Convergence analysis of space-time Jacobi spectral collocation method for solving time-fractional Schrödinger equations

Yin Yang\textsuperscript{a,}\textsuperscript{*}, Jindi Wang\textsuperscript{a}, Shangyou Zhang\textsuperscript{b}, Emran Tohidi\textsuperscript{c}

\textsuperscript{a} Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Key Laboratory of Intelligent Computing \\& Information Processing of Ministry of Education, School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China
\textsuperscript{b} Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
\textsuperscript{c} Department of Mathematics, Kosar University of Bojnord, Bojnord, P. O. Box 9415615458, Iran

\section*{Article Info}

\begin{itemize}
  \item Article history:
  \item Available online 5 July 2019
\end{itemize}

MSC:
33C45
35Q40
41A25
65M70

Keywords:
Convergence analysis
Time-fractional Schrödinger equation
Jacobi spectral-collocation method
Gauss-type quadrature

\section*{Abstract}

In this paper, the space-time Jacobi spectral collocation method (JSC Method) is used to solve the time-fractional nonlinear Schrödinger equations subject to the appropriate initial and boundary conditions. At first, the considered problem is transformed into the associated system of nonlinear Volterra integro partial differential equations (PDEs) with weakly singular kernels by the definition and related properties of fractional derivative and integral operators. Therefore, by collocating the associated system of integro-PDEs in both of the space and time variables together with approximating the existing integral in the equation using the Jacobi-Gauss-Type quadrature formula, then the problem is reduced to a set of nonlinear algebraic equations. We can consider solving the system by some robust iterative solvers. In order to support the convergence of the proposed method, we provided some numerical examples and calculated their $L^\infty$ norm and weighted $L^2$ norm at the end of the article.

\section*{1. Introduction}

The Schrödinger equation is a significant development in the theory of quantum mechanics [1]. This equation is a powerful differential structure that is related to the quantum systems changes over the temporal when the effects of quantum are considerable.

It should be noted that, the classic variant of this equation is stated in terms of the integer first order temporal and second order spatial partial derivatives. Because of non-locality of fractional differential operators, the fractional variant of Schrödinger equation can better describe the physical and chemical events in real world applications [2,3]. The notion of fractional Schrödinger equation was first introduced by Laskin [4], in which the Feynman path integral is extended. Because of nonlinearity, complexity and non-locality of the fractional Schrödinger equations, the classical methods for solving PDEs are not efficient. On the other hand, solution of such these equations has considerable importance for researchers to have a deterministic behavior for simulating the events accurately. Therefore, numerical and analytical schemes should be explored and extended to compute the solution of fractional Schrödinger equations successfully.

\textsuperscript{*} Corresponding author.
E-mail address: yangyinxu@xtu.edu.cn (Y. Yang).

https://doi.org/10.1016/j.amc.2019.06.003
0096-3003/© 2019 Elsevier Inc. All rights reserved.
In recent years, in order to solve the fractional Schrödinger equation, scientists proposed many analytical methods and numerical methods. Among the analytical methods, one can point out to the homotopy analysis method (HAM) [5]. Analytical methods are very straightforward for solving any nonlinear PDE, which do not need to discretization or linearization process. But, one of the disadvantages of these methods is being time consuming. Since, in these approaches the integration and differentiation processes are performed symbolically. Meanwhile, in numerical methods, we have two robust tools such as operational matrices of differentiation and Gaussian quadrature rules which accelerate (and reduce the computational time of) the process of differentiation and integration, respectively. Moreover, in some of the numerical approaches such as Krylov subspace methods [6], the no smooth solutions may be computed without any regularization tool, since the solution of these schemes depend on (and keep the behavior of) the right hand side functions of operator equations. Therefore, numerical methods are more attractive to be implemented by researchers. Among the numerical methods, one can point out to low order numerical methods such as finite elements [7], local discontinuous Galerkin (DG) [8,9] and reproducing kernel [10] techniques which was proposed for solving time-fractional Schrödinger equations recently. It should be noted that, since the fractional differential operators are global, it is better to use some global numerical approaches to solve the considered equations. Taking into account that, if for solving a PDE, space variables are discretized by global methods such as radial basis function collocation technique and time variable is localized by local schemes such as Galerkin finite element methods (FEMs) to solve nonlinear time fractional parabolic problems [11] and finite difference methods (FDMs), this may results to an unbalanced numerical scheme [12–14] which has spectral accuracy in space variable and low order algebraic accuracy in time variable. Therefore, it is desirable to propose a balanced numerical scheme that has spectral accuracy in both time and space directions.

Fractional differential equations (FDEs) are easy to be implemented with respect to the spectral Galerkin methods specially for solving nonlinear FDEs [15]. These methods are successfully applied for solving nonlinear fractional boundary value problems (BVPs) [16,18] and integral equations (IEs) [19,20] with a rigorous convergence analysis and also regularized for solving FDEs with nonsmooth solutions [21]. Also, in [22,23] the author solved fractional Schrödinger equations just in numerical implementation point of view via using spectral collocation method also used in [17]. In [24,25] a linearized L1-Galerkin finite element method and linearized compact alternating direction implicit (ADI) schemes are proposed to solve the multidimensional nonlinear time-fractional Schrödinger equation, respectively. To the authors’ knowledge, space-time Jacobi spectral collocation methods (that supported by a rigorous convergence analysis) for solving nonlinear time-fractional Schrödinger equations have had few results. This motivate us to propose a Jacobi spectral collocation scheme together with a full convergence analysis for solving time-fractional Schrödinger equations which is a balanced approach and has spectral accuracy in both time and space directions.

The time-fractional PDE we will considered as follows:

$$i\frac{\partial^\mu \psi(x, y, t)}{\partial t^\mu} = a_1 \frac{\partial^2 \psi(x, y, t)}{\partial x^2} + a_2 \frac{\partial^2 \psi(x, y, t)}{\partial y^2} + \gamma |\psi(x, y, t)|^2 \psi(x, y, t) + \delta R(x, y, t),$$

$$0 < \mu < 1, \quad (x, y, t) \in \Omega_1 \times \Omega_2 \times \Omega_3,$$  

(1.1)

where $\Omega_1 = [0, L_1]$, $\Omega_2 = [0, L_2]$ and $\Omega_3 = [0, T]$, with the initial time conditions,

$$\psi(x, y, 0) = \zeta_1(x, y), \quad (x, y) \in \Omega_1 \times \Omega_2,$$

(1.2)

and two-dimensional boundary space conditions,

$$\psi(0, y, t) = \zeta_2(y, t), \quad \psi(L_1, y, t) = \zeta_3(y, t), \quad (y, t) \in \Omega_2 \times \Omega_3,$$

$$\psi(x, 0, t) = \zeta_4(x, t), \quad \psi(x, L_2, t) = \zeta_5(x, t), \quad (x, t) \in \Omega_1 \times \Omega_3,$$

while $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$ and $R(x, y, t)$ are given functions.

It should be noted that the definition of Caputo fractional derivative as follows,

$$\frac{\partial^\mu}{\partial t^\mu} (g(t)) = \begin{cases} \frac{\partial^m g(t)}{\partial t^m}, & \text{if } m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\mu)} \int_0^t \frac{g(m)(s)}{(t-s)^{m-\mu+1}} ds, & 0 < m < \mu < m, \end{cases}$$

where $\frac{\partial^m}{\partial t^m} (\cdot)$ denotes the of order $\mu$ derivative.

The definition of Riemann-Liouville (R-L) fractional integral as follows, indicated by $I^\mu_t$,

$$I^\mu_t (g(t)) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} g(s) ds, \quad s > 0.$$  

Moreover,

$$I^\mu_t \left( \frac{\partial^\mu}{\partial t^\mu} g(t) \right) = g(t) - \sum_{i=0}^{m-1} g^{(i)}(0) \frac{t^i}{i!}, \quad m - 1 < \mu < m.$$  

In the next section, we proposed space-time Jacobi spectral collocation method to solve the Eq. (1.1). Some useful lemmas and preliminaries are provided in Section 3, such as the error estimation of interpolation by Jacobi-Gauss points. Rigorous
convergence analysis associated to the used numerical method in weighted $L^2$ norms and $L^\infty$ norms is provided in Section 4. In Section 5, algorithm implementation and numerical results are given. Finally, in the last section, some concluding remarks and considered problems for the future research works are stated.

2. Jacobi spectral collocation methods

We first split all of the known and unknown functions into real and imaginary parts and transform the basic equation into a system of coupled time-fractional PDEs in Caputo sense. In the next step, we impose the Riemann-Liouville fractional integral operator on both sides of the equations to change this system into a nonlinear system of Volterra integro-PDEs with weakly singular kernels that contains initial conditions. Finally, both of the space and time variables are collocated and the existing integrals are approximated by the powerful Gaussian quadrature rules to change the considered problem to a set of nonlinear algebraic equations. We consider solving the system by some robust iterative solvers.

Now, one can split all of the known and unknown functions into real and imaginary parts as follows

\[
\psi = u + iv, \quad R = f + ig, \quad \zeta_1 = g_1 + ig_2, \\
\zeta_2 = g_3 + ig_4, \quad \zeta_3 = g_5 + ig_6, \\
\zeta_4 = g_7 + ig_8, \quad \zeta_5 = g_9 + ig_{10},
\]

(2.1)

where, $u$, $v$, $f$, $g$, $g_1$, $g_2$, $g_3$, $g_4$, $g_5$, $g_6$, $g_7$, $g_8$, and $g_{10}$ are the real functions.

By considering assumptions of (2.1), the basic Eq. (1.1) can be rewritten in the following form

\[
i \left( \frac{\partial^\mu u}{\partial t^\mu} - a_1 \frac{\partial^2 u}{\partial x^2} - a_2 \frac{\partial^2 v}{\partial y^2} - \gamma (u^2 + v^2)v - \delta g \right) - \left( \frac{\partial^\mu v}{\partial t^\mu} + a_1 \frac{\partial^2 v}{\partial x^2} + a_2 \frac{\partial^2 u}{\partial y^2} + \gamma (u^2 + v^2)u + \delta f \right) = 0.
\]

(2.2)

Therefore, the aforementioned complex equation can be transformed into the following coupled real-time-fractional PDEs

\[
\begin{align*}
\frac{\partial^\mu u}{\partial t^\mu} &= a_1 \frac{\partial^2 v}{\partial x^2} + a_2 \frac{\partial^2 v}{\partial y^2} + \gamma (u^2 + v^2)v + \delta g, \\
-\frac{\partial^\mu v}{\partial t^\mu} &= a_2 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial y^2} + \gamma (u^2 + v^2)u + \delta f,
\end{align*}
\]

(2.3)

the initial boundary conditions became

\[
\begin{align*}
&u(x, y, 0) = g_1(x, y), \quad v(x, y, 0) = g_2(x, y), \quad (x, y) \in \Omega_1 \times \Omega_2, \\
u(0, y, t) = g_3(y, t), \quad v(0, y, t) = g_4(y, t), \quad (y, t) \in \Omega_2 \times \Omega_3, \\
u(0, y, t) = g_5(y, t), \quad v(0, y, t) = g_6(y, t), \quad (y, t) \in \Omega_2 \times \Omega_3, \\
u(x, 0, t) = g_7(x, t), \quad v(x, 0, t) = g_8(x, t), \\
u(x, 1, t) = g_9(x, t), \quad v(x, 1, t) = g_{10}(y, t). \quad (x, t) \in \Omega_1 \times \Omega_3.
\end{align*}
\]

(2.4)

Taking into account that, Eqs. (2.3) and (2.4) are equal to Eqs. (1.1) and (1.2). So, instead of solving (1.1) and (1.2) numerically, we compute numerical solution of (2.3) and (2.4) using the Jacobi spectral collocation method. Before the space-time collocation method, first using the R-L fractional integral of order $\mu$, we will transform (2.3) into the associated system of weakly singular Volterra integro-PDEs,

\[
\begin{align*}
&u(x, y, t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} k_1(x, y, \tau, u(x, y, \tau), v(x, y, \tau)) d\tau + \delta \tilde{g}(x, y, t) + g_1(x, y), \\
v(x, y, t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} k_2(x, y, \tau, u(x, y, \tau), v(x, y, \tau)) d\tau + \delta \tilde{f}(x, y, t) + g_2(x, y),
\end{align*}
\]

(2.5)

where

\[
\begin{align*}
k_1(x, y, \tau, u(x, y, \tau), v(x, y, \tau)) &= a_1 \frac{\partial^2 v(x, y, \tau)}{\partial x^2} + a_2 \frac{\partial^2 v(x, y, \tau)}{\partial y^2} + \gamma (u^2(x, y, \tau) + v^2(x, y, \tau))v(x, y, \tau), \\
k_2(x, y, \tau, u(x, y, \tau), v(x, y, \tau)) &= -a_1 \frac{\partial^2 u(x, y, \tau)}{\partial x^2} - a_2 \frac{\partial^2 u(x, y, \tau)}{\partial y^2} - \gamma (u^2(x, y, \tau) + v^2(x, y, \tau))u(x, y, \tau), \\
\tilde{g}(x, y, t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} g(x, y, \tau) d\tau, \\
\tilde{f}(x, y, t) &= -\frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} f(x, y, \tau) d\tau.
\end{align*}
\]

(2.6)

Since (2.5) has weakly singular kernels around $t = 0^+$, the numerical process of (2.5) may be difficult. To apply the orthogonal Jacobi polynomials for solving equations of (2.5), we should consider the variable transformations as follows,
with following conditions

\[
x = \frac{L_1}{2} (1 + \bar{x}), \quad \bar{x} = \frac{2x}{L_1} - 1, \quad \bar{x} \in [-1, 1],
\]

\[
y = \frac{L_2}{2} (1 + \bar{y}), \quad \bar{y} = \frac{2y}{L_2} - 1, \quad \bar{y} \in [-1, 1],
\]

\[
t = \frac{T}{2} (1 + \tilde{t}), \quad \tilde{t} = \frac{2t}{T} - 1, \quad t \in [-1, 1],
\]

\[
\tau = \frac{T}{2} (1 + s), \quad s = \frac{2\tau}{T} - 1, \quad s \in [-1, t].
\]

For the sake of simplicity, we still use \( x, y, t \) to indicate \( \bar{x}, \bar{y}, \tilde{t} \), therefore, equations of (2.5) should be rewritten with the following form

\[
\bar{u}(x, y, t) = \int_{-1}^{t} (t - s)^{\mu - 1} \tilde{k}_1(x, y, s, \bar{u}(x, y, s), \bar{v}(x, y, s)) ds + \delta \bar{g}(x, y, t) + \bar{g}_1(x, y),
\]

\[
\bar{v}(x, y, t) = \int_{-1}^{t} (t - s)^{\mu - 1} \tilde{k}_2(x, y, s, \bar{u}(x, y, s), \bar{v}(x, y, s)) ds + \delta \bar{f}(x, y, t) + \bar{g}_2(x, y),
\]

(2.7)

with the boundary conditions

\[
\bar{u}(-1, y, t) = \bar{g}_3(y, t), \quad \bar{u}(1, y, t) = \bar{g}_5(y, t),
\]

\[
\bar{u}(x, -1, t) = \bar{g}_7(x, t), \quad \bar{u}(x, 1, t) = \bar{g}_9(x, t),
\]

\[
\bar{v}(-1, y, t) = \bar{g}_4(y, t), \quad \bar{v}(1, y, t) = \bar{g}_6(y, t),
\]

\[
\bar{v}(x, -1, t) = \bar{g}_8(x, t), \quad \bar{v}(x, 1, t) = \bar{g}_{10}(y, t).
\]

where

\[
\bar{u}(x, y, t) = u\left(\frac{L_1}{2} (1 + x), \frac{L_2}{2} (1 + y), \frac{T}{2} (1 + t)\right),
\]

\[
\bar{v}(x, y, t) = v\left(\frac{L_1}{2} (1 + x), \frac{L_2}{2} (1 + y), \frac{T}{2} (1 + t)\right),
\]

\[
\bar{g}(x, y, t) = \bar{g}\left(\frac{L_1}{2} (1 + x), \frac{L_2}{2} (1 + y), \frac{T}{2} (1 + t)\right),
\]

\[
\bar{f}(x, y, t) = \bar{f}\left(\frac{L_1}{2} (1 + x), \frac{L_2}{2} (1 + y), \frac{T}{2} (1 + t)\right),
\]

\[
\bar{g}_i(x, y) = g_i\left(\frac{L_1}{2} (1 + x), \frac{L_2}{2} (1 + y)\right), \quad i = 1, 2,
\]

\[
\bar{g}_i(y, t) = g_i\left(\frac{L_2}{2} (1 + y), \frac{T}{2} (1 + t)\right), \quad i = 3, 4, 5, 6,
\]

\[
\bar{g}_i(x, t) = g_i\left(\frac{L_1}{2} (1 + x), \frac{T}{2} (1 + t)\right), \quad i = 7, 8, 9, 10,
\]

\[
\tilde{k}_1(x, y, s, \bar{u}(x, y, s), \bar{v}(x, y, s)) = \frac{1}{\Gamma(\mu)} \left(\frac{T}{2}\right)^{\mu} \left(a_1\left(\frac{2}{L_1}\right)^2 \frac{\partial^2 \bar{u}}{\partial x^2} + a_2\left(\frac{2}{L_2}\right)^2 \frac{\partial^2 \bar{v}}{\partial y^2} + \gamma (\bar{u}^2 + \bar{v}^2) \bar{v}\right),
\]

\[
\tilde{k}_2(x, y, s, \bar{u}(x, y, s), \bar{v}(x, y, s)) = \frac{1}{\Gamma(\mu)} \left(\frac{T}{2}\right)^{\mu} \left(-a_1\left(\frac{2}{L_1}\right)^2 \frac{\partial^2 \bar{u}}{\partial x^2} - a_2\left(\frac{2}{L_2}\right)^2 \frac{\partial^2 \bar{v}}{\partial y^2} - \gamma (\bar{u}^2 + \bar{v}^2) \bar{u}\right).
\]

Now, we can collocate the variable \( t \) of equations of (2.7) at the Jacobi Gauss nodes corresponding to \( \theta = \vartheta = -\mu \), in which \( a^\vartheta (t) = (1 - t)^{-\mu} (1 + t)^{-\mu} = (1 - t^2)^{-\mu} \), for \( 0 \leq t \leq M \),

\[
\bar{u}(x, y, t_i) = \int_{-1}^{t_i} (t_i - s)^{\mu - 1} \tilde{k}_1(x, y, s, \bar{u}(x, y, s), \bar{v}(x, y, s)) ds + \delta \bar{g}(x, y, t_i) + \bar{g}_1(x, y),
\]

\[
\bar{v}(x, y, t_i) = \int_{-1}^{t_i} (t_i - s)^{\mu - 1} \tilde{k}_2(x, y, s, \bar{u}(x, y, s), \bar{v}(x, y, s)) ds + \delta \bar{f}(x, y, t_i) + \bar{g}_2(x, y),
\]

(2.8)

We should apply the following change of variables for implementing Jacobi Gauss quadrature rule

\[
s(\theta) = s_t(\theta) = \frac{1 + t_i}{2} \vartheta + \frac{t_i - 1}{2}, \quad 0 \leq l \leq M, \quad \theta \in [-1, 1].
\]

(2.9)
Therefore,
\[
\bar{u}(x, y, t_l) = \left( \frac{1 + t_l}{2} \right)^\mu \int_{l-1}^1 (1 - \theta)^{\mu-1} \tilde{k}_1(x, y, s(\theta), \bar{u}(x, y, s(\theta))), \bar{v}(x, y, s(\theta))) d\theta + \delta\bar{g}(x, y, t_l) + \bar{g}_1(x, y),
\]
\[
\bar{v}(x, y, t_l) = \left( \frac{1 + t_l}{2} \right)^\mu \int_{l-1}^1 (1 - \theta)^{\mu-1} \tilde{k}_2(x, y, s(\theta), \bar{u}(x, y, s(\theta))), \bar{v}(x, y, s(\theta))) d\theta + \delta\bar{f}(x, y, t_l) + \bar{g}_2(x, y). \tag{2.10}
\]
Applying Gaussian quadrature formula to change the integral parts of the above formulas to a summation form,
\[
\int_{l-1}^1 (1 - \theta)^{\mu-1} \tilde{k}_1(x, y, s(\theta), \bar{u}(x, y, s(\theta))), \bar{v}(x, y, s(\theta))) d\theta \approx \sum_{k=0}^1 \tilde{k}_1(x, y, s(\theta_k), \bar{u}(x, y, s(\theta_k))), \bar{v}(x, y, s(\theta_k))) \omega_k^{\mu-1}, \tag{2.11}
\]
\[
\int_{l-1}^1 (1 - \theta)^{\mu-1} \tilde{k}_2(x, y, s(\theta), \bar{u}(x, y, s(\theta))), \bar{v}(x, y, s(\theta))) d\theta \approx \sum_{k=0}^1 \tilde{k}_2(x, y, s(\theta_k), \bar{u}(x, y, s(\theta_k))), \bar{v}(x, y, s(\theta_k))) \omega_k^{\mu-1},
\]
where \(\{\theta_k\}_{k=0}^L\) are Jacobi-Gauss-Lobatto nodes and \(\{\omega_k^{\mu-1}\}_{k=0}^L\) is the weight function in \([-1, 1]\) and \(L \geq M\), \(\omega^{\mu-1}(t) = (1 - t)^{\mu-1}\).

Let \(\tilde{u}(x, y)\) and \(\tilde{v}(x, y)\) denote to the \(\bar{u}(x, y, t_l)\) and \(\bar{v}(x, y, t_l)\), respectively. One can approximate \(\bar{u}(x, y, t)\) and \(\bar{v}(x, y, t)\) by their Lagrange interpolation polynomials in the following form
\[
\bar{u}(x, y, t_l) \approx \tilde{u}(x, y, t_l) = \sum_{l=0}^M \tilde{u}^l(x, y) F_l(t),
\]
\[
\bar{v}(x, y, t_l) \approx \tilde{v}(x, y, t_l) = \sum_{l=0}^M \tilde{v}^l(x, y) F_l(t),
\]
where \(F_l(t)\) is the lth Lagrange polynomial associated to the \(t_l\) for \(0 \leq l \leq M\).

By using the aforementioned approximations together with the implemented Gauss quadrature rule, (2.10) can be reduced as follows \((0 \leq l \leq M)\),
\[
\tilde{u}(x, y) = \left( \frac{1 + t_l}{2} \right)^\mu \sum_{k=0}^L \tilde{k}_1(x, y, s(\theta_k), \bar{u}(x, y, s(\theta_k))), \bar{v}(x, y, s(\theta_k))) \omega_k^{\mu-1} + \delta\tilde{g}(x, y, t_l) + \tilde{g}_1(x, y),
\]
\[
\tilde{v}(x, y) = \left( \frac{1 + t_l}{2} \right)^\mu \sum_{k=0}^L \tilde{k}_2(x, y, s(\theta_k), \bar{u}(x, y, s(\theta_k))), \bar{v}(x, y, s(\theta_k))) \omega_k^{\mu-1} + \delta\tilde{f}(x, y, t_l) + \tilde{g}_2(x, y). \tag{2.12}
\]
For applying JSC method to the space variable, we use the Legendre-Gauss-Lobatto points \(\{\xi_i\}_{i=0}^{N_1}, \{\eta_j\}_{j=0}^{N_2}\) on the interval \([-1, 1]\) corresponding to \(\omega^{0,0}(x, y) = (1 - x)^{0}(1 + x)^{0}(1 - y)^{0}(1 + y)^{0} = 1\). Then, (2.12) has the following form \((1 \leq i \leq N_1 - 1, 1 \leq j \leq N_2 - 1)\),
\[
\tilde{u}(x_i, y_j) = \left( \frac{1 + t_l}{2} \right)^\mu \sum_{k=0}^L \tilde{k}_1(x_i, y_j, s(\theta_k), \bar{u}(x_i, y_j, s(\theta_k))), \bar{v}(x_i, y_j, s(\theta_k))) \omega_k^{\mu-1} + \delta\tilde{g}(x_i, y_j, t_l) + \tilde{g}_1(x_i, y_j),
\]
\[
\tilde{v}(x_i, y_j) = \left( \frac{1 + t_l}{2} \right)^\mu \sum_{k=0}^L \tilde{k}_2(x_i, y_j, s(\theta_k), \bar{u}(x_i, y_j, s(\theta_k))), \bar{v}(x_i, y_j, s(\theta_k))) \omega_k^{\mu-1} + \delta\tilde{f}(x_i, y_j, t_l) + \tilde{g}_2(x_i, y_j). \tag{2.13}
\]
Again, let \(\tilde{u}_{ij}^l\) and \(\tilde{v}_{ij}^l\) denote to the \(\tilde{u}(x_i, y_j, t_l)\) and \(\tilde{v}(x_i, y_j, t_l)\), respectively. One can consider the Lagrange interpolation polynomial for both of the functions \(\tilde{u}\) and \(\tilde{v}\) and write
\[
\tilde{u}(x, y, t_l) \approx \tilde{u}_{N_1N_2}(x, y, t_l) = \sum_{l=0}^M \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \tilde{u}_{ij}^l H_i(x) H_j(y) F_l(t) = \sum_{l=0}^M \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \tilde{u}_{ij}^l H_i(x) H_j(y) F_l(t) + \sum_{l=0}^M \sum_{i=0}^{N_1} \tilde{g}_4 H_0(x) H_i(x) F_l(t)
\]
\[
+ \tilde{g}_4 H_0(x) H_i(x) F_l(t) + \sum_{l=0}^M \sum_{i=0}^{N_1} \tilde{g}_6 H_0(x) H_i(x) F_l(t).
\]
\[
\tilde{v}(x, y, t_l) \approx \tilde{v}_{N_1N_2}(x, y, t_l) = \sum_{l=0}^M \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \tilde{v}_{ij}^l H_i(x) H_j(y) F_l(t) = \sum_{l=0}^M \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \tilde{v}_{ij}^l H_i(x) H_j(y) F_l(t) + \sum_{l=0}^M \sum_{i=0}^{N_1} \tilde{g}_6 H_0(x) H_i(x) F_l(t)
\]
\[
+ \tilde{g}_6 H_0(x) H_i(x) F_l(t) + \sum_{l=0}^M \sum_{i=0}^{N_1} \tilde{g}_10 H_0(x) H_i(x) F_l(t). \tag{2.14}
\]
where $H_i(x)$ is the $i$th Lagrange polynomial corresponding to $x_i$ for $0 \leq i \leq N_1$, where $H_j(y)$ is the $j$th Lagrange polynomial corresponding to $y_j$ for $0 \leq j \leq N_2$.

Therefore, the full discrete system of algebraic equations arising from the space-time Jacobi spectral collocation method to solve (1.1) and (1.2) can be stated as

$$(1.1) \quad \sum_{k=0}^{l} \tilde{u}_{ij}(k) + \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \sum_{m=0}^{M} \tilde{u}_{ij}(n_1, n_2, (x_i)H_{n_1}(y_j)E_m(s(\theta_k))) = \delta \tilde{g}(x_i, y_j, t_i) + \tilde{g}_1(x_i, y_j),$$

$$(1.2) \quad \sum_{k=0}^{l} \tilde{v}_{ij}(k) + \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \sum_{m=0}^{M} \tilde{v}_{ij}(n_1, n_2, (x_i)H_{n_1}(y_j)E_m(s(\theta_k))) = \delta \tilde{f}(x_i, y_j, t_i) + \tilde{g}_2(x_i, y_j).$$

We can solve the question about the above-mentioned system of nonlinear algebraic equations can be solved using Newton Raphson iterative method. The practical implementation can be done by a well-known command "fsolve" in matlab or maple softwares.

3. Some preliminaries and useful Lemmas

In the next section, we will introduction the convergence analysis associated to the proposed space-time JSC method to solve (1.1) and (1.2). Therefore, we need to some definitions and lemmas for stabilishing the proof of the main theorem. These lemmas include error of the Gauss quadrature rules, estimation of the interpolation errors, Lebesgue constant corresponding to the Legendre series, and finally the Gronwall inequality.

**Definition 3.1** [26]. Let $I$ be a bounded interval in $\mathbb{R}$ and $L^p(I)$ is a measurable function space, where $1 \leq p < \infty$, that is, for $\forall u \in I \to \mathbb{R}$, if $\int_a^b |u(x)|^p dx < \infty$, we can define its norm as

$$\|u\|_{L^p(I)} = \left( \int_a^b |u(x)|^p dx \right)^{\frac{1}{p}}. \quad (3.1)$$

Obviously, it is a Banach space.

**Definition 3.2** [26]. Let $I$ be a bounded interval in $\mathbb{R}$, one can define $H^m(I)$, if it satisfied that for $u \in L^2(I)$, we can always find the function $v \in L^2(I)$ so that $\frac{d^ku}{dx^k} = v$. In other words,

$$H^m(I) = \{ u \in L^2(I) : \frac{d^k u}{dx^k} \in L^2(I), \quad \text{for} \quad 0 \leq k \leq m \}. \quad (3.2)$$

define the inner product in $H^m(I)$ as,

$$(u, v)_m = \sum_{k=0}^{m} \int_a^b \frac{d^k u}{dx^k}(x) \frac{d^k v}{dx^k}(x) dx, \quad (3.3)$$

obviously, $H^m(I)$ is a Hilbert space. Norm of function can be given,

$$\|v\|_{H^m(I)} = \left( \sum_{k=0}^{m} \left| \frac{d^k u}{dx^k} \right|^2 \right)^{\frac{1}{2}}. \quad (3.4)$$

**Lemma 3.1** [27]. (Integration error) Assume that a set of quadrature nodes of three types of Gaussian formulas include Gauss or Gauss-Radau or Gauss-Lobatto, for the function $u$, $\phi$, where $u \in H^m(I)$, $\phi \in \mathcal{P}_N$ (the set of all algebraic polynomials of degree $\leq n$), and for some $m \geq 1$, with $I := (-1, 1)$, we can find a constant $C$ independent of $N$ such that

$$\left| \int_{-1}^{1} u(x) \phi(x) dx - (u, \phi)_N \right| \leq C N^{-m} \| u \|_{H^m(I)} \| \phi \|_{L^2(I)}. \quad (3.5)$$
where
\[ |u|_{H^m,N}(I) = \left( \sum_{j=\min(m,N+1)}^{m} \|u^{(j)}\|_{L^2(I)}^{2} \right)^{\frac{1}{2}}, \]
\[(u, \phi)_N = \sum_{j=0}^{N} \omega_j u(x_j) \phi(x_j). \]

**Lemma 3.2** [27]. For \( u \in H^{m,N}(I) \) and by \( I_N^{\alpha,\beta} u \) denote its interpolation polynomial, where \( \{x_i\}_{i=0}^{N} \) is a set of Jacobi-Gauss points, namely,
\[ I_N^{\alpha,\beta} u = \sum_{i=0}^{N} u(x_i) F_i(x), \]
then, the following estimates hold
\[ \|u - I_N^{\alpha,\beta} u\|_{\alpha,\beta}^{m,N}(I) \leq CN^{-m} |u|_{H^{m,N}(I)}, \]
\[ \|u - I_N^{\alpha,\beta} u\|_{L^\infty(I)} \leq \begin{cases} CN^\frac{1}{2} - m |u|_{H^{m,N}(I)}, & -1 \leq \alpha, \beta < -\frac{1}{2}, \\ CN^{1+\epsilon - m} \log N |u|_{H^{m,N}(I)}, & \text{otherwise}, \end{cases} \]
where \( F_i(x) \) \( i = 0, 1, \ldots, N \), are the Lagrange interpolation basis functions associated with the Jacobi collocation points \( \{x_i\}_{i=0}^{N} \),
where \( \epsilon = \max(\alpha, \beta) \) and \( \alpha^c = \alpha^{1/2} - \frac{1}{2} \) denotes the Chebyshev weight function.
For \( u \in H^{m,N}(I) \) and \( I_N u \) denote its interpolation polynomial, where \( \{x_i\}_{i=0}^{N} \) is a set of Legendre-Gauss points. Namely,
\[ I_N u = \sum_{i=0}^{N} u(x_i) F_i(x). \]

Then, the following estimates hold [20]
\[ \|u - I_N u\|_{L^2(I)} \leq CN^{-m} |u|_{H^{m,N}(I)}, \]
\[ \|u - I_N u\|_{L^\infty(I)} \leq CN^{\frac{1}{2} - m} |u|_{H^{m,N}(I)}. \]

**Lemma 3.3** [26]. For every function \( u \), bounded, we can find a constant \( C \), independent of \( u \), satisfied
\[ \sup_N \left\| \sum_{i=0}^{N} u(x_i) F_i(x) \right\|_{L^2(I)} \leq \max_{x \in [-1,1]} |u(x)|, \]
where \( F_i(x) \) is the Lagrange interpolation basis function, and \( \{x_i\}_{i=0}^{N} \) is the Jacobi collocation points.

From Mastroianni and Occorso [28], we get the following results related to the Lebesgue constant, for the Lagrange interpolation polynomial associated with the zero of the Jacobian polynomial,

**Lemma 3.4** [29]. Suppose \( \{F_i(x)\}_{i=0}^{N} \) is the N Lagrangian polynomial taken at the Gaussian node of the Jacobi polynomial, then,
\[ \|I_N^{\alpha,\beta}\|_{L^\infty(I)} \leq \max_{x \in [-1,1]} \sum_{j=0}^{N} |F_j(x)| \]
\[ = \begin{cases} O(\log N), & -1 < \alpha, \beta < \frac{1}{2}, \\ O(N^\epsilon), & \epsilon = \max(\alpha, \beta), \text{ otherwise} \end{cases} \]

The following lemma is a generalization of the singular kernel of the Gronwall lemma, which is very important in partial differential equations, and can be found in the references, for example, in [30], to a large extent affected the main results of our research.

**Lemma 3.5** [31]. Suppose \( u \) and \( v \) are non-negative local integrable functions defined on the interval \([-1,1] \), \( L \geq 0, 0 < \mu < 1 \). In addition,
\[ u(x) \leq v(x) + \int_{-1}^{x} (x - \tau)^{-\mu} u(\tau) d\tau, \]
we can find a constant \( C = C(\mu) \) satisfied,
\[ u(x) \leq v(x) + CL \int_{-1}^{x} (x - \tau)^{-\mu} v(\tau) d\tau, \text{ for } -1 \leq x < 1. \]
If $E(x)$ satisfies
\[ E(x) \leq L \int_{-1}^{x} E(s)ds + J(x), \quad -1 < x < 1, \]
where $E(x)$ is a nonnegative integrable function, $J(x)$ is an integrable function, then
\[ \|E\|_{L^\infty([-1,1])} \leq C\|J\|_{L^\infty([-1,1])}, \]
\[ \|E\|_{L^r_{\omega,\beta}([-1,1])} \leq C\|J\|_{L^r_{\omega,\beta}([-1,1])}. \]

Lemma 3.6 [32]. \( \forall \nu \in C^{r}[−1,1], \) you can always find the polynomial function \( T_{\nu} \nu \in \mathcal{P}_{N} \), making it for non-negative integers \( r \) and \( \kappa \in (0, 1) \), satisfying a constant \( C_{r, \kappa} > 0 \),
\[ \|v - T_{\nu} \nu\|_{L^\nu(\tau)} \leq C_{r, \kappa} N^{-\kappa} \|v\|_{r, \kappa}, \]
the standard norm in which we define \( C^{r}[−1,1] \) is \( \| \cdot \|_{r, \kappa} \), and \( T_{\nu} \) is a linear operator from \( C^{r}[−1,1] \) to \( \mathcal{P}_{N} \).

Lemma 3.7 [32]. Let \( \kappa \in (0, 1) \) and defined that
\[ (\mathcal{M}v)(x) = \int_{-1}^{x} (x - \tau)^{-\kappa} K(x, \tau) v(\tau) d\tau. \]
Then, for \( \forall v \in C[−1,1] \), you can always find a positive constant \( C \) such that
\[ \left| (\mathcal{M}v(x')) - (\mathcal{M}v(x'')) \right| \leq C \max_{x \in [-1,1]} |v(x)|, \]
assume that \( 0 < \kappa < 1 - \mu \), for any \( x', x'' \in [-1,1] \) and \( x' \neq x'' \). This established that
\[ \|\mathcal{M}v\|_{0, \kappa} \leq C \max_{x \in [-1,1]} |v(x)|, \quad 0 < \kappa < 1 - \mu. \]

4. Convergence analysis of Jacobi spectral collocation method

The main purpose of this section is to make a specific convergence analysis for the numerical scheme presented in the previous article. According to the convergence result, we show that the Jacobi collocation method we use is the approximate solution obtained in (2.15) exponentially approximates the exact solution. First, we will derive the error estimate for the function in \( L^\infty \)-norm.

Here, we assume that the kernel function \( K(x, s, U(s)) \) has the two following properties which are required for the proof of the convergence analysis:
1. The Lipschitz property, in other words
\[ |K(x, s, \tilde{U}(s)) - K(x, s, U(s))| \leq L_{K} |\tilde{U} - U(s)|, \quad \forall \tilde{U}, U \in [-1,1] \]
where \( U(s) = [\bar{u}(s), v(s)]^T \) and \( \tilde{U}(s) = [\tilde{u}(s), \tilde{v}(s)]^T; \)
2. \( K(x, s, 0) = 0_{2 \times 1} \).

Since the one-dimensional and two-dimensional convergence methods and results of the space are the same, here we make the proof process concise, and the following is only for the rigor of the one-dimensional Schrödinger equation convergence analysis.

Theorem 1. Suppose \( U(x, t) \) is an exact solution of the Eq. (2.7) and sets \( U^{M}(x, t) = \sum_{j=0}^{M} U^{j}(x)F_{j}(t) \) is the discrete solution of the Eq. (2.12) in the time direction, \( \epsilon_{t} = \max \{|\alpha_{t}|, \beta_{t}| \}. \) Then when \( M \) is big enough,
\[ \|U(x, t) - U^{M}(x, t)\|_{L^\infty} \leq \begin{cases} CM^{-\kappa}[\log MK^{*} + CM^{1+\epsilon-\kappa} \log MU^{*}], & -1 < \alpha \beta \leq -\frac{1}{2}, \\
CM^{\frac{\epsilon}{\kappa}} - M^{-\kappa}K^{*} + CM^{1+\epsilon-s-\kappa} \log MU^{*}, & -\frac{1}{2} \leq \epsilon < \mu - \frac{1}{2}. \end{cases} \]
\[ \|U(x, t) - U^{M}(x, t)\|_{L^{1}} \leq \begin{cases} CM^{-\kappa}(K^{*} + U^{*}) + CM^{1+\epsilon-\kappa} \log MK^{*} + CM^{1+\epsilon-s-\kappa} \log MU^{*}, & -1 < \alpha \beta \leq -\frac{1}{2}, \\
CM^{-\kappa}(K^{*} + U^{*}) + CM^{1+\epsilon-s-\kappa} \log TU K^{*} + CM^{1+\epsilon-\kappa-s} \log MU^{*}, & -\frac{1}{2} \leq \epsilon < \mu - \frac{1}{2}. \end{cases} \]

Where
\[ K^{*} = \max_{x \in [1]} |K(x, s(\theta), U^{M}(z, s(\theta)))|_{H_{\infty}^{M}(i)}, \quad U^{*} = |U(x, t)|_{H_{\infty}^{M}(i)}. \]

Proof. According to Lemma 1
\[ (K(x, s(\theta), U^{M}(x, s(\theta))))_{N, s} := \sum_{j=0}^{N} K(x, s(\theta), U^{M}(x, s(\theta))) \omega^{j}k^{j-1.0}, \]
\[ (K(x, s(\theta), U^{M}(x, s(\theta))))_{N, s} := \sum_{j=0}^{N} K(x, s(\theta), U^{M}(x, s(\theta))) \omega^{j}k^{j-1.0}, \]
\[ (K(x, s(\theta), U^{M}(x, s(\theta))))_{N, s} := \sum_{j=0}^{N} K(x, s(\theta), U^{M}(x, s(\theta))) \omega^{j}k^{j-1.0}, \]
then, the numerical schemes (2.12) can be written as
\[ U^t(x) - \left( 1 + \frac{t_j}{2} \right)^{\mu - 1} \int_{t_j-1}^{t_j-1} (1 - \theta)^{\mu - 1} K(x, s(\theta), U^M(x, s(\theta))) d\theta = \delta G(x, t_j) + F(x), \] (4.4)
which gives
\[ U^t(x) - \left( 1 + \frac{t_j}{2} \right)^{\mu - 1} \int_{t_j-1}^{t_j-1} (1 - \theta)^{\mu - 1} K(x, s(\theta), U^M(x, s(\theta))) d\theta = \delta G(x, t) + F(x) + J_1(x). \] (4.5)
where
\[ J_1(x) = \left( 1 + \frac{t_j}{2} \right)^{\mu - 1} \int_{t_j-1}^{t_j-1} (1 - \theta)^{\mu - 1} K(x, s(\theta), U^M(x, s(\theta))) d\theta. \] (4.6)
then, we have
\[ \| J_1(x) \| \leq C N^{-m} \left( \left( 1 + \frac{t_j}{2} \right)^{\mu - 1} \right) |K(x, s(\theta), U^M(x, s(\theta)))|_{H^m_{\omega}(\theta)} \leq C N^{-m} |K(x, s(\theta), U^M(x, s(\theta)))|_{H^m_{\omega}(\theta)}. \] (4.7)
On the other hand, (4.5) can be rewritten as follows:
\[ U^t(x) - \int_{t_j}^{t_j-1} (t_j - s)^{\mu - 1} K(x, s, U^M(x, s)) ds = \delta G(x, t_j) + F(x). \] (4.8)
Multiplying \( f_i(t) \) on the both sides of (4.8) and summing up from 0 to \( M \) yield,
\[ U^M(x, t) - I_M \left( \int_{t_j}^{t_j-1} (t_j - s)^{\mu - 1} K(x, s, U^M(x, s)) ds \right) = \delta I_M G(x, t) + I_M F(x) + I_M J_1(x). \] (4.9)
which can be restated in the following form:
\[ \delta I_M G(x, t) + F(x) + I_M J_1(x) = U^M(x, t) - I_M \left( \int_{t_j}^{t_j-1} (t_j - s)^{\mu - 1} K(x, s, U(x, s)) ds \right), \] (4.10)
where the interpolation operator \( I_M \) is defined by (3.7). It follows from (4.10) and (2.7), that
\[ \delta I_M G(x, t) + F(x) + I_M J_1(x) = U^M(x, t) - I_M \left( (t_j - s)^{\mu - 1} K(x, s, U^M(x, s)) - K(x, s, U^M(x, s)) \right), \] (4.11)
Let \( e(x, t) = U^M(x, t) - U(x, t) \), \( x \in [-1, 1] \), denote the error function. Then, we have
\[ I_M J_1(x) = e(x, t) + (U - I_M U)(x, t) - I_M \left( \int_{t_j}^{t_j-1} (t_j - s)^{\mu - 1} K(x, s, U^M(x, s)) - K(x, s, U^M(x, s)) ds \right), \] (4.12)
consequently,
\[ e(x, t) = \int_{t_j}^{t_j-1} (t_j - s)^{\mu - 1} K(x, s, U^M(x, s)) - K(x, s, U^M(x, s)) ds + I_M J_1(x) + J_2(x, t) + J_3(x, t), \] (4.13)
where
\[ J_2(x, t) = I_M U(x, t) - U(x, t), \]
\[ J_3(x, t) = -I_M \left( \int_{t_j}^{t_j-1} (t_j - s)^{\mu - 1} K(x, s, U^M(x, s)) - K(x, s, U^M(x, s)) ds \right) + I_M \left( \int_{t_j}^{t_j-1} (t_j - s)^{\mu - 1} \left[ K(x, s, U^M(x, s)) - K(x, s, U(x, s)) \right] ds \right). \]
According to the Lipschitz property of the kernel \( K \), we have
\[ |e(x, t)|_{L^1(t)} \leq \left( \int_{t_j}^{t_j-1} |(t_j - s)^{\mu - 1} K(x, s, U^M(x, s)) - K(x, s, U(x, s))| ds \right) + |I_M J_1(x) + J_2(x, t) + J_3(x, t)| \]
\[ \leq L_k \int_{t_j}^{t_j-1} |U^M(x, s) - U(x, s)| ds + |I_M J_1(x) + J_2(x, t) + J_3(x, t)| \]
\[ = L_k \int_{t_j}^{t_j-1} |e(x, s)| ds + |I_M J_1(x) + J_2(x, t) + J_3(x, t)|. \] (4.14)
Using the Gronwall inequality yield,
\[
\|e(x)\|_{L^2(t)} \leq C(\|I_M(J_1)\|_{L^2(t)} + \|I_2(x)\|_{L^2(t)} + \|J_3(x)\|_{L^2(t)}).
\]
(4.15)
\[
\|e(x)\|_{L^\infty(t)} \leq C(\|I_M(J_1)\|_{L^\infty(t)} + \|I_2(x)\|_{L^\infty(t)} + \|J_3(x)\|_{L^\infty(t)}).
\]

From (4.7) and lemma 5, we have
\[
\|I_M(J_1)\|_{L^2(t)} \leq \max\limits_{t \in I} |J_1| \leq CM^{-m}K^*,
\]
\[
\|I_M(J_1)\|_{L^\infty(t)} \leq CM^{-m} \max\limits_{x \in \Omega} |K(x, s(\theta), U^M(x, s(\theta)))|_{H^\infty_w(I)} \times \max\limits_{j = 0}^M |F_j(t)|
\leq CM^{-m}K^* \times \begin{cases} O(\log M), & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ O(M^{\alpha + \frac{1}{2}}), & \text{otherwise}. \end{cases}
\]
(4.16)

Using \(L^2\)– error and \(L^\infty\)– error bounds for the interpolation polynomials gives
\[
\|I_2\|_{L^2(t)} = \|I_M U(x, t) - U(x, t)\|_{L^2(t)} 
\leq CM^{-m}\|U(x, t)\|_{H^\infty_w(I)},
\]
\[
\|I_2\|_{L^\infty(t)} = \|I_M U(x, t) - U(x, t)\|_{L^\infty(t)} 
\leq \begin{cases} CM^\frac{1}{2} -m\|U(x, t)\|_{H^\infty_w(I)}, & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ CM^{1 + \epsilon - m}\log M\|U(x, t)\|_{H^\infty_w(I)}, & \text{otherwise}. \end{cases}
\]
(4.17)

By letting \(m = 1, 2, \ldots, n\), we have
\[
\|I_3\|_{L^2(t)} = \|(I_M - I) \int_{t_{j-1}}^{t_j} (t_j - s)^{-1}(K(x, s, U^M(x, s)) - K(x, s, U(x, s))ds\|_{L^2(t)} 
\leq \|I_M - I\|_{L^2(t)} \int_{t_{j-1}}^{t_j} (t_j - s)^{-1} ds\|_{L^2(t)} 
\leq \|I_M - I\| \|e(s)\|_{L^2(t)} 
= \|I_M - I\| |Me\|_{L^2(t)} 
= \|I_M - I\| (|Me - TMe|)\|_{L^2(t)} 
\leq I_M (|Me - TMe|)\|_{L^2(t)} + \|I_M - TMe|\|_{L^\infty(t)} 
\leq C\|Me - TMe\|_{L^\infty(t)} 
\leq C M^{-k}\|Me\|_{\infty, \kappa} 
\leq C M^{-k}\|e\|_{L^\infty(t)} 
\]
\[
\|I_3\|_{L^\infty(t)} = \|(I_M - I)Me\|_{L^\infty(t)} 
= \|(I_M - I)Me - TMe\|_{L^\infty(t)} 
\leq (1 + \|I_M\|_{L^\infty(t)}) CM^{-k}\|Me\|_{\infty, \kappa}, \quad k \in (0, \mu) 
\leq \begin{cases} CM^{-k}\|e\|_{L^\infty(t)}, & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ CM^{1 + \epsilon - k}\|e\|_{L^\infty(t)}, & -\frac{1}{2} \leq \epsilon < \mu - \frac{1}{2}. \end{cases}
\]
(4.18)

Considering all above estimates together, yield
\[
\|e\|_{L^2(t)} \leq CM^{\frac{1}{2} - m}K* + CM^{-m}\|U(x, t)\|_{H^\infty_w(I)} + CM^{-k}\|e\|_{L^\infty(t)} 
\]
\[
\|e\|_{L^\infty(t)} \leq \begin{cases} CM^{-m}\log MK* + CM^{\frac{1}{2} - m}\|U(x, t)\|_{H^\infty_w(I)} + CM^{-k}\log M\|e\|_{L^\infty(t)}, \\ CM^{\frac{1}{2} - m}K* + CM^{1 + \epsilon - m}\log M\|U(x, t)\|_{H^\infty_w(I)}CM^{\frac{1}{2} + \epsilon - k}\|e\|_{L^\infty(t)} 
\leq \begin{cases} -1 < \alpha, \beta \leq -\frac{1}{2}, \\ \frac{1}{2} \leq \epsilon < \mu - \frac{1}{2}. \end{cases}
\]
(4.19)

\[\square\]

**Theorem 2.** Suppose \(U^M(x, t) = \sum_{j=0}^M U_j(x)F_j(t)\) is the discrete solution in the time direction of the Eq. (2.12), and \(U_N^M(x, t) = \sum_{j=0}^N \sum_{j=0}^M u_j^i H_i(x)F_j(t)\) is the Eq. (2.15) at the completely discrete solution of time space, then for large enough \(N\), we can get the
following error estimate:

\[
\|U^M - U^M_N\|_{L^2} \leq \begin{cases} 
\text{ClogMN}^{\frac{1}{2} - m}U^*, \\
\text{CM}^{\frac{1}{2} - m}U^*, 
\end{cases}
\]

\[
\|U^M - U^M_N\|_{L^2} \leq \text{CM}^{\frac{1}{2} - m}U^*. 
\]  

(4.20)

**Proof.** By subtracting (2.15) from (2.13), using the Lipschitz condition, we obtain

\[
U_j^j(x_i) - U_j^i = \frac{1}{\Gamma(\mu)} \left( \int_{t_j}^{t_j} (t_j - s)^{\mu - 1} \left( K(x_i, s, U^M_j(x_i, s)) - K(x_i, s, U^M_N_j(x_i, s)) \right) ds \right)
\]

\[
\leq L_k \int_{t_j}^{t_j} (t_j - s)^{\mu - 1} |U^M_j(x_i, s) - U^M_N_j(x_i, s)| ds
\]

\[
\leq L_k \int_{t_j}^{t_j} (t_j - s)^{\mu - 1} \left( \sum_{m=0}^{M} \sum_{n=0}^{N} \left| U^m_j(x_i) - U^m_n H_n(x_i) \right| F_n(s) \right) ds
\]

\[
= L_k \sum_{m=0}^{M} C_m^j \left( U^m_j(x_i) - \sum_{n=0}^{N} U^m_n H_n(x_i) \right)
\]

(4.21)

where

\[
C_m^j = \int_{t_j}^{t_j} (t_j - s)^{\mu - 1} F_n(s) ds.
\]

Multiplying by \(H_i(z)\) on the both sides of (4.21) and summing from 0 to \(N\) for \(i\), then

\[
\sum_{i=0}^{N} U_j^j(x_i) H_i(x) - \sum_{i=0}^{N} U_j^i H_i(x) \leq L_k \sum_{i=0}^{N} \left( \sum_{m=0}^{M} C_m^j \left( U^m_j(x_i) - \sum_{n=0}^{N} U^m_n H_n(x_i) \right) \right) H_i(x),
\]

so,

\[
I_0 U_j^j(x) - \sum_{i=0}^{N} U_j^i H_i(x) \leq L_k \sum_{m=0}^{M} C_m^j \left[ I_0 U^m_j(x) - I_0 \left( \sum_{n=0}^{N} U^m_n H_n(x) \right) \right],
\]

(4.22)

where,

\[
I_0 U^m_j(x) = \sum_{i=0}^{N} U^m_j(x_i) H_i(x),
\]

\[
I_0 \left( \sum_{n=0}^{N} U^m_n H_n(x) \right) = \sum_{i=0}^{N} \sum_{n=0}^{N} U^m_n H_n(x_i) H_i(x).
\]

(4.23)

Let \(e(x, t_j) = U^j_j(x) - \sum_{i=0}^{N} U^i_j H_i(x)\), we can rewrite (4.23) as,

\[
\max_{t_i \in [-1, 1]} \|e(x, t)\| \leq C|J_4|,
\]

(4.24)

where

\[
J_4 = I_0 U^j_j(x) - U^j_j(x),
\]

(4.25)

According to lemma 2, yield,

\[
\|J_4\|_{L^2} = \|I_0 U^j_j(x) - U^j_j(x)\|_{L^2} \leq \text{CN}^{\frac{1}{2} - m} |u|_{H^{m,n}(I)}.
\]

(4.26)

\[
\|J_4\|_{L^\infty} \leq \text{CN}^{\frac{1}{2} - m} |u|_{H^{m,n}(I)}.
\]

(4.27)

On the other hand,

\[
U^M - U^M_N = \sum_{j=0}^{M} \left( U^j_j(x) - \sum_{i=0}^{N} U^i_j H_i(x) \right) F_j(t)
\]

\[
= \sum_{j=0}^{M} e(x, t_j) F_j(t) = I_{10} e(x, t).
\]

(4.28)
Using lemma 4, we have
\[
\|U^M - U_M^N\|_{L^\infty(I)} \leq \begin{cases} 
C \log M \max_{t \in [-1,1]} |e(x,t)|, & -1 < \alpha, \beta < -\frac{1}{2}, \\
CM^{1+\epsilon} \max_{t \in [-1,1]} |e(x,t)|, & \text{otherwise}.
\end{cases} \tag{4.29}
\]
Using lemma 3, we have
\[
\|U^M - U_M^N\|_{L^2(I)} \leq C \log M \max_{t \in [-1,1]} |e(x,t)|. \tag{4.30}
\]
Thus, combine (4.25), (4.27), (4.29) with (4.30), we can compute the results showed in (4.20). \qed

**Theorem 3.** Suppose \( U(x,t) \) is an exact solution of the Eq. (2.7) and sets \( U_N^N(x,t) = \sum_{i=0}^N \sum_{j=0}^M u_i H_i(x) F_j(t) \) is the entire discrete solution of (2.15) and satisfies the initial conditions and boundary conditions. If \( U(x,t) \in H^\infty_{\omega,\alpha,\beta} \) then for big enough \( M \) and \( N \), there are the following error estimates:

\[
|U(x,t) - U_N^M(x,t)|_{L^\infty(I)} \leq \begin{cases} 
CM^{-m} \log M K^* + CM^{\frac{1}{2} - m} U^* + C \log M N^{\frac{1}{2} - m} U^*, & -1 < \alpha, \beta \leq -\frac{1}{2}, \\
CM^{\frac{1}{2} - m} K^* + CM^{1+\epsilon - m} \log MU^* + CM^{1+\epsilon} N^{\frac{1}{2} - m} U^*, & -\frac{1}{2} \leq \epsilon < \mu - \frac{1}{2}.
\end{cases}
\tag{4.31}
\]

\[
\|U(x,t) - U_N^M(x,t)\|_{L^2(I)} \leq \begin{cases} 
CM^{-m} (K^* + U^*) + CM^{-m-k} \log M K^* + CM^{\frac{1}{2} - m-k} U^* + CM^{1+\epsilon - m-k} \log MU^* + CN^{-m} U^*, & -1 < \alpha, \beta \leq -\frac{1}{2}, \\
CM^{-m} (K^* + U^*) + CM^{\frac{1}{2} - m-k} K^* + CM^{1+\epsilon - m-k} \log MU^* + CN^{-m} U^*, & -\frac{1}{2} \leq \epsilon < \mu - \frac{1}{2}.
\end{cases}
\tag{4.31}
\]
**Proof.** Using the triangle inequality

\[ |U(x, t) - U_M(x, t)| \leq |U(x, t) - U^M(x, t)| + |U^M(x, t) - U_M(x, t)|, \]

and combine Theorems 1 and 2, leads to estimation (4.31). □

5. Algorithm implementation and numerical results

5.1. Example 1. 1D linear equation.

The domain is \((x, t) \in (-1, 1) \times (0, 2)\).

\[
i \frac{\partial^{3/4}}{\partial t^{3/4}} \psi(x, t) + \frac{1}{\pi} \frac{\partial^2}{\partial x^2} \psi(x, t) = f_1(x, t), \quad (x, t) \in (-1, 1) \times (0, 2),
\]

\[
\psi(x, 0) = 0, \quad x \in (-1, 1),
\]

\[
\psi(-1, t) = \psi(1, t) = -t^2, \quad t \in (0, 2),
\]

where

\[
f_1 = \frac{16 \sqrt{2}}{5 \pi} \Gamma\left(\frac{3}{4}\right) t^{5/4} (i \cos \pi x - \sin \pi x) + t^2 (-\pi^2 \cos \pi x - i \pi^2 \sin \pi x).
\]

The exact solution is

\[
\psi(x, t) = t^2 (\cos \pi x + i \sin \pi x).
\]

In the space direction \(x\), we use \(P_{M+2}\) Lagrange-Gauss-Lobatto orthogonal polynomials, where the node \(x_0 = -1\) and \(x_{M+1} = 1\). In the time direction \(t\), we use \(P_N\) Jacobi orthogonal polynomials with the index \((\mu - 1, 0) = (-1/4, 0)\), i.e., \((1 + s)^{-1/4}(1 - s)^0\), where \(s = t - 1\). The number of collocation points is \(MN\), instead of \((M+2)N\), as the boundary values are
Fig. 3. The solution and the error for (5.3), $\mu = 7/8$.

Table 1

Example 1. The $L^\infty$ error and the $L^2$ error at $t = 2$, for (5.1).

<table>
<thead>
<tr>
<th>$M, N$</th>
<th>$|\psi - \psi_h|_{L^\infty}$</th>
<th>$|\psi - \psi_h|_{L^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4, 4</td>
<td>2.2628E-001</td>
<td>6.3147E-002</td>
</tr>
<tr>
<td>6, 6</td>
<td>9.1637E-003</td>
<td>1.2706E-003</td>
</tr>
<tr>
<td>8, 8</td>
<td>2.5137E-004</td>
<td>2.1386E-005</td>
</tr>
<tr>
<td>10, 10</td>
<td>5.4994E-006</td>
<td>1.8660E-006</td>
</tr>
<tr>
<td>12, 12</td>
<td>8.5810E-007</td>
<td>8.3305E-007</td>
</tr>
</tbody>
</table>

given. The numerical solution and the exact solution at time $t = 2$ are plotted in Fig. 1, for $M = 4$ and $N = 4$. We list the numerical $L^\infty$ error and the $L^2$ error at time $t = 2$ in Table 1, from [26]. The method converges exponentially.

5.2. Example 2. 1D nonlinear equation.

Consider 1D nonlinear equation

$$i \frac{\partial^{7/8} \psi}{\partial t^{7/8}} + \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = f_2, \quad (x, t) \in (-1, 1) \times (0, 2),$$

$$\psi(x, 0) = 0, \quad x \in (-1, 1),$$

$$\psi(-1, t) = \psi(1, t) = 0, \quad t \in (0, 2),$$

(5.2)

where $f_2$ is defined such that the exact solution is

$$\psi(x, t) = t^{15/8} \left( \frac{1}{8} \sin 2\pi x + \frac{i}{12} \sin \pi x \right).$$
5.3. Example 3. 2D nonlinear equation.

We solve the following time-fractional Schrödinger equation.

\[
\frac{i}{\mu} \frac{\partial^{7/8} \psi}{\partial t^{7/8}} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + |\psi|^2 \psi = f_3, \quad (x, y, t) \in (-1, 1)^2 \times (0, 2),
\]

\[
\psi (x, y, 0) = 0, \quad (x, y) \in (-1, 1)^2,
\]

\[
\psi (\pm 1, \pm 1, t) = 0, \quad t \in (0, 2).
\]

(5.3)
where

$$f_3 = \Gamma(\frac{7}{8})t\left(-\frac{35}{256}\sin(\pi x)\sin(2\pi y) + i\frac{105}{512}\sin(2\pi x)\sin(\pi y)\right) + t^{15/8}\left(-\frac{5}{8}\sin(2\pi x)\sin(\pi y) - i\frac{5}{12}\sin(\pi x)\sin(2\pi y)\right)$$

$$+ t^{45/8}\left(\frac{1}{64}\sin^2(2\pi x)\sin^2(\pi y) + \frac{1}{144}\sin^2(\pi x)\sin^2(2\pi y)\right) - \left(\frac{1}{8}\sin(2\pi x)\sin(\pi y) + i\frac{1}{12}\sin(\pi x)\sin(2\pi y)\right).$$

The exact solution is

$$\psi(x, y, t) = t^{15/8}\left(\frac{1}{8}\sin(2\pi x)\sin(\pi y) + i\frac{1}{12}\sin(\pi x)\sin(2\pi y)\right).$$

In the space direction, we use 2D ten $P_{K+2}$ Lagrange-Gauss-Lobatto orthogonal polynomials. In the time direction $t$, we use $P_N$ Jacobi orthogonal polynomials with the weight index $(\mu - 1, 0) = (-1/8, 0)$, i.e., $(1 + s)^{-1/8}(1 - s)^0$, $s = t - 1$. The number of collocation points is $M^2N$. The exact solution and the error of the numerical solution at time $t = 2$ are plotted in Figs. 3 and 4, for $M = 10$ and $N = 10$. We list the numerical $L^\infty$ error and the $L^2$ error at time $t = 2$ in Table 3. On the first few levels, the numerical solution converges exponentially.

### 6. Conclusions

In this paper, space-time JSC method has been implemented to solve 1D and 2D time-fractional Schrödinger equations with the appropriate initial and boundary conditions. In this regard, we first transform the basic equation into the associated system of nonlinear weakly singular integro-PDEs and then apply the collocation scheme together with approximating the existing integrals with high accurate Gaussian quadrature rules, which reduce the basic equation into the corresponding system of nonlinear algebraic equations. This system of algebraic equations can be solved by robust iterative solvers such as Newton Raphson iterative method. The presented method has two basic advantages with respect to recent numerical methods. The proposed method is a balanced technique that has spectral accuracy in both of the spatial and temporal variables. Moreover, a rigorous convergence analysis is provided to support the proposed numerical idea theoretically. From the results of numerical experiments, one can conclude that, even by using small number of collocation points, high accurate solutions are computed by the proposed spectral method. At the current time, this approach is considered to be implemented and extended to solve other time-fractional PDEs. But, some modifications are need to be applied.

### Acknowledgments

The author would like to thank the referees for the helpful suggestions. The work was supported by NSFC Project (11671342, 91430213, 11671157, 11771369), Project of Scientific Research Fund of Hunan Provincial Science and Technology Department (2018JJ2374, 2018WK4006) and Key Project of Hunan Provincial Department of Education (17A210).

### References


