galerkin finite element space associated with $T_h$ defined as follows

$$V_h = \{ v = \{v_0, v_b\} : v_0|_I \in P_k(I), \ v_b|_x \in P_0(x), \ x \subset \partial I, \ I \in T_h, \ v_b|_{\partial\Omega} = 0 \}. $$

We like to emphasize that $v_b$ takes single value on $x_i$ for $i = 0, \cdots, N$.

For $v \in V_h$, a weak derivative $D_w v$ is a piecewise polynomial such that on each $I_i$, $D_w v \in P_{k+1}(I_i)$ satisfies

$$\begin{align*}
(D_w v, q)_{I_i} &= -(v_0, D_q q)_{I_i} + (v_b, q)_{\partial I_i} \quad \forall q \in P_{k+1}(I_i),
\end{align*}$$

where $\langle v, w \rangle_{\partial I_i} = v(x_i)w(x_i) - v(x_{i-1})w(x_{i-1})$. For simplicity, we adopt the following notations,

$$\begin{align*}
\langle v, w \rangle_{T_h} &= \sum_{i=1}^{N} \langle v, w \rangle_{I_i} = \sum_{i=1}^{N} \int_{I_i} vwdx, \\
\langle v, w \rangle_{\partial T_h} &= \sum_{i=1}^{N} \langle v, w \rangle_{\partial I_i} = \sum_{i=1}^{N} (v(x_i)w(x_i) - v(x_{i-1})w(x_{i-1})).
\end{align*}$$

Algorithm 1. A SFWG method for (2.1)-(2.2) seeks $u_h \in V_h$ satisfying the following equation:

$$\begin{align*}
(D_w u_h, D_w v) = (f, v) \quad \forall v = \{v_0, v_b\} \in V_h.
\end{align*}$$

For any $v \in V_h$, we define two semi-norms,

$$\begin{align*}
\|v\|_2 &= (D_w v, D_w v), \\
\|v\|_{1,h}^2 &= \sum_{I \in T_h} \left( \|Dv_0\|_{L^2(I)}^2 + h_I^{-1}\|v_0 - v_b\|_{L^2(I)}^2 \right).
\end{align*}$$

The following lemma indicates that the two semi-norms are equivalent.

Lemma 2.1. There exist two positive constants $C_1$ and $C_2$ independent of $h$, such that for any $v \in V_h$, we have

$$\begin{align*}
C_1\|v\|_{1,h} \leq \|v\| \leq C_2\|v\|_{1,h}.
\end{align*}$$
Proof. We prove the upper bound first. By the definition of weak derivative (2.4), letting \( q = D_w v \), we get
\[
\|v\|^2 = \sum_{I \in T_h} (Dv_0, D_w v)_I + \langle v_b - v_0, D_w v \rangle_{\partial I}
\]
\[
\leq \sum_{I \in T_h} (Dv_0, D_w v)_I + \|v_b - v_0\|_{\partial I} \|D_w v\|_{\partial I}
\]
\[
\leq \sum_{I \in T_h} \left( \|Dv_0\|_I + C h^{-1/2} \|v_b - v_0\|_{\partial I} \right) \|D_w v\|_I
\]
\[
\leq C \|v\|_{1, 1, h} \|v\|,
\]
where we used the finite dimensional norm equivalence and the scaling argument.

The upper bound is proved.

To prove the lower bound, we choose a special \( q \) so that the above inequality can be reversed. Let \( q \in P_{k+1}(I) \) on each \( I \) be such that
\[
q(x_{i-1}) = h^{-1} (-v_b(x_{i-1}) + v_0(x_{i-1})),
\]
\[
q(x_i) = h^{-1} (v_b(x_{i-1}) - v_0(x_{i-1})),
\]
\[
(q, p_{k-1})_I = (Dv_0, p_{k-1})_I \quad \forall p_{k-1} \in P_{k-1}(I).
\]
By the finite dimensional norm equivalence and the scaling argument,
\[
\|q\|_0 \leq C \|v\|_{1, 1, h}.
\]
By the definition of weak derivative (2.4), with above \( q \), we get
\[
\|v\|^2_{1, 1, h} = (D_w v, q)_I \leq \|D_w v\|_0 \|q\|_0 \leq \|v\| \|v\|_{1, 1, h}.
\]
The proof is complete.

It is easy to see that \( \|v\|_{1, 1, h} \) defines a norm in \( V_h \). The above lemma implies that the semi-norm \( \| \cdot \| \) is also a norm in \( V_h \). Therefore the SFWG method is well posed.

3. Error Equation

We start this section with the following lemma. First let \( \Pi_j \) be the elementwise defined \( L^2 \) projection onto \( P_j(I) \) for \( I \in T_h \) and \( Q_h u = \{ \Pi_k u, u \} \in V_h \).

Lemma 3.1. Let \( \phi \in H^1(\Omega) \). Then on any \( I \in T_h \), we have
\[
D_w(Q_h \phi) = \Pi_{k+1}(D\phi).
\]

Proof. Using (2.4) and integration by parts, we have that for any \( q \in P_{k+1}(I) \)
\[
(D_w(Q_h \phi), q)_I = - (\Pi_k \phi, Dq)_I + \langle \phi, q \rangle_{\partial I}
\]
\[
= - (\phi, Dq)_I + \langle \phi, q \rangle_{\partial I}
\]
\[
= (D\phi, q)_I = (\Pi_{k+1}(D\phi), q)_I,
\]
which implies the equation (3.1).

Let \( e_h = Q_h u - u_h \in V_h \). Next we derive an error equation that \( e_h \) satisfies.
Lemma 3.2. For any \( v \in V_h \), the following error equation holds true
\[
(D_w e_h, D_w v) = \ell(u, v),
\]
where
\[
\ell(u, v) = \langle Du - \Pi_{k+1} Du, v_0 - v_b \rangle_{\partial \Omega}.
\]

Proof. For \( v = \{v_0, v_b\} \in V_h \), testing (2.1) by \( v_b \) and using the fact that \( \langle Du, v_b \rangle_{\partial \Omega} = 0 \), we arrive at
\[
-(u'', v_0) = (Du, Dv_0)_{\partial \Omega} - (Du, v_0 - v_b)_{\partial \Omega} = (f, v_0).
\]
It follows from integration by parts, (2.4) and (3.1) that
\[
(Du, Dv_0)_{\partial \Omega} = (\Pi_{k+1} Du, Dv_0)_{\partial \Omega} = -(v_0, D(\Pi_{k+1} Du))_{\partial \Omega} + (v_0, \Pi_{k+1} Du)_{\partial \Omega}
\]
\[
= (\Pi_{k+1} Du, D_w v)_{\partial \Omega} + (v_0 - v_b, \Pi_{k+1} Du)_{\partial \Omega}
\]
\[
= (D_w (Q_h u), D_w v) + (v_0 - v_b, \Pi_{k+1} Du)_{\partial \Omega}.
\]
The equations (3.3) and (3.4) yield
\[
-(u'', v_0) = (D_w Q_h u, D_w v) - \ell(u, v) = (f, v_0),
\]
which gives
\[
(D_w Q_h u, D_w v) = \ell(u, v) + (f, v_0).
\]
The error equation follows from subtracting (2.5) from (3.6),
\[
(D_w e_h, D_w v) = \ell(u, v) \quad \forall v \in V_h.
\]
This completes the proof of the lemma. \( \square \)

4. Error Estimates in Energy Norm

For any function \( \varphi \in H^1(I_i) \) with \( I_i = [x_{i-1}, x_i] \), the following trace inequality holds true:
\[
|\varphi(x_j)|^2 \leq C (h_I^{-1} \|\varphi\|_{L^2(I_i)}^2 + h_I \|D\varphi\|_{L^2(I_i)}^2), \quad j = i - 1, i.
\]
Next we will bound the term \( \ell(u, v) \).

Lemma 4.1. For any \( w \in H^{k+3}(\Omega) \) and \( v = \{v_0, v_b\} \in V_h \), we have
\[
|\ell(w, v)| \leq C h^{k+2} |w|_{k+3} \|v\|.
\]

Proof. Using the Cauchy-Schwarz inequality, the trace inequality (4.1), and (2.8), we have
\[
|\ell(w, v)| = \left| \sum_{I \in \mathcal{T}_h} \langle Dw - \Pi_{k+1} Dw, v_0 - v_b \rangle_{\partial I} \right|
\]
\[
\leq C \sum_{I \in \mathcal{T}_h} \|Dw - \Pi_{k+1} Dw\|_{\partial I} \|v_0 - v_b\|_{\partial I}
\]
\[
\leq C \left( \sum_{I \in \mathcal{T}_h} h_I \|Dw - \Pi_{k+1} Dw\|_{\partial I}^2 \right)^{\frac{1}{2}} \left( \sum_{I \in \mathcal{T}_h} h_I^{-1} \|v_0 - v_b\|_{\partial I}^2 \right)^{\frac{1}{2}}
\]
\[
\leq C h^{k+2} |w|_{k+3} \|v\|.
\]
We have proved the lemma. \( \square \)
Theorem 4.1. Let \( u_h \in V_h \) be the SFWG finite element solution of (2.5). There exists a constant \( C \) such that
\[
\| Q_h u - u_h \| \leq C h^{k+2} |u|_{k+3}.
\]

Proof. By letting \( v = e_h \) in (3.2) and using (4.2), we have
\[
\| e_h \|^2 = (D_w e_h, D_w e_h) = |\ell(u, e_h)|
\leq C h^{k+2} |u|_{k+3} \| e_h \|.
\]
which implies (4.3). This completes the proof. \( \square \)

5. Error Estimates in the \( L^2 \) Norm

Recall \( e_h = \{e_0, e_h\} = Q_h u - u_h \). The considered dual problem seeks \( \Phi \in H^1_0(\Omega) \) satisfying
\[
-\Phi'' = e_0 \quad \text{in} \ \Omega.
\]
Assume that the following \( H^2 \)-regularity holds
\[
\| \Phi \|_2 \leq C \| e_0 \|.
\]

Theorem 5.1. Let \( u_h \in V_h \) be the SFWG finite element solution of (2.5). Assume that (5.2) holds true. Then there exists a constant \( C \) such that
\[
\| \Pi_k u - u_0 \| \leq C h^{k+3} |u|_{k+3}.
\]

Proof. Testing (5.1) by \( e_0 \) yields
\[
\| e_0 \|^2 = - (\Phi'', e_0).
\]
Letting \( u = \Phi \) and \( v = e_h \) in (3.5), we obtain
\[
- (\Phi'', e_0) = (D_w Q_h \Phi, D_w e_h) - \ell(\Phi, e_h).
\]
It follows from (5.4) and (5.5) that
\[
\| e_0 \|^2 = (D_w Q_h \Phi, D_w e_h) - \ell(\Phi, e_h).
\]
Using (5.6) and (3.2), we have
\[
\| e_0 \|^2 = \ell(u, Q_h \Phi) - \ell(\Phi, e_h).
\]
Using the Cauchy-Schwarz inequality and the trace inequality (4.1), we have
\[
|\ell(u, Q_h \Phi)| = \left| \sum_{I \in T_h} \langle Du - \Pi_{k+1} Du, \Pi_k \Phi - \Phi \rangle_{\partial I} \right|
\leq C \sum_{I \in T_h} \| Du - \Pi_{k+1} Du \|_{\partial I} \| \Pi_k \Phi - \Phi \|_{\partial I}
\leq C \left( \sum_{I \in T_h} h_I \| Du - \Pi_{k+1} Du \|_{\partial I} \right)^{1/2} \left( \sum_{I \in T_h} h_I^{-1} \| \Pi_k \Phi - \Phi \|_{\partial I}^2 \right)^{1/2}
\leq C h^{k+3} |u|_{k+3} \| \Phi \|_2.
\]
To estimate \( \ell(\Phi, e_h) \), we use (4.2) to obtain
\[
|\ell(\Phi, e_h)| \leq C h |\Phi|_2 \| e_h \|.
\]
By (5.9) and (5.3), we have
\begin{equation}
|\ell(\Phi, e_h)| \leq C h^{k+3} |u|_{k+3} \|\Phi\|_2.
\end{equation}
Combining (5.7), (5.8) and (5.10) yields
\begin{equation}
\|e_0\|^2 \leq C h^{k+3} |u|_{k+3} \|\Phi\|_2,
\end{equation}
which, combined with the regularity assumption (5.2), gives the desired optimal order error estimate (5.3).

6. A locally lifted $P_{k+2}$ solution

As the SFWG solution is two-order superconvergent, we lift the solution to a $P_{k+2}$ solution which converges at the optimal order.

Elementwise we compute a solution $\hat{u}_h \in P_{k+2}(I)$ by
\begin{align}
(D\hat{u}_h - D_w u_h, Dp) &= 0 \quad \forall p \in P_{k+2}(I) \setminus P_0(I), \\
(\hat{u}_h - u_0, p) &= 0 \quad \forall p \in P_{k+2}(I).
\end{align}
The square linear system of equations (6.1)–(6.2) has a unique solution because of (6.2), which, combined with the regularity assumption (5.2), gives the desired optimal order.

\begin{theorem}
Let $u \in H^1_0(\Omega) \cap H^{k+3}$ be the exact solution of (2.1)–(2.2). Let $u_h \in V_h$ be the SFWG finite element solution of (2.5). Let $\hat{u}_h \in P_{k+2}(I)$ be locally lifted solution of (6.1)–(6.2). Then there exists a constant $C$ such that
\begin{equation}
\|u - \hat{u}_h\| \leq C h^{k+3} |u|_{k+3}.
\end{equation}
\end{theorem}

\begin{proof}
(6.2) means that
$$
\Pi_0 \hat{u}_h = \Pi_0 u_h.
$$
We separate the error in to two parts.
$$
\|u - \hat{u}_h\|_0 \leq \|\Pi_0 (u - \hat{u}_h)\|_0 + \|(I - \Pi_0)(u - \hat{u}_h)\|_0.
$$
For the $P_0$ error, by (5.3), we have
$$
\|\Pi_0 (u - \hat{u}_h)\|_0 = \|\Pi_0 (\Pi_k u - u_h)\|_0 \leq \|\Pi_k u - u_h\|_0 \leq C h^{k+3} |u|_{k+3}.
$$
For the $P_0$-orthogonal error, we separate it further into two.
$$
\|(I - \Pi_0)(u - \hat{u}_h)\|_0 \leq C h \|\Pi_k u - u_h\|_0 + C h \|\Pi_k^2 u - \hat{u}\|_0 \\
\leq C h^{k+3} |u|_{k+3} + C h \|\Pi_k^2 u - \hat{u}\|_0.
$$
By (3.1), (6.1) and (4.3),
\begin{equation}
\|D(\Pi_k^2 u - \hat{u}_h)\|_0^2 \\
= (D(\Pi_k^2 u - u), D(\Pi_k^2 u - \hat{u}_h)) + (D(\Pi_k^2 u - \hat{u}_h), D(\Pi_k^2 u - u) + (D_w Q_h u - D_w u_h, D(\Pi_k^2 u - \hat{u}_h)) \\
\leq \left(\|D(\Pi_k^2 u - u)\|_0 + \|D(\Pi_k^2 u - u)\|_0 + \|Q_h u - u_h\|ight) \\
\cdot \|D(\Pi_k^2 u - \hat{u}_h)\|_0 \\
\leq C h^{k+3} |u|_{k+3} \|\Pi_k^2 u - \hat{u}_h\|_0.
\end{equation}
Combining above three inequalities, (6.3) is proved.

\[ \square \]

7. Numerical Experiments

We solve the 1D elliptic equation (2.1)-(2.2) on the domain \( \Omega = (0, 1) \). The function \( f \) is chosen so that the exact solution is

\[ u(x) = \sin(\pi x). \]

We let the first level grid consist of one interval. Each subsequent level of grid is the refinement by cutting each previous interval into two. We compute this example by 5 SFWG finite elements. The results are listed in Table 7.1. For all cases, we get two orders of superconvergence in both \( L^2 \) and \( H^1 \) norms.

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<th>( | \Pi_k u - u_0 |_0 )</th>
<th>rate</th>
<th>( | Q_k u - u_h | )</th>
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We then lift the \( P_k \) weak Galerkin finite element solution to a \( P_{k+2} \) solution. The results are listed in Table 7.2. In all cases, the lifted solution converges two orders higher than that of the original \( P_k \) solution.

Data Availability Statement

There is no data used in this research.
Table 7.2. Error profiles and convergence rates for solution (7.1).

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<td>by the $P_5$ solution and its lift</td>
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