Weak Galerkin and continuous Galerkin coupled finite element methods for the Stokes-Darcy interface problem

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Abstract. We consider a model of coupled free and porous media flow governed by Stokes equation and Darcy’s law with the Beavers-Joseph-Saffman interface condition. In this paper, we propose a new numerical approach for the Stokes-Darcy system. The approach employs the classical finite element method for the Darcy region and the weak Galerkin finite element method for the Stokes region. We construct corresponding discrete scheme and prove its well-posedness. The estimates for the corresponding numerical approximation are derived. Finally, we present some numerical experiments to validate the efficiency of the approach for solving this problem.

AMS subject classifications: 65N30, 65N15, 65N12, 76N20

Key words: finite element methods, weak Galerkin finite element methods, weak gradient, Stokes equations, Darcy’s equation.

1 Introduction

The coupled model of fluid flow with porous media flow has gained increasing attention in recent years with many applications, such as groundwater flows \cite{10, 14, 26}, industrial filtrations \cite{17}, flow in vuggy porous media \cite{23} etc. In the model, the Stokes equation is used to describe the free flow and the Darcy’s law is used to describe the flow in porous media. Due to the different mechanism of flow in subdomains, it is necessary to introduce appropriate interface conditions on the interface between the two different flows. Firstly, the normal velocity is continuous, which is the result of mass conservation. Secondly, the normal force on the interface is balanced. Thirdly, the slip speed is proportional to the tangential tensor. The third condition is generally referred to as the Beavers-Joseph interface condition, which was first observed through experiments \cite{7}. In this paper, we

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utilize the simplified case: the Beavers-Joseph-Saffman (BJS) interface condition \cite{21, 22, 37}. In this Beavers-Joseph-Saffman condition, the contribution of the tangential velocity of the porous media flow is neglected.

Various numerical methods have been proposed for the Stokes-Darcy system. In \cite{28}, continuous finite element method and mixed finite method are considered in the Stokes and Darcy subdomains, respectively. The authors also proved existence and uniqueness of weak solutions to the problem. Continuous finite elements used in both regions are studied in \cite{14}. In \cite{35}, discontinuous Galerkin (DG) method is employed in the Stokes region while the mixed finite element method is employed in Darcy region. The analysis of DG methods in two subdomains can be found in \cite{33}. In \cite{16, 20}, HDG methods are considered to deal with the problem. Domain decomposition approaches make it possible to break up the coupled problem into two single problems, which might be solved in parallel. The related work can be found in \cite{11, 15}. The results in \cite{4, 31} demonstrate that the multi-grid approach can be applied to this problem. In recent years, people pay attention to numerical schemes on general meshes because of its flexibility in practice. Corresponding works for Stokes-Darcy system include virtual element method \cite{43}, weak Galerkin finite element methods \cite{12, 23, 24} and so on.

The weak Galerkin (WG) finite element method is first proposed by Wang and Ye for solving the second order elliptic equation in \cite{39}. The key idea of this method is that the classical derivatives were substituted by weakly defined derivatives for totally discontinuous functions. The continuity of the approximating function is ensured by adding a stabilizer. In form, WG method is closely related to the framework of HDG method. The relationship of WG method and HDG method has been discussed in \cite{9, 42} so we will not extend here. In recent years the WG method is widely used because of its flexibility, such as Stokes equation \cite{34, 41, 46, 48}, Brinkman equation \cite{30, 47, 49}, Darcy equation \cite{25, 27, 29}, convection problems \cite{18, 19}, elasticity problems \cite{44, 45}, elliptic interface problems \cite{13, 32, 33}. Recently, WG methods for the Stokes-Darcy system is researched. In \cite{12}, the authors consider the mixed formulation in the Darcy region, both the Stokes equation and Darcy equation involves the flux and the pressure. In addition, the formulation of the Stokes equations involves symmetric full stress tensor. In \cite{23}, authors used the Stokes equations in velocity -pressure formulation and the Darcy equation in primal pressure formulation, the approximations of the velocity, hydraulic and pressure are piecewise constant. The same formulation of the Stokes-Darcy system was discussed in \cite{24}, the authors use different degrees of polynomials on the arbitrary polygonal meshes and gave a series of numerical experiments.

Continuous finite elements have the benefit of simplicity and there are many codes available \cite{38}. Employing the continuous finite element for the Darcy equation in the second order primal formulation makes the porous medium simulators highly optimized \cite{35} and does not create additional degrees of freedom. The WG numerical scheme for the Stokes equations, as mentioned in \cite{41}, has a few advantages in comparison with the traditional continuous finite element methods. First, the construction of the WG numerical scheme for the Stokes equations is straightforward. It is easy to implement when
using standard polynomials as approximation functions. Secondly, the WG finite element partition can be arbitrary polygons in two dimensional space or polyhedral in three dimensional space with certain shape regularity, which makes the numerical approximation and mesh generation flexible. Thirdly, there is no need to choose a large parameter to satisfy the stability in the WG formulation. However, these advantages are not achieved by simply combining two discrete formulation together. In order to impose the interface condition, the discrete functions on both sides of the interface need to be consistent. The employment of WG method in the Stokes region allows us to select the same local discrete space as the global discrete space in the Darcy region, and for the remaining region of the Stokes side, we can use different combination of polynomial spaces for the numerical scheme. Moreover, like [12], because of the full stress tensor involved in Stokes equation, we need the Korn’s inequality in different case to illustrate the well-posedness of the scheme.

The paper is organized as follows. In Section 2, the model problem, notations, and the fundamental definitions are presented. In Section 3, the numerical scheme is constructed. In Section 4, we discuss the solvability and stability of the scheme. In Section 5, we derive an error equation. We obtain the estimates for the corresponding numerical approximation. Finally, in Section 6, some numerical experiments are presented.

2 Preliminaries

In this section, we describe in detail the coupled model problem, finite element partitions and the related finite dimensional function spaces.

2.1 A Coupled Stokes-Darcy Problem

Let $\Omega$ be a bounded domain in $\mathbb{R}^d, d=2, 3$, which is divided into two subdomains, a fluid region $\Omega_S$ and a porous medium region $\Omega_D$. In this article, $\Omega_i$ ($i=S, D$) refers to either $\Omega_S$ or $\Omega_D$, and it is the same for the other symbols with subscript $i$. We assume that both $\Omega_S$ and $\Omega_D$ have Lipschitz continuous boundaries. Let $\Sigma$ be the interface $\Sigma = \partial \Omega_S \cap \partial \Omega_D$. Define $\Gamma_S = \partial \Omega_S \setminus \Sigma$, $\Gamma_D = \partial \Omega_D \setminus \Sigma$. Moreover, $n$ is the unit outward vector of $\Omega_S$ on $\partial \Omega_S$ and the unit outward normal vector of $\Omega_D$ on $\Gamma_D$, $\tau_j, j=1, \ldots, d-1$ are unit tangent vectors on $\Sigma$.

In the free flow region $\Omega_S$, let $u_S$ denote the fluid velocity, $p_S$ the kinematic pressure, $s$ the external body force density, and $\mu$ the kinematic viscosity of the fluid. In addition, we use $T(u_S, p_S) = 2\mu D(u_S) - p_S \mathbb{I}$ to denote the stress tensor, where $D(u_S) = \frac{1}{2}(\nabla u_S + \nabla^T u_S)$ and $\mathbb{I}$ denotes the identity matrix. Then the free flow in $\Omega_S$ satisfies the Stokes system

\begin{align*}
-\nabla \cdot T(u_S, p_S) &= s \quad \text{in} \quad \Omega_S, \quad (2.1) \\
\nabla \cdot u_S &= 0 \quad \text{in} \quad \Omega_S. \quad (2.2)
\end{align*}

In the porous medium region $\Omega_D$, let $u_D$ denote the fluid discharge rate, and $f_D$ the source term, $p_D$ the hydraulic head. Then the flow in the porous medium satisfies the
following Darcy’s law

\[
\begin{align*}
\nabla \cdot u_D &= f_D & \text{in } \Omega_D, \\
u_D &= -K \nabla p_D & \text{in } \Omega_D.
\end{align*}
\]  

(2.3)

Eliminating \( u_D \), we obtain

\[-\nabla \cdot (K \nabla p_D) = f_D \text{ in } \Omega_D, \]  

(2.4)

where \( K \) is the hydraulic conductivity tensor, assumed to be a constant and bounded below.

The following boundary conditions are considered on \( \Gamma_S \) and \( \Gamma_D \)

\[-K \nabla p_D \cdot n = 0 \text{ on } \Gamma_D, \]  

(2.5)

\[u_S = 0 \text{ on } \Gamma_S. \]  

(2.6)

On the interface, we first have the transmission conditions

\[u_S \cdot n = -K \nabla p_D \cdot n \text{ on } \Sigma, \]  

(2.7)

\[-T(u_S, p_S) n \cdot n = p_D \text{ on } \Sigma. \]  

(2.8)

Then, the Beavers-Joseph-Saffman law reads

\[-T(u_S, p_S) n \cdot \tau_j = \alpha K^{-1/2} u_S \cdot \tau_j, \text{ on } \Sigma. \]  

(2.9)

Without loss of generality, we also assume that \( K = \lambda I \), where \( I \) is the identity matrix.

Let \( K \) be an open bounded domain in \( \mathbb{R}^d \), and \( s \) be a positive integer. We will follow the standard notations of Sobolev spaces \([1]\). As usual, we use \( \| \cdot \|_{m,K} \), \( | \cdot |_{m,K} \), \( (\cdot, \cdot)_{m,K} \) to
represent the norm, seminorm and inner product of Sobolev space $H^m(K)$ respectively. We shall drop the subscript $K$ if $K = \Omega$ and drop the subscript $m$ if $m = 2$. The space $H(div; K)$ defines as follows,

$$H(div; K) = \{ \mathbf{v} : \mathbf{v} \in \mathbb{L}^2(K)^d, \nabla \cdot \mathbf{v} \in \mathbb{L}^2(K) \}.$$  

### 2.2 Notations for partitions

In the following paper, $\Omega_i$ refers to either $\Omega_S$ or $\Omega_D$, and it is the same for the other symbols with subscript $i$. Denote by $T_h$ the union of $T_{S,h}$ and $T_{D,h}$, where $T_{S,h}$ refers to the WG-regular partition of $\Omega_S$ and $T_{D,h}$ refers to the FEM partition of $\Omega_D$. $T_i$ represents the element in $\Omega_i$. Denote the edges or flat faces in $T_h$ by $\mathcal{E}_h$, and define $e_i$ the edge or flat face on $\partial T_i$. Especially, the partition $T_{S,h}$ and $T_{D,h}$ are not necessary to be consistent at the interface $\Sigma$, and denote the mesh size of $T_{S,h}$ and $T_{D,h}$ by $h_S$ and $h_D$, respectively.

To define the WG method, we first introduce the definition of weak function definition on $T_S$,

$$\mathbf{v}_S = \begin{cases} \mathbf{v}_{S,0}, & \text{in } T_S, \\ \mathbf{v}_{S,b}, & \text{on } \partial T_S. \end{cases} \quad (2.10)$$

The weak function is formed by two parts, the internal function $\mathbf{v}_{S,0}$ and the boundary function $\mathbf{v}_{S,b}$. In general, $\mathbf{v}_{S,b}$ may not necessarily be related to the trace of $\mathbf{v}_{S,0}$ on $\partial T_S$, we can write $\mathbf{v}_{S,b}$ as $\{ \mathbf{v}_{S,0}, \mathbf{v}_{S,b} \}$ in short without confusion. Note that $\mathbf{v}_{S,b}$ takes single value on $\partial T_S$. Then we define some discrete spaces. Denote by $\rho \in P_k(T_i)$ that $\rho|_{T_i}$ is polynomial with degree no more than $k_i$, and the piecewise function space $P_k(e_i)$ is similar. We define the finite dimensional function spaces as follows.

- $V_h = \{ (\mathbf{v}_{S,0}, \mathbf{v}_{S,b}) : \mathbf{v}_{S,0} \in [P_{k_0}(T_S)]^d, \mathbf{v}_{S,b} \in [P_{k_b}(e_S)]^d, \forall T_S \in T_{S,h} \}$
- $V_h^0 = \{ (\mathbf{v}_{S,0}, \mathbf{v}_{S,b}) \in V_h : \mathbf{v}_{S,b} = 0 \text{ on } \Gamma_S \}$
- $W_{S,h} = \{ q_{S,h} : q_{S,h} \in \mathbb{L}^2(\Omega_S), q_{S,h} \in P_{k_S}(T_S), \forall T_S \in T_{S,h} \}$
- $W_{D,h} = \{ q_{D,h} : q_{D,h} \in H^1(\Omega_D) \cap \mathbb{L}^2(\Omega_D), q_{D,h}|_{T_D} \in P_{k_D}(T_D), \forall T_D \in T_{D,h} \}$

where

$$k_0 \geq 1, k_D \geq 1, k_b \geq 1$$

are nonnegative integers denoting the degree of polynomials, and $k = k_D$. We also assume that

$$k_b - 1 \leq k_S \leq k_b \leq k_0 \leq k_b + 1.$$ 

Two possible selections are listed as follows.

Case I: $k_0 = k_b = k_D = k$, where $k \geq 1$.
Case II: $k_0 = k_b = k_D = k$, $k_b = k - 1$, where $k \geq 2$.

The discrete weak gradient and divergence of vector valued functions are defined as follows.
Definition 2.1. For any \( \mathbf{v}_{\text{S},h} \in V_h \), the discrete weak gradient \( \nabla_w \mathbf{v}_{\text{S},h}|_{T_s} \in [P_{k-1}(T_s)]^{d \times d} \) satisfies
\[
(\nabla_w \mathbf{v}_{\text{S},h}, \tau)|_{T_s} = -(\mathbf{v}_{\text{S,0}}, \nabla \cdot \tau)|_{T_s} + (\mathbf{v}_{\text{S},h}, \tau \cdot \mathbf{n})_{\partial T_s}, \quad \forall \tau \in [P_{k-1}(T_s)]^{d \times d}. \tag{2.11}
\]

Denote
\[
D_w(\mathbf{v}) = \frac{1}{2}(\nabla_w \mathbf{v} + \nabla_w \mathbf{v}^T).
\]

Analogously, we can define the discrete weak divergence.

Definition 2.2. For any \( \mathbf{v}_{\text{S},h} \in V_h \), the discrete weak divergence \( \nabla_w \cdot \mathbf{v}_{\text{S},h}|_{T_s} \in P_{k-1}(T_s) \) satisfies
\[
(\nabla_w \cdot \mathbf{v}_{\text{S},h}, q)|_{T_s} = -(\mathbf{v}_{\text{S,0}}, \nabla q)|_{T_s} + (\mathbf{v}_{\text{S},h}, q \mathbf{n})_{\partial T_s}, \quad \forall q \in P_{k-1}(T_s). \tag{2.12}
\]

3 Numerical Scheme

Before describing the WG numerical algorithm, we introduce several \( L^2 \) projection operators.

Define \( Q_h = \{ Q_0, Q_h \} \), which means that \( Q_0 \) is the \( L^2 \) projection to \( [P_k(T_s)]^d \) in \( T_s \); \( Q_h \) is the \( L^2 \) projection to \( [P_k(e_s)]^d \) on \( e_s \), and the \( L^2 \) projection to \( [P_k(e)]^d \) on \( e \in \Sigma \). Furthermore, we define \( Q_h \) as the \( L^2 \) projection from \( H^1(\Omega) \cap L_0^2(\Omega_D) \) to \( W_{D,h,k} \). \( Q_h \) is the \( L^2 \) projection to \( [P_{k-1}(T_s)]^{d \times d} \) in \( T_s \), \( Q_h \) the \( L^2 \) projection to \( P_{k-1}(T_s) \) in \( T_s \), and \( \tau_h \) the \( L^2 \) projection to \( P_{k-1}(T_s) \) in \( T_s \).

We introduce the bilinear forms \( a_S(\cdot, \cdot): V_h^0 \times V_h^0 \rightarrow \mathbb{R} \), \( a_D(\cdot, \cdot): W_{D,h} \times W_{D,h} \rightarrow \mathbb{R} \), \( b(\cdot, \cdot): V_h^0 \times W_{S,h} \rightarrow \mathbb{R} \) and cross terms \( a_\Sigma(\cdot, \cdot): (V_h^0 \times W_{D,h}; (V_h^0 \times W_{D,h}) \Omega_h) \rightarrow \mathbb{R} \),
\[
a_S(\mathbf{v}_{\text{S},h}, \mathbf{w}_{\text{S},h}) = (2\mu D_w(\mathbf{v}_{\text{S},h}), D_w(\mathbf{w}_{\text{S},h}))_{\Omega_h} + \sum_{j=1}^{d-1} (a K^{-1/2} \mathbf{v}_{\text{S},h} \cdot \mathbf{\tau}_j, \mathbf{w}_{\text{S},h} \cdot \mathbf{\tau}_j)_{\Sigma} + s(\mathbf{v}_{\text{S},h}, \mathbf{w}_{\text{S},h}),
\]
\[
a_\Sigma(\mathbf{v}_{\text{S},h}, q_D; \mathbf{w}_{\text{S},h}, r_D) = (g q_D, \mathbf{w}_{\text{S},h} \cdot \mathbf{n})_{\Sigma} - (g r_D, \mathbf{v}_{\text{S},h} \cdot \mathbf{n})_{\Sigma},
\]
\[
a_D(p_{D,h}, q_{D,h}) = \langle g K \nabla q_D, \nabla r_D \rangle_{\Omega_h},
\]
\[
b(\mathbf{v}_{\text{S},h}, q_{S,h}) = (\nabla_w \cdot \mathbf{v}_{\text{S},h}, q_{S,h})_{\Omega_h},
\]
where
\[
s(\mathbf{v}_{\text{S},h}, \mathbf{w}_{\text{S},h}) = \sum_{T_s \in T_{S,h}} \rho S h_T^{-1} \langle Q_b \mathbf{v}_{\text{S},h} - \mathbf{v}_{\text{S},h}, Q_h \mathbf{w}_{\text{S},h} - \mathbf{w}_{\text{S},h} \rangle_{\partial T_s}.
\]

The bilinear form \( s(\cdot, \cdot) \) is called stabilizer or smoother, which helps to enhance the continuity of weak functions. The weak Galerkin method is independent of the stabilization parameter \( \rho_S \), we set \( \rho_S = 1 \) in this paper.

With these preparations we can define the numerical scheme as follows.
Algorithm 1. The numerical solution of (2.1) - (2.9) can be obtained by seeking $u_{S,h} \in V^0_h$, $p_{S,h} \in W_{S,h}$, $p_{D,h} \in W_{D,h}$ such that

$$a_S(u_{S,h}, v_{S,h}) + a_D( p_{D,h}, q_{D,h} ) + a_\Sigma(u_{S,h}, p_{D,h}, v_{S,h}, q_{D,h}) - b(v_{S,h}, p_{S,h})$$

$$= (s, v_{S,0})_{\Omega_s} + (g f_D, q_{D,0})_{\Omega_d},$$

where

$$b(u_{S,h}, q_{S,h}) = 0,$$

hold for all $v_{S,h} \in V^0_h$, $q_{S,h} \in W_{S,h}$ and $q_{D,h} \in W_{D,h}$.

Remark 3.1. When $k_0 = 1$ and $k_b = 0$, we can establish a similar algorithm by redefining the function space $V_h$, which is detailed in Section 4.

4 Existence and Uniqueness

We devote most of this section to checking two important properties of the numerical scheme: the boundedness of the sum of the bilinears $a_S(\cdot, \cdot)$, $a_D(\cdot, \cdot)$, and $a_\Sigma(\cdot, \cdot)$, then the inf-sup condition related to $b(\cdot, \cdot)$.

First, we need to equip the space $V_h, W_{D,h}$ with proper norms.

Definition 4.1. For $v_{S,h} \in V_h$,

$$|||v_{S,h}|||^2 = a_S(v_{S,h}, v_{S,h}) = ||(2\mu)^{1/2} D_w(v_{S,h})||_{\Omega_s}^2 + \sum_{j=1}^{d-1} |||a^{1/2} K^{-1/4} v_{S,b} \cdot \tau_j|||^2_{\Omega_s} + \sum_{T_s \in T_h} h_T^{-1} ||Q_b v_{S,0} - v_{S,b}||^2_{\partial T_s},$$

and for $q_{D,h} \in W_{D,h}$,

$$||q_{D,h}||^2_{L^2} = a_D( q_{D,h}, q_{D,h} ) = (g \kappa \nabla q_{D,h}, \nabla q_{D,h})_{\Omega_d}.$$

Obviously both $|||\cdot|||$ and $||\cdot||_{1,h}$ are seminorms, and we can estimate the discrete $H^1$ norm by the trip-bar norm using the Korn’s inequality.

Lemma 4.1. For any $v_{S,h} \in V_h$, we have

$$\sum_{T_s \in T_h} ||\nabla v_{S,0}||_{T_s} \leq C |||v_{S,h}|||.$$

Proof. As proved in [6], there is the following discrete Korn’s inequality

$$\leq C \left( \sum_{T_s \in T_h} ||D(v_{S,0})||_{T_s}^2 + \sup_{m \in RM, 0 m ||_{T_s} = 1} \left( \frac{1}{||v_{S,0}||_{T_s}^2} \right) + \sum_{e \in E_h} ||\pi_e v_{S,0}||_e^2 \right),$$
where $RM$ is the space of rigid motions, $\pi_e$ is the projection operator from $[L^2(\varepsilon)]^d$ onto $[P_1(\varepsilon)]^d$. We need to estimate the three terms separately.

We can deduce from the integration by parts and the definition (2.11) that in each element $T_S \in T_{S,h}$

\[(D(v_{s,0}), D(v_{s,0}))_{T_S} = \langle -v_{s,0}, \nabla \cdot D(v_{s,0}) \rangle_{T_S} + \langle v_{s,0}, D(v_{s,0}) \cdot n \rangle_{\partial T_S} + \langle Q_b v_{s,0} - v_{s,0}, D(v_{s,0}) \cdot n \rangle_{\partial T_S}\]

Sum over all element $T_S$ and apply the trace inequality (A.1) and the inverse inequality (A.2), then it follows that

\[
\|D(v_{s,0})\|_{T_S}^2 \leq C(\|\nabla_w v_{s,b,h}\|_{T_S}, \|D(v_{s,0})\|_{T_S} + \|Q_b v_{s,0} - v_{s,0}\|_{\partial T_S}, \|D(v_{s,0})\|_{\partial T_S}) \leq C(\|\nabla_w v_{s,0}\|_{T_S} + h^{-1}_T \|Q_b v_{s,0} - v_{s,0}\|_{\partial T_S}, \|D(v_{s,0})\|_{T_S})
\]

therefore,

\[
\sum_{T_S \in T_{S,h}} \|D(v_{s,0})\|_{T_S} \leq C\|v_{s,b,h}\|.
\]

For the second and the third terms, from $k_b \geq 1$ and the boundary condition (2.6) we can get that

\[
\sup_{m \in RM, \|m\|_{T_S} = 1, m_{T_S} = 0} \left( \int_{T_S} v_{s,0} \cdot m ds \right) \leq \sup_{m \in RM, \|m\|_{T_S} = 1, m_{T_S} = 0} \left( \int_{T_S} (Q_b v_{s,0} - v_{s,0}) \cdot m ds \right) \leq C\|v_{s,b,h}\|
\]

which completes the proof. \(\square\)

**Lemma 4.2.** $\|\cdot\|$ and $\|\cdot\|_{1,h}$ provide a norm in $V^0_h$ and $W_{D,h}$ respectively.

**Proof.** Notice that if $\|v_{s,b,h}\| = 0$, then

\[
a_S(v_{s,b,h}, v_{s,b,h}) = 0,
\]

which means that $D_w(v_{s,b}) = 0$ in all $T_S \in T_{S,h}$, $Q_b v_{s,0} = v_{s,b}$ on $\partial T_S$, $v_{s,b} \cdot n = 0$ on $\Sigma$. From the Lemma 4.1, we know that $\nabla v_{s,0} = 0$ in all $T_S$, so that $v_{s,0}$ is piecewise constant. Moreover, $Q_b v_{s,0} = v_{s,b}$ yields $v_{s,b}$ is a constant in $\Omega_S$. Combining with the fact that $v_{s,b} = 0$ on $\Gamma_S$, we know that $v_{s,b} = 0$. Therefore, $\|\cdot\|$ defines a norm in $V^0_h$.

Similarly, if $\|q_{D,h}\|_{1,h} = 0$, we know that $\nabla q_{D,h} = 0$ in $\Omega_D$, so $q_{D,h}$ is a constant in $\Omega_D$. It follows from the fact that $q_{D,h} \in L^2_0(\Omega_D)$ that $q_{D,h} = 0$. Therefore, $\|\cdot\|_{1,h}$ defines a norm in $W_{D,h}$. \(\square\)
Remark 4.1. When \( k_0 = 1 \) and \( k_b = 0 \), the space \( V_h \) needs to be adjusted as follow

\[
V_h = \{(v_{S,0}, v_{S,n}, v_{S,T_j}) : v_{S,0} \in [P_1(T_S)]^d, v_{S,n} \in P_1(e_S), v_{S,T_j} \in P_0(e_S), v_{S,n} \in P_k(e \cap \Sigma), v_{S,T_j} \in P_k(e \cap \Sigma) \}
\]

where \( j = 1 \) to \( d - 1 \), \( k = k_D \). The weak gradient and divergence should satisfy that \( \forall m \in [P_0(T_S)]^{d \times d} \), \( q_{S,h} \in P_0(T_S) \),

\[
\begin{align*}
(\nabla w v_{S,h}, m)_{T_S} & = - (v_{S,0}, \nabla \cdot m)_{T_S} + \langle v_{S,n}, mn \cdot ne \rangle_{\partial T_S} + \sum_{j=1}^{d-1} \langle v_{S,T_j}, m n \cdot \tau_j \rangle_{\partial T_S}, \\
(\nabla w \cdot v_{S,h} q_{S,h})_{T_S} & = - (v_{S,0}, \nabla q_{S,h})_{T_S} + \langle v_{S,n}, (n \cdot ne), q_{S,h} \rangle_{\partial T_S},
\end{align*}
\]

where \( n_e \) and \( \tau_j \) are given normal vector and tangent vector on each edge/face.

For any \( v_{S,h}, w_{S,h} \in V^0_h \), \( q_{D,h}, r_{D,h} \in W_{D,h} \), \( q_{S,h} \in W_{S,h} \), the corresponding bilinear forms are

\[
am\,(v_{S,h}, w_{S,h}) = (2\mu D_w(v_{S,h}), D_w(w_{S,h}))_{\Omega_S} + \sum_{j=1}^{d-1} \langle \alpha \kappa^{-1/2} v_{S,h} \cdot \tau_j, w_{S,h} \cdot \tau_j \rangle_{\Sigma} + s(v_{h,w}),
\]

\[
a_D(q_{D,h}, r_{D,h}) = (\kappa \nabla q_{D,h}, \nabla r_{D,h})_{\Omega_D},
\]

\[
a_G(v_{S,h}, q_{D,h}, w_{S,h}, r_{D,h}) = \langle \kappa \nabla q_{D,h}, v_{S,h} \cdot n \rangle_{\Sigma} - \langle \kappa \nabla q_{S,h}, v_{S,h} \cdot n \rangle_{\Sigma},
\]

\[
s(v_{S,h}, w_{S,h}) = \sum_{T_S \in T_{S,h}} \|Q_n(v_{S,0} - w_{S,0}) - v_{S,n} Q_n|_{T_S} - w_{S,n}|_{T_S} \|_{\partial T_S},
\]

\[
+ \sum_{T_S \in T_{S,h}} \sum_{j=1}^{d-1} h_{T_S}^{-1} \langle Q_{\tau_j}(v_{S,0} - w_{S,0}) - v_{S,T_j} Q_{\tau}(w_{S,0} - w_{S,0}) - w_{S,T_j} Q_{\tau}(w_{S,0} - w_{S,0}) \rangle_{\partial T_S},
\]

\[
b(v_{S,h}, q_{S,h}) = (\nabla w \cdot v_{S,h}, q_{S,h})_{\Omega_S}.
\]

where \( s(\cdot, \cdot) \) is the stabilizer, \( Q_n \) and \( Q_{\tau} \) are corresponding \( L^2 \) projections.

If the partition of \( \Omega \) is a standard finite element triangulation, the estimate

\[
\sum_{T_S \in T_{S,h}} \|\nabla v_{S,0}\|_{T_S} \leq C \|v_{S,h}\|
\]

follows the Korn’s inequality established in [6]. The proof is similar to Lemma 4.1. The proof and conclusions in the rest part of this article can be applied to this case trivially, which is left for the interested readers.

Lemma 4.3. The projection operators \( Q_h, \bar{Q}_h \) and \( Q_h \) satisfy the following commutative properties

\[
\nabla w(Q_h v) = Q_h(\nabla v), \quad \forall v \in [H^1(\Omega_S)]^d,
\]

\[
\nabla w(\cdot)(Q_h v) = \bar{Q}_h(\nabla \cdot v), \quad \forall v \in [H(div, \Omega_S)]^d.
\]
Proof. Take any \( \tau \in [P_k(T_S)]^{d \times d} \), it follows that

\[
(\nabla w(Q_h \mathbf{v}), \tau)_{\Omega_S} = -(Q_0 \mathbf{v}, \nabla \cdot \tau)_{\Omega_S} + \sum_{T_S \in T_S} (Q_h \mathbf{v}, \tau \cdot \mathbf{n})_{\partial T_S}
\]

\[
= -(\mathbf{v}, \nabla \cdot \tau)_{\Omega_S} + \sum_{T_S \in T_S} (\mathbf{v}, \tau \cdot \mathbf{n})_{\partial T_S}
\]

\[
= (\nabla \mathbf{v}, \tau)_{\Omega_S} = (Q_h \nabla \mathbf{v}, \tau)_{\Omega_S}.
\]

Analogously, in the divergence case for any \( q \in P_k(T_S) \), we have

\[
(\nabla w \cdot (Q_h \mathbf{v}), q)_{\Omega_S} = -(Q_0 \mathbf{v}, \nabla q)_{\Omega_S} + \sum_{T_S \in T_S} (Q_h \mathbf{v}, q \cdot \mathbf{n})_{\partial T_S}
\]

\[
= -(\mathbf{v}, \nabla q)_{\Omega_S} + \sum_{T_S \in T_S} (\mathbf{v}, q \cdot \mathbf{n})_{\partial T_S}
\]

\[
= (\nabla \cdot \mathbf{v}, q)_{\Omega_S} = (Q_h (\nabla \cdot \mathbf{v}), q)_{\Omega_S}.
\]

\[\square\]

From definition of \( \| \cdot \| \) and \( \| \cdot \|_{1,h} \), we have the following lemma.

**Lemma 4.4.** For any \( \mathbf{v}_h, \mathbf{w}_h \in V_h^d \), \( p_{D,h}, r_{D,h} \in W_{D,h} \), we have

\[
a_S(\mathbf{v}_h, \mathbf{w}_h) + a_D(p_{D,h}, r_{D,h}) + a_E(\mathbf{v}_h, p_{D,h}, \mathbf{w}_h, r_{D,h}) = \| \mathbf{v}_h \|_{}^2 + \| p_{D,h} \|_{1,h}^2.
\]

For the bilinear form \( b(\mathbf{v}_h, \hat{\rho}_{S,h}) \) in the numerical scheme, if we confine \( \hat{\rho}_{S,h} \) to \( L^2_0(\Omega_S) \), we can establish the inf-sup condition.

**Lemma 4.5.** There exists a positive constant \( \beta \) independent of \( h \) such that

\[
\sup_{\mathbf{v}_S,h \in V_h^d} \frac{b(\mathbf{v}_S,h, \hat{\rho}_{S,h})}{\| \mathbf{v}_S,h \|} \geq \beta \| \hat{\rho}_{S,h} \|_{\Omega_S},
\]

for all \( \hat{\rho}_{S,h} \in W_{S,h} \cap L^2_0(\Omega_S) \).

**Proof.** It is well known [5] that for any \( \hat{\rho}_{S,h} \in W_{S,h} \cap L^2_0(\Omega_S) \), we can find \( \mathbf{w} \in [H_0]^d(\Omega_S) \) such that \( \nabla \cdot \mathbf{w} = \hat{\rho}_{S,h} \) and \( \| \mathbf{w} \|_{1,\Omega_S} \leq C \| \hat{\rho}_{S,h} \|_{\Omega_S} \).

Consider \( \mathbf{v}_{S,h} = Q_h \mathbf{w} \). From the Lemma 4.3, we have that

\[
(\nabla \cdot \mathbf{v}_{S,h}, \hat{\rho}_{S,h}) = (\nabla \cdot Q_h \mathbf{w}, \hat{\rho}_{S,h}) = (Q_h (\nabla \cdot \mathbf{w}), \hat{\rho}_{S,h}) = (\nabla \cdot \mathbf{w}, \hat{\rho}_{S,h}) = \| \hat{\rho}_{S,h} \|_{T_S}^2.
\]

Note that

\[
\| D_w(\mathbf{v}_{S,h}) \|_{\Omega_S} \leq \| \nabla \mathbf{v}_{S,h} \|_{\Omega_S} = \| Q_h (\nabla \mathbf{w}) \|_{\Omega_S} \leq \| \nabla \mathbf{w} \|_{\Omega_S}.
\]
Using the trace inequality (\[A.1\]), we have
\[
\sum_{j=1}^{d-1} \| (2\alpha)^{\frac{1}{2}} v_{S,j} \cdot \tau_j \|_\Sigma^2 \leq C \| Q_h w \|_\Sigma^2 \leq C \| w \|_\Sigma^2 \leq C \| \nabla w \|_{\Omega_S}^2
\]
and
\[
\sum_{T_S \in T_{S,h}} h_{T_S}^{-1} \| Q_h v_{S,0} - v_{S,h} \|_{T_S} = \sum_{T_S \in T_{S,h}} h_{T_S}^{-1} \| Q_0 w - Q_h w \|_{T_S} \leq C \sum_{T_S \in T_{S,h}} (h_{T_S}^{-1} \| Q_0 w - w \|_{T_S} + \| \nabla (Q_0 w - w) \|_{T_S}) \leq C \| \nabla w \|_{\Omega_S}.
\]
Thus \( \| v_{S,h} \| \leq C \| \nabla w \|_{\Omega_S} \). It follows that
\[
\sup_{v_{S,h} \in V_0^h} \frac{b(v_{S,h}, \rho_{S,h})}{\| v_{S,h} \|} \geq \frac{(\nabla_{v_{S,h}} \cdot \rho_{S,h})}{\| v_{S,h} \|}
= C \frac{(\nabla \cdot w, \rho_{S,h})}{\| \nabla w \|_{\Omega_S}} \geq \beta \| \rho_{S,h} \|_{\Omega_S},
\]
which completes the proof.

**Lemma 4.6.** The weak Galerkin finite element method scheme (3.1)- (3.2) has one and only one unique solution.

**Proof.** Since (3.1)-(3.2) is linear system of equations, we only need to prove the uniqueness. Consider the homogeneous case, i.e. \( s = 0, f_D = 0 \).

Take \( v_{S,h} = u_{S,h} q_{D,h} = p_{D,h}, q_{S,h} = p_{S,h} \), and add the equations (3.1)-(3.2) together. Then we can get
\[
\| u_{S,h} \|^2 + \| q_{D,h} \|^2 = a_s(u_{S,h}, u_{S,h}) + a_D(q_{D,h}, q_{D,h}) + a_s(u_{S,h}, q_{D,h}, u_{S,h}, q_{D,h}) - b(v_{S,h}, p_{S,h}) = 0,
\]
which leads to \( u_{S,h} = 0, q_{D,h} = 0, \) and \( b(v_{S,h}, p_{S,h}) = 0 \) for any \( v_{S,h} \in V_0^h \).
Denote $p_{S,h} = \hat{p}_{S,h} + \bar{p}_{S,h}$, where $\hat{p}_{S,h} = \int_{\Omega} p_{S,h} dx / |\Omega|$, $\hat{p}_{S,h} \in W_{S,h} \cap L^2_0(\Omega)$. It is well known that there exists $\check{\nu}_S \in H^1_0(\Omega)$ such that $\nabla \cdot \check{\nu}_S = \hat{p}_{S,h}$. Take $\nu_{S,h} = Q_h \check{\nu}_S$ to obtain

$$0 = b(\nu_{S,h}, p_{S,h}) = (\nabla \cdot Q_h \nu_{S,h}, \hat{p}_{S,h}) + (\nabla \cdot Q_h \check{\nu}_S, \bar{p}_{S,h}) = (\hat{Q}_h \nabla \cdot (\check{\nu}_S), \hat{p}_{S,h}) + (\hat{Q}_h \nabla \cdot (\check{\nu}_S), \bar{p}_{S,h})$$

so that $\bar{p}_{S,h} = 0$. Then taking any $\int_{\Omega} \nabla \cdot \nu_{S,h} dx \neq 0$ yields $\hat{p}_{S,h} = 0$, which completes the proof.

\section{Error Analysis}

In the following part, our goal is to obtain equations for the errors $e_{S,h} = Q_h u_s - u_{S,h}$, $e_p = p_D - p_{D,h}$, $\epsilon_{D,h} = Q_h \epsilon_D - D_h$, and $\epsilon_{S,h} = \tau_h p_s - p_{S,h}$, which is critical to the convergence analysis for numerical scheme (3.1) - (3.2).

As verified in [11], the following properties can be derived from the definition of weak operators (2.11) - (2.12) and the integration by parts.

\textbf{Lemma 5.1.} For any $w_S \in H^1(\Omega)$, $\rho_S \in H^1(\Omega)$, and $v_{S,h} \in V^0_h$, it follows that

\begin{align*}
(\nabla w, \nabla v_{S,h})_{\Omega} &= (\nabla w_S, \nabla v_{S,h})_{\Omega} - \sum_{T_S \in T_{S,h}} \langle v_{S,h} - v_{S,0}, (Q_h \nabla w_S) \cdot n \rangle_{\partial T_S}, \tag{5.1}
(\nabla \cdot v_{S,h}, \tau_h \rho_S)_{\Omega} &= (\nabla \cdot v_{S,0}, \rho_S)_{\Omega} - \sum_{T_S \in T_{S,h}} \langle v_{S,0} - v_{S,h}, (\tau_h \rho_S) \cdot n \rangle_{\partial T_S}. \tag{5.2}
\end{align*}

With the above lemma, we can establish the error equations.

\textbf{Lemma 5.2.} Let $u_S, p_S$ and $p_D$ be the exact solutions of equations (2.1) - (2.9), $u_{S,h}, p_{S,h}$ and $p_{D,h}$ be the numerical solution of the scheme (3.1) - (3.2). Then the following equations

\begin{align*}
a_S(e_{S,h}, v_{S,h}) + a_D(e_D, Q_{D,h}) &= a_S(e_{S,h}, \epsilon_D) + a_D(e_D, v_{S,h} - q_{D,h}) \tag{5.1} \\
b(e_{S,h}, e_{S,h}) &= b(u_s, p_s, (v_{S,h}) , \tag{5.2} \\
b(e_{S,h}, q_{S,h}) &= 0, \tag{5.3}
\end{align*}

hold for all $v_{S,h} \in V^0_h$, $q_{S,h} \in W_{S,h}$ and $q_{D,h} \in W_{D,h}$. Where

$$\varphi_{u_{S,h}, p_S}(v_{S,h}) = \sum_{T_S \in T_{S,h}} \langle 2p(v_{s,0} - v_{S,h}), D(u_S) \cdot n - (Q_h D(u_S)) \cdot n \rangle_{\partial T_S}$$

$$- \sum_{T_S \in T_{S,h}} \langle v_{S,0} - v_{S,h}, (p_S - \tau_h p_s) n \rangle_{\partial T_S} + s(Q_h u_S, v_{S,h}).$$
Proof. Testing the problem (2.1) and (2.4) by \( \mathbf{v}_{S,0} \) and \( q_{D,h} \) yields

\[
- (\nabla \cdot T(\mathbf{u}_S, p_S), \mathbf{v}_{S,0})_{\Omega_S} = (\mathbf{s}, \mathbf{v}_{S,0})_{\Omega_S},
- (\nabla \cdot (K \nabla p_D), q_{D,h})_{\Omega_D} = (f_D, q_{D,h})_{\Omega_D}.
\]

From the integration by parts, it reaches that

\[
(s, v_{S,0})_{\Omega_S} = \sum_{T_S \in T_{Sh}} (2\mu D(\mathbf{u}_S), \nabla v_{S,0})_{T_S} - \sum_{T_S \in T_{Sh}} (\nabla \cdot v_{S,0})_{T_S} p_S
- \sum_{T_S \in T_{Sh}} (2\mu \mathbf{v}_{S,0} \cdot D(\mathbf{u}_S) \cdot n)_{T_S} + \sum_{T_S \in T_{Sh}} (\mathbf{v}_{S,0} \cdot p_S n)_{T_S}
- \sum_{e \in \Sigma} \langle \mathbf{v}_{S,0} \cdot T(\mathbf{u}_S, p_S) n \rangle_e,
\]

and

\[
(g_{f_D}, q_{D,h})_{\Omega_D} = (gK \nabla p_D, \nabla q_{D,h})_{\Omega_D} - \langle g q_{D,h} K \nabla p_D \cdot n' \rangle_{\partial \Omega_D}
= (gK \nabla p_D, \nabla q_{D,h})_{\Omega_D} - \langle g q_{D,h} K \nabla p_D \cdot n' \rangle_{\Sigma},
\]

where, \( n' \) is the unit outward vector of \( \Omega_D \) on \( \partial \Omega_D \), we know that \( -n = n' \) on \( \Sigma \). From the interface conditions (2.8)-(2.9), we know that

\[
- \sum_{e \in \Sigma} \langle \mathbf{v}_{S,b} \cdot T(\mathbf{u}_S, p_S) n \rangle_e + \sum_{e \in \Sigma} \langle g q_{D,h} K \nabla p_D \cdot n \rangle_e
- \sum_{e \in \Sigma} \langle \mathbf{v}_{S,b} \cdot \nabla \mathbf{u}_S \rangle e - \sum_{j=1}^{d-1} \sum_{e \in \Sigma} \langle \mathbf{v}_{S,b} \cdot \mathbf{\tau}_j \cdot T(\mathbf{u}_S, p_S) \mathbf{n} \cdot \mathbf{\tau}_j \rangle e
- \sum_{e \in \Sigma} \langle g q_{D,h} K \nabla p_D \cdot n \rangle e
= \sum_{e \in \Sigma} \langle g p_D, \mathbf{v}_{S,b} \cdot n \rangle e - \sum_{e \in \Sigma} \langle g q_{D,h} Q_h \mathbf{u}_S \cdot n \rangle e + \sum_{j=1}^{d-1} \sum_{e \in \Sigma} \langle a \mathbf{u}_S \cdot \mathbf{\tau}_j \cdot \mathbf{v}_{S,b} \cdot \mathbf{\tau}_j \rangle e.
\]
Applying Lemma 5.1 yields
\[
(s, v_S, 0)_{\Omega_S} + (g \mathbf{f}_D, q_{D,h})_{\Omega_D} = \\
\sum_{T_S \in \mathcal{T}_{S,h}} (2 \mu D_w(Q_{h} u_S), D_w(v_{S,h}))_{T_S} - \sum_{T_S \in \mathcal{T}_{S,h}} \langle \nabla w \cdot v_{S,h}, \tau_h p_S \rangle_{T_S} \\
+ \langle g \mathbf{K} \nabla p_D, \nabla q_{D,h} \rangle_{\Omega_D} - \sum_{T_S \in \mathcal{T}_{S,h}} \langle 2\mu (v_{S,0} - v_{S,h}), D(u_S) \cdot n - (Q_{h} D(u_S)) \cdot n \rangle_{\partial T_S} \\
+ \sum_{T_S \in \mathcal{T}_{S,h}} \langle v_{S,0} - v_{S,h}, (p_S - \tau_h p_S) n \rangle_{\partial T_S} + \sum_{e \in \Sigma} \langle g p_D, v_{S,h} \cdot n \rangle_e \\
- \sum_{e \in \Sigma} \langle g q_{D,h}, Q_{h} u_S \rangle_{e} + \sum_{j=1}^{d-1} \sum_{e \in \Sigma} \langle a Q_{h} u_S \cdot \tau_j, v_{S,h} \cdot \tau_j \rangle_e
\]
\[
= a_S(Q_{h} u_S, v_{S,h}) + a_D(p_{D,h}, q_{D,h}) + a_{\Sigma}(Q_{h} u_S, p_D, v_{S,h}, q_{D,h}) - b(v_{S,h}, p_S) - \varphi u_{s,h} p_s(v_{S,h}).
\]
Notice that from the numerical scheme (3.1) we have
\[
(a_S(u_S, v_S, 0)_{\Omega_S} + (g \mathbf{f}_D, q_{D,h})_{\Omega_D})_{\Omega_D},
\]
so that
\[
a_S(e_{S,h}, v_{S,h}) + a_D(e_{D,h}, q_{D,h}) + a_{\Sigma}(e_{S,h}, e_{D,h}, v_{S,h}, q_{D,h}) - b(e_{S,h}, e_{S,h}) = \varphi u_{s,h} p_s(v_{S,h}).
\]
As to formula (5.2), testing (2.2) by any $q_{S,h} \in \mathcal{W}_{S,h}$ and then it follows from Lemma 4.3 that
\[
\langle \nabla w \cdot Q_{h} u_S, q_{S,h} \rangle_{\Omega_S} = \langle Q_{h} (\nabla \cdot u_S), q_{S,h} \rangle_{\Omega_S} = \langle \nabla \cdot u_S, q_{S,h} \rangle_{\Omega_S} = 0.
\]
From the numerical scheme (3.2) we have
\[
\langle \nabla w \cdot u_{S,h}, q_{S,h} \rangle_{\Omega_S} = 0,
\]
which leads to
\[
b(e_{S,h}, q) = \langle \nabla w \cdot (Q_{h} u_S - u_{S,h}), q_{S,h} \rangle_{\Omega_S} = 0.
\]
Thus, we complete the proof.

Now we are in the position to give the error estimate. Some useful technical tools are introduced in Appendix A. In this section, we shall use these tools to derive the optimal error estimate.
Theorem 5.1. Let $u_S \in [H^{k_0+1}(\Omega_S)]^d$, $p_S \in H^{k_0}(\Omega_S)$ and $p_D \in L^2(\Omega_D) \cap H^{k_0+1}(\Omega_D)$ be the solution of the problem (2.1)-(2.9), $u_{S,h}, p_{S,h} \in V_{h}^0$, $p_{D,h} \in W_{D,h}$ be the solution of the weak Galerkin scheme (3.1)-(3.2). Then the following error estimate holds true
\[
\|\varepsilon_{S,h}\|=\|\varepsilon_{S,h}\|_{\Omega_S}+\|p_D-p_{D,h}\|_{1,\Omega_D},
\]
(5.4)
Proof. For simplicity, we use $\delta$ to represent $h^{k+1}_S\|\nabla u\|_{k+1,\Omega_S}+h^{k+1}_S\|p\|_{k+1,\Omega_S}+h^{k}_S\|u\|_{k+1,\Omega_S}$. Let $v_{S,h} = e_{S,h}$, $q_{D,h} = e_{D,h}$, and $q_{S,h} = e_{h}$. Adding equations (5.1)-(5.2) together, we have
\[
\|\varepsilon_{S,h}\|^2+a_D(e_D,e_{D,h}) = \varepsilon_{D,h}(e_{D,h}e_D;e_{S,h};e_{D,h}).
\]
Adding $a_D(e_D,p_D-Q_{h}p_D)$ to both sides of the equation above gives
\[
\|\varepsilon_{S,h}\|^2+a_D(e_D,e_D) = a_D(e_D,Q_{h}p_D-p_D)+\varepsilon_{S,h}(e_{S,h}e_{D,h};e_{D,h}).
\]
Using Cauchy inequality and Lemma (A.1) we have
\[
a_D(e_D,Q_{h}p_D-p_D) = \langle gK\nabla e_D, \nabla(Q_{h}p_D-p_D) \rangle \leq \|gK\nabla e_D\|_{\Omega_S} \|\nabla(Q_{h}p_D-p_D)\|_{\Omega_S} \leq C h^{k} \|\nabla e_D\|_{k+1}.
\]
Next, we will estimate $\varepsilon_{S,h}(e_{S,h})$. It follows from Lemma (A.3) that
\[
\varepsilon_{S,h}(e_{S,h}) \leq C \delta \|e_{S,h}\|.
\]
By using the trace inequality (A.1) and the definition of $\|\cdot\|$, inverse inequality (A.2) and Lemma 4.1
\[
\|e_{D,h}\|_0 \leq \left( \frac{1}{2} \sum_{c \in T} \|g(p_D-Q_{h}p_D)\|_{c}^{2} \right)^{\frac{1}{2}} \left( \sum_{c \in T} h_{S,c}^{-\frac{1}{2}} \|e_{S,c}-Q_{h}e_{S,0}\|_{c}^{2} + h_{S}^{-\frac{1}{2}} \|Q_{h}e_{S,0}\|_{c}^{2} \right)^{\frac{1}{2}} \leq \frac{1}{2} C h^{k} \|p_D\|_{k+1} \|e_{S,h}\| + \|e_{S,h}\|_{T_{S,h}} \leq C h^{k+1} \|p_D\|_{k+1} \|e_{S,h}\|.
\]
(5.5)
Using the Young inequality and the Cauchy inequality we have
\[
\|\|e_{s,h}\| + \|\nabla e_D\| \leq C(\delta + h^{k_p} \|p_D\|_{k_D} + \Omega_D).
\]

Denote \(\varepsilon_{s,h} = \varepsilon_{s,h} + \varepsilon_{s,h}'\), where \(\varepsilon_{s,h} = \int_{\Omega_S} \varepsilon_{s,h} \, dx / |\Omega_S|\) and \(\varepsilon_{s,h} \in L^2_0(\Omega_S)\). From the error equation (5.1), we obtain
\[
b(\mathbf{v}_h, \varepsilon_h) = a(s(e_{s,h}', \mathbf{v}_{s,h})) + a_D(e_{s,h}, q_{s,h}) + a_G(e_{s,h}, \mathbf{v}_{s,h}, q_{s,h}) - \phi_{u_p}(\mathbf{v}_{s,h}).
\]

Taking \(\mathbf{v}_{s,h} = Q_h \mathbf{v}\) in Lemma 4.5 and taking \(q_{D,h} = 0\) in error equation (5.1), we have
\[
|b(\mathbf{v}_{s,h}, \varepsilon_{s,h})| = |(\nabla \cdot \mathbf{v}, \varepsilon_{s,h})| = |b(\mathbf{v}_{s,h}, \varepsilon_{s,h})| = a(s(e_{s,h}', \mathbf{v}_{s,h})) - \phi_{u_p}(\mathbf{v}_{s,h}) \leq \|e_{s,h}\| \|\mathbf{v}_{s,h}\| + |\phi_{u_p}(\mathbf{v}_{s,h})| \leq C_\delta \|\mathbf{v}_{s,h}\|,
\]

From Lemma 4.5 we know that
\[
\beta \|\varepsilon_{s,h}\|_{\Omega_S} \leq \sup_{\mathbf{v}_{s,h} \in V_h^0} \left|\frac{b(\mathbf{v}_{s,h}, \varepsilon_h)}{\|\mathbf{v}_{s,h}\|}\right| = (5.6)
\]

Combine the estimate (5.6) with (5.7), we have
\[
\|\varepsilon_{s,h}\|_{\Omega_S} \leq C_\delta.
\]

Next we consider the estimate for \(\varepsilon_h\). Take a smooth function \(\mathbf{v} \in [C_0^1(\Omega_S)]^d\) dependent on the domain \(\Omega_S\) such that
\[
\int_{\Omega_S} \nabla \cdot \mathbf{v} \, dx = 1.
\]

Write \(\gamma = \|\mathbf{v}\|_{1,\Omega_S}\) and let \(\mathbf{v}_{s,h} = Q_h \mathbf{v}\), from Lemma 4.5 we have
\[
\int_{\Omega_S} \nabla \cdot \mathbf{v}_{s,h} \, dx = \int_{\Omega_S} \mathbf{Q}_h \nabla \cdot \mathbf{v} \, dx = \int_{\Omega_S} \nabla \cdot \mathbf{v} \, dx = 1,
\]

\[
\|\mathbf{v}_{s,h}\| + \|\nabla \cdot \mathbf{v}_{s,h}\|_{\Omega_S} \leq C_0 \gamma + \|\mathbf{Q}_h \nabla \cdot \mathbf{v}\|_{\Omega_S} \leq (C_0 + 1) \gamma.
\]

From the error equation, we have
\[
|\varepsilon_{s,h}| = \frac{b(\mathbf{v}_{s,h}, \varepsilon_h)}{\int_{\Omega_S} \nabla \cdot \mathbf{v}_{s,h} \, dx} = a(s(e_{s,h}', \mathbf{v}_{s,h})) - \phi_{u_p}(\mathbf{v}_{s,h}) - b(\mathbf{v}_{s,h}, \varepsilon_h) \leq \|e_{s,h}\| \|\mathbf{v}_{s,h}\| + |\phi_{u_p}(\mathbf{v}_{s,h})| + \|\varepsilon_h\|_{\Omega_S} \|\nabla \cdot \mathbf{v}_{s,h}\|_{\Omega_S} \leq C_\delta,
\]

which completes the proof.
6 Numerical experiments

In this section, we conduct several numerical experiments to numerically validate the efficiency of the weak Galerkin and continuous Galerkin coupled finite elements. In the first numerical test, we solve the following coupled problem on \( \{ \Omega_S = (0, \pi) \times (0, \pi) \} \cup \{ \Omega_D = (0, \pi) \times (-\pi, 0) \} \):

\[
\begin{align*}
-\nabla \cdot (\nabla u_S + \nabla^T u_S) + \nabla p_S &= f_S \quad \text{in } \Omega_S, \\
-\nabla \cdot (\nabla p_D) &= 0 \quad \text{in } \Omega_D, \\
\begin{bmatrix}
\mathbf{u}_S \cdot \mathbf{n} \\
(\nabla \mathbf{u}_S - \nabla^T \mathbf{u}_S + p_S I) \mathbf{n} \cdot \mathbf{n} \\
(\nabla \mathbf{u}_S - \nabla^T \mathbf{u}_S + p_S I) \mathbf{n} \cdot \mathbf{t}
\end{bmatrix} &= \begin{bmatrix}
-\nabla p_D \cdot \mathbf{n} \\
p_D \\
\mathbf{u}_S \cdot \mathbf{t}
\end{bmatrix} \quad \text{on } \Gamma,
\end{align*}
\]

with Dirichlet boundary conditions

\[
\begin{align*}
\mathbf{u}_S &= \begin{bmatrix} 2\sin y \cos y \cos x \\ (\sin^2 y - 2) \sin x \end{bmatrix} \quad \text{on } \Gamma_S, \\
p_D &= e^y - e^{-y} \sin x \quad \text{on } \Gamma_D.
\end{align*}
\]

The source function in (6.1) is defined by

\[
f_S = \begin{bmatrix} \sin y \cos x (5 \cos y + 1) \\ \sin x (-\cos^2 y + \frac{3}{2} \sin^2 y - 1 + \cos y) \end{bmatrix}.
\]

The exact solutions are

\[
\begin{align*}
\mathbf{u}_S &= \begin{bmatrix} 2\sin y \cos y \cos x \\ (\sin^2 y - 2) \sin x \end{bmatrix} \quad \text{in } \Omega_S, \\
p_S &= \sin x \sin y \quad \text{in } \Omega_S, \\
p_D &= (e^y - e^{-y}) \sin x \quad \text{in } \Omega_D.
\end{align*}
\]

On the interface, (6.2) is satisfied as

\[
\begin{bmatrix}
2\sin x \\
0
\end{bmatrix} = \begin{bmatrix}
2\sin x \\
0
\end{bmatrix} \quad \text{on } \Sigma.
\]

We plot the velocity field \( (\mathbf{u}_S \& \mathbf{u}_D = -\nabla p_D) \) in Figure 2.

In the computation, the first level grid consists of four triangles, cutting each of two rectangles (see Figure 2) into two triangles by a north-west to south-east diagonal line. Then, each subsequent grid is a bi-sectional refinement. We apply first the weak Galerkin \( P_1 \) finite element method for \( \mathbf{u}_S \) and \( p_S \) and the continuous Galerkin \( P_1 \) finite element method for \( p_D \) in solving (6.3). The errors and the orders of convergence for the variables...
in various norms are reported in Table 1. We can see, as proved theoretically, all numerical solutions converge at the optimal order. In particular, the continuous $P_1$ element has an $H^1$-superconvergence, the same as in solving the standard (uncoupled) elliptic problems.

In Tables 2–5 we solve problem (6.1) with exact solution (6.3) by $P_2$ to $P_5$ weak Galerkin and continuous Galerkin coupled finite elements. The optimal order of convergence is achieved in every case. Further, the usual superconvergence in both $L^2$ and $H^1$ norms also appears here in the Darcy region for the $P_2$ continuous Galerkin finite element.

In the second numerical example, we solve the coupled problem (6.3) on domain \{\(\Omega_S = (0,1) \times (1,2)\) \(\cup \\Omega_D = (0,1) \times (0,1)\)\}. The exact solutions are

\[
\begin{align*}
  \mathbf{u}_S &= \begin{pmatrix} -\cos(\pi x) \sin(\pi y) \\ \sin(\pi x) \cos(\pi y) \end{pmatrix} \quad \text{in} \ \Omega_S, \\
p_S &= \sin(\pi x) \quad \text{in} \ \Omega_S, \\
p_D &= y \sin(\pi x) \quad \text{in} \ \Omega_D. 
\end{align*}
\] (6.5)

On the interface \(\Gamma = (0,1) \times \{1\}\), the interface condition (6.2) is reduced to

\[
\begin{pmatrix} \sin \pi x \\
\sin \pi x \\
0 \end{pmatrix} = \begin{pmatrix} \sin \pi x \\
\sin \pi x \\
0 \end{pmatrix} \quad \text{on} \ \Sigma. 
\] (6.7)

The velocity field \((\mathbf{u}_S, \mathbf{u}_D)\) is plotted in Figure 6. We list the order of convergence in
Table 1: The errors and the orders of convergence ($n$ in the $O(h^n)$ error), by the $P_1$ WG elements and $P_1$ CG elements, for (6.3).

<table>
<thead>
<tr>
<th>level</th>
<th>$|Q_h u_S - u_{S,h}|_0$</th>
<th>$k$</th>
<th>$|Q_h u_S - u_{S,h}|_0$</th>
<th>$k$</th>
<th>$|Q_h p_S - p_{S,h}|_0$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$0.1787E+00$</td>
<td>1.7</td>
<td>$0.9329E+00$</td>
<td>0.7</td>
<td>$0.3213E+00$</td>
<td>0.8</td>
</tr>
<tr>
<td>5</td>
<td>$0.4960E-01$</td>
<td>1.8</td>
<td>$0.4967E+00$</td>
<td>0.9</td>
<td>$0.1384E+00$</td>
<td>1.2</td>
</tr>
<tr>
<td>6</td>
<td>$0.1286E-01$</td>
<td>1.9</td>
<td>$0.2516E+00$</td>
<td>1.0</td>
<td>$0.5846E-01$</td>
<td>1.2</td>
</tr>
<tr>
<td>7</td>
<td>$0.3247E-02$</td>
<td>2.0</td>
<td>$0.1260E+00$</td>
<td>1.0</td>
<td>$0.2702E-01$</td>
<td>1.1</td>
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<table>
<thead>
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<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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<td>$0.2986E+00$</td>
<td>1.9</td>
</tr>
<tr>
<td>5</td>
<td>$0.6046E-01$</td>
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<td>$0.7693E-01$</td>
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</tr>
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<td>6</td>
<td>$0.1525E-01$</td>
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<td>$0.1940E-01$</td>
<td>2.0</td>
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<tr>
<td>7</td>
<td>$0.3821E-02$</td>
<td>2.0</td>
<td>$0.4860E-02$</td>
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</tr>
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</table>

Table 1 by $P_2$, $P_3$ and $P_5$ WG-CG coupled finite element methods. The optimal order convergence is obtained in all cases, confirming the theory.
Table 2: The errors and the orders of convergence ($k$ in the $O(h^k)$ error), by the $P_2$ WG elements and $P_2$ CG elements, for \(6.3\).

<table>
<thead>
<tr>
<th>level</th>
<th>$|Q_h u_S - u_{S,h}|_0$</th>
<th>$k$</th>
<th>$|Q_h u_S - u_{S,h}|$</th>
<th>$k$</th>
<th>$|Q_h p_S - p_{S,h}|_0$</th>
<th>$k$</th>
</tr>
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<td>0.2252E+00</td>
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<td>0.4999E-01</td>
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</tr>
<tr>
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<td>0.4274E-03</td>
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<td>0.1664E-01</td>
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<td>0.2692E-02</td>
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</tr>
<tr>
<td>7</td>
<td>0.5478E-04</td>
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<td>0.4241E-02</td>
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<td>0.6472E-03</td>
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<table>
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</tr>
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<td>0.2203E-01</td>
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<tr>
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<td>0.3085E-02</td>
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<tr>
<td>6</td>
<td>0.8877E-05</td>
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<td>0.4340E-03</td>
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</tr>
<tr>
<td>7</td>
<td>0.5803E-06</td>
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<td>0.6326E-04</td>
<td>2.8</td>
</tr>
</tbody>
</table>

Table 3: The errors and the orders of convergence ($k$ in the $O(h^k)$ error), by the $P_3$ WG elements and $P_3$ CG elements, for \(6.3\).

<table>
<thead>
<tr>
<th>level</th>
<th>$|Q_h u_S - u_{S,h}|_0$</th>
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<th>$|Q_h u_S - u_{S,h}|$</th>
<th>$k$</th>
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<tbody>
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<td>3.5</td>
<td>0.1913E+00</td>
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<td>0.2971E-01</td>
<td>2.7</td>
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<tr>
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<td>0.1355E-02</td>
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<td>0.2799E-01</td>
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<td>0.3366E-02</td>
<td>3.1</td>
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<tr>
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<td>0.9254E-04</td>
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<td>0.3680E-03</td>
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</table>

<table>
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<th>$|I_h p_D - p_{D,h}|$</th>
<th>$k$</th>
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<tr>
<td>4</td>
<td>0.2382E-03</td>
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<td>0.5399E-02</td>
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<td>5</td>
<td>0.1489E-04</td>
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<td>0.7037E-03</td>
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</tr>
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<td>6</td>
<td>0.9317E-06</td>
<td>4.0</td>
<td>0.8981E-04</td>
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</tr>
</tbody>
</table>
Table 4: The errors and the orders of convergence ($k$ in the $O(h^k)$ error), by the $P_4$ WG elements and $P_4\ CG$ elements, for (6.3).

<table>
<thead>
<tr>
<th>Level</th>
<th>$|Q_h u_S - u_{S,h}|_0$</th>
<th>$k$</th>
<th>$|Q_h u_S - u_{S,h}|$</th>
<th>$k$</th>
<th>$|Q_h p_S - p_{S,h}|_0$</th>
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<td>0.1582E-04</td>
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<tr>
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<td>0.6320E-07</td>
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<table>
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<tr>
<th>Level</th>
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<th>$|I_h p_D - p_{D,h}|_1$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.2951E-03</td>
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<td>0.3700E-02</td>
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</tr>
<tr>
<td>4</td>
<td>0.9371E-05</td>
<td>5.0</td>
<td>0.2424E-03</td>
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</tr>
<tr>
<td>5</td>
<td>0.2917E-06</td>
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<td>0.1542E-04</td>
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</tr>
<tr>
<td>6</td>
<td>0.9062E-08</td>
<td>5.0</td>
<td>0.9709E-06</td>
<td>4.0</td>
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</table>

Table 5: The errors and the orders of convergence ($k$ in the $O(h^k)$ error), by the $P_5$ WG elements and $P_5\ CG$ elements, for (6.3).

<table>
<thead>
<tr>
<th>Level</th>
<th>$|Q_h u_S - u_{S,h}|_0$</th>
<th>$k$</th>
<th>$|Q_h u_S - u_{S,h}|$</th>
<th>$k$</th>
<th>$|Q_h p_S - p_{S,h}|_0$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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<td>0.4723E-02</td>
<td>4.9</td>
<td>0.4216E-03</td>
<td>4.7</td>
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<td>4</td>
<td>0.3123E-05</td>
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<td>0.1544E-03</td>
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<td>0.1364E-04</td>
<td>5.0</td>
</tr>
<tr>
<td>5</td>
<td>0.4998E-07</td>
<td>6.0</td>
<td>0.4914E-05</td>
<td>5.0</td>
<td>0.4294E-06</td>
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<td>0.7883E-09</td>
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<td>0.1545E-06</td>
<td>5.0</td>
<td>0.1483E-07</td>
<td>4.9</td>
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<table>
<thead>
<tr>
<th>Level</th>
<th>$|I_h p_D - p_{D,h}|_0$</th>
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<th>$|I_h p_D - p_{D,h}|_1$</th>
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<tbody>
<tr>
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<td>0.8608E-08</td>
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</table>
Figure 3: The numerical solution for \((u_S, u_D)_1\) and its error by \(P_4\) elements on level 4, for (6.3).

Figure 4: The numerical solution for \((u_S, u_D)_2\) and its error by \(P_4\) elements on level 4, for (6.3).
Figure 5: The numerical solution for \((p_S,p_D)\) and its error by \(P_4\) elements on level 4, for (6.3).

Figure 6: The velocity field of the second example, (6.5).
Table 6: The errors and the orders of convergence, for (6.5).

<table>
<thead>
<tr>
<th>level</th>
<th>$|Q_hu_S - u_{S,h}|_0$</th>
<th>$k$</th>
<th>$|Q_hp_S - p_{S,h}|_0$</th>
<th>$k$</th>
<th>$|I_hp_D - p_{D,h}|_0$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.1283E-01</td>
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<td>0.4895E-01</td>
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<tr>
<td>4</td>
<td>0.1682E-02</td>
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<td>0.1261E-01</td>
<td>2.0</td>
<td>0.4920E-04</td>
<td>3.7</td>
</tr>
<tr>
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By coupled $P_2$ weak Galerkin and continuous Galerkin element.

<table>
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<tr>
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<th>$|Q_hu_S - u_{S,h}|_0$</th>
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</table>

By coupled $P_3$ weak Galerkin and continuous Galerkin element.

<table>
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<tr>
<th>level</th>
<th>$|Q_hu_S - u_{S,h}|_0$</th>
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<th>$|Q_hp_S - p_{S,h}|_0$</th>
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<tr>
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<td>0.3185E-11</td>
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</tr>
</tbody>
</table>

By coupled $P_5$ weak Galerkin and continuous Galerkin element.
A Some technique tools

In order to give the error estimates, we need to estimate the remainder \( \varphi_{u,p_s}(v_{S,h}) \) with the help of the trace theorem, the inverse theorem and the classical interpolation theory. In this section we shall introduce these techniques.

Lemma A.1. Let \( T_h \) be a finite element partition of domain \( K \) satisfying the shape regularity assumptions as specified in [40] and \( w \in [H^{r+1}(K)]^d \) and \( \rho \in H^m(K) \). Then, for \( 0 \leq s \leq 1 \) we have

\[
\sum_{T \in T_h} h_T^{2s} \| w - Q_0 w \|_{T, \delta}^2 \leq C h^{2(r+1)} \| w \|_{r+1}^2, \quad 1 \leq r \leq k_0,
\]
\[
\sum_{T \in T_h} h_T^{2s} \| \nabla w - Q_h(\nabla w) \|_{T, \delta}^2 \leq C h^{2(r+1)} \| \nabla w \|_{r+1}^2, \quad 1 \leq r \leq k_b,
\]
\[
\sum_{T \in T_h} h_T^{2s} \| \rho - \pi_h \rho \|_{T, \delta}^2 \leq C h^{2(m+1)} \| \rho \|_{r+1}^2, \quad 1 \leq m \leq k_S.
\]

Here \( C \) denotes a generic constant independent of the meshsize \( h \) and the functions in the estimates.

Let \( T \) be an element satisfying the assumption verified in [40] with \( e \) as a face. For any function \( g \in H^1(T) \), the following trace inequality has been proved in [40]

\[
\| g \|_e^2 \leq C (h_T^{-1} \| g \|_T + h_T \| \nabla g \|_T^2).
\] (A.1)

Particularly, if \( g \) is polynomial in \( T \) we have the inverse inequality [40]

\[
\| \nabla g \|_T^2 \leq C h_T^{-2} \| g \|_T^2,
\] (A.2)

where \( C \) is a constant only related to the degree of polynomial and the dimension. Combining with the trace inequality we can get further that

\[
\| \nabla g \|_T^2 \leq C h_T^{-1} \| g \|_T^2.
\] (A.3)

The vector version of the trace theorem and the inverse theorem are trivial.

Lemma A.2. For any \( v_{S,h} \in V_{h}^0 \), we have

\[
\sum_{T \in T_{S,h}} \| v_{S,0} - v_{S,0b} \|_{\partial T} \leq C h^{1/2} \| v_{S,h} \|.
\] (A.4)

Proof. When \( k_0 = k_b \), (A.4) is obvious. So we only need to discuss the case that \( k_b = k_0 - 1 \). We only consider the vector valued function \( v_{S,h} \). From Lemma A.1 we have

\[
\sum_{T \in T_{S,h}} \| \nabla v_{S,0} \|_{T} \leq C \| v_{S,h} \|.
\] (A.5)
Using the trace inequality (A.1) and Poincare’s inequality, we can obtain that
\[
\sum_{T_s \in T_{SA}} \| \mathbf{v}_{s,0} - \mathbf{v}_{s,b} \|_{\partial T_s} \leq \sum_{T_s \in T_{SA}} \left( \| \mathbf{v}_{s,0} - \mathbf{Q}_b \mathbf{v}_{s,0} \|_{\partial T_s} + \sum_{T_s \in T_{SA}} \| \mathbf{Q}_b \mathbf{v}_{s,0} - \mathbf{v}_{s,b} \|_{\partial T_s} \right)
\]
\[
\leq C \sum_{T_s \in T_{SA}} h_{T_s} \| \nabla \mathbf{v}_{s,b} \|_{\partial T_s} + \sum_{T_s \in T_{SA}} \| \mathbf{Q}_b \mathbf{v}_{s,0} - \mathbf{v}_{s,b} \|_{\partial T_s}
\]
\[
\leq Ch_s^2 \sum_{T_s \in T_{SA}} \| \nabla \mathbf{v}_{s,b} \|_{T_s} + h_s^2 \sum_{T_s \in T_{SA}} h_{T_s}^{-1} \| \mathbf{Q}_b \mathbf{v}_{s,0} - \mathbf{v}_{s,b} \|_{\partial T_s}
\]
\[
\leq Ch_s^2 \| \mathbf{v}_{s,b} \|_{\Omega_s},
\]
which completes the proof. □

**Lemma A.3.** Assume that \( \mathbf{w}_s \in [H^{k+1}_0(\Omega_s)]^d, \rho_s \in H^{k+1}_0(\Omega_s), \) then
\[
\sum_{T_s \in T_{SA}} \| D(\mathbf{w}_s) - \mathbf{Q}_b D(\mathbf{w}_s) \|_{\partial T_s} \leq Ch_s^{k+\frac{1}{2}} \| \mathbf{w}_s \|_{k+1,\Omega_s}, \tag{A.6}
\]
\[
\sum_{T_s \in T_{SA}} \| \rho_s - \pi_h \rho_s \|_{\partial T_s} \leq Ch_s^{k+\frac{1}{2}} \| \rho_s \|_{k+1,\Omega_s}, \tag{A.7}
\]

Thus,
\[
| \varphi_{\mathbf{w}_s, \rho_s}(\mathbf{v}_{s,b}) | \leq C(Ch_s^{k+\frac{1}{2}} \| \nabla \mathbf{w}_s \|_{k+1,\Omega_s} + h_s^{k+1} \| \rho_s \|_{k+1,\Omega_s}) \| \mathbf{v}_{s,b} \|_{\Omega_s}. \tag{A.8}
\]

**Proof.** Applying the trace inequality, we can get that
\[
\sum_{T_s \in T_{SA}} \| D(\mathbf{w}_s) - \mathbf{Q}_b D(\mathbf{w}_s) \|_{\partial T_s}
\]
\[
\leq C \sum_{T_s \in T_{SA}} (h_{T_s}^{-1} \| \nabla \mathbf{w}_s - \mathbf{Q}_b \nabla \mathbf{w}_s \|_{T_s} + h_{T_s}^2 \| \nabla \mathbf{w}_s - \mathbf{Q}_b \nabla \mathbf{w}_s \|_{1,T})
\]
\[
\leq Ch_s^{k+\frac{1}{2}} \| \nabla \mathbf{w}_s \|_{k+1,\Omega_s},
\]
and for \( \rho_s \) we have
\[
\sum_{T_s \in T_{SA}} \| \rho_s - \pi_h \rho_s \|_{\partial T_s}
\]
\[
\leq C \sum_{T_s \in T_{SA}} (h_{T_s}^{-\frac{1}{2}} \| \rho_s - \pi_h \rho_s \|_{T_s} + h_{T_s}^{\frac{1}{2}} \| \rho_s - \pi_h \rho_s \|_{1,T})
\]
\[
\leq Ch_s^{k+\frac{1}{2}} \| \rho_s \|_{k+1,\Omega_s}.
\]
Combining (A.4)–(A.7) together yields

\[ |q_{w_s}(v_{s,b})| = \sum_{T_S \in T_{S,b}} (2\mu(v_{s,\partial} - v_{s,b}), D(w_S) \cdot n - (Q_h D(w_s)) \cdot n)_{\partial T_S} \]

\[ - \sum_{T_S \in T_{S,b}} (v_{s,\partial} - v_{s,b}(p_{s,\partial} - \pi_{w_s} p_{s})n)_{\partial T_S} + s(Q_h w_S, v_{s,b}) \]

\[ \leq C \left( \sum_{T_S \in T_{S,b}} \|v_{s,b} - v_{s,\partial}\|_{\partial T_S}^2 \right)^{1/2} \left( \sum_{T_S \in T_{S,b}} \|D(w_s) \cdot n - Q_h D(w_s) \cdot n\|_{\partial T_S}^2 \right)^{1/2} \]

\[ + C \left( \sum_{T_S \in T_{S,b}} \|v_{s,\partial} - v_{s,b}\|_{\partial T_S}^2 \right)^{1/2} \left( \sum_{T_S \in T_{S,b}} \|p_{s} - \pi_{w_s} p_{s}\|_{\partial T_S}^2 \right)^{1/2} \]

\[ + C \left( \sum_{T_S \in T_{S,b}} h^{-1}_T \|Q_h v_{s,\partial} - v_{s,b}\|_{\partial T_S}^2 \right)^{1/2} \left( \sum_{T_S \in T_{S,b}} h^{-1}_T \|Q_h 0 w_S - Q_h w_S\|_{\partial T_S}^2 \right)^{1/2} \]

\[ \leq C(h^{k+2}_S + \|\nabla w_S\|_{k+1,\Omega_S} + h^{k+1}_S \|w_S\|_{k+1,\Omega_S} + h^{k+1}_S \|p_S\|_{k+1,\Omega_S}) \|v_{s,b}\| \] 

which completes the proof.

**B The table of notations**

There are a great number of necessary notations referred in this paper, which may be a barrier for readers. So in this appendix, we present a table of notations.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega )</td>
<td>bounded domain</td>
</tr>
<tr>
<td>( \Omega_S )</td>
<td>fluid region</td>
</tr>
<tr>
<td>( \Omega_D )</td>
<td>porous medium region</td>
</tr>
<tr>
<td>( \partial \Omega_S )</td>
<td>boundary of ( \Omega_S )</td>
</tr>
<tr>
<td>( \partial \Omega_D )</td>
<td>boundary of ( \Omega_D )</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>interface</td>
</tr>
<tr>
<td>( \Gamma_S )</td>
<td>( \partial \Omega_S \setminus \Sigma )</td>
</tr>
<tr>
<td>( \Gamma_D )</td>
<td>( \partial \Omega_D \setminus \Sigma )</td>
</tr>
<tr>
<td>( n )</td>
<td>unit normal outward vector</td>
</tr>
<tr>
<td>( \mathbf{o}_j, j = 1, \ldots, d - 1 )</td>
<td>unit tangent vectors</td>
</tr>
<tr>
<td>( u_S )</td>
<td>fluid velocity</td>
</tr>
<tr>
<td>( p_S )</td>
<td>kinematic pressure</td>
</tr>
<tr>
<td>( f_S )</td>
<td>external body force density</td>
</tr>
<tr>
<td>( T(u_S, p_S) )</td>
<td>stress tensor</td>
</tr>
<tr>
<td>( D(u_S) )</td>
<td>strain tensor</td>
</tr>
<tr>
<td>( I )</td>
<td>identity matrix</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>$\mu$</td>
<td>kinematic viscosity of the fluid</td>
</tr>
<tr>
<td>$u_D$</td>
<td>fluid discharge rate</td>
</tr>
<tr>
<td>$K$</td>
<td>hydraulic conductivity tensor</td>
</tr>
<tr>
<td>$p_D$</td>
<td>hydraulic head</td>
</tr>
<tr>
<td>$f_D$</td>
<td>source term</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$z$</td>
<td>relative depth from an arbitrary fixed reference height</td>
</tr>
<tr>
<td>$K$</td>
<td>open bounded domain</td>
</tr>
<tr>
<td>$H^m(K)$</td>
<td>Sobolev space</td>
</tr>
<tr>
<td>$| \cdot |_m$</td>
<td>norm of $H^m(K)$</td>
</tr>
<tr>
<td>$(\cdot, \cdot)_m$</td>
<td>inner product of $H^m(K)$</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$T_{S,h}$</td>
<td>WG-regular partition of $\Omega_S$</td>
</tr>
<tr>
<td>$T_{D,h}$</td>
<td>FEM partition of $\Omega_D$</td>
</tr>
<tr>
<td>$T_h$</td>
<td>union of $T_{S,h}$ and $T_{D,h}$</td>
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<tr>
<td>$\mathcal{E}_h$</td>
<td>edges or flat faces in $T_h$</td>
</tr>
<tr>
<td>$T_i$</td>
<td>element of $T_{i,h}$</td>
</tr>
<tr>
<td>$e_i$</td>
<td>edge or flat face on $\partial T_i$</td>
</tr>
<tr>
<td>$h_S$</td>
<td>mesh size of $T_{S,h}$</td>
</tr>
<tr>
<td>$h_D$</td>
<td>mesh size of $T_{D,h}$</td>
</tr>
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<td>$v_{S,h}$</td>
<td>weak function on $T_S$</td>
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<tr>
<td>$v_{S,0}$</td>
<td>internal part of $v_{S,h}$</td>
</tr>
<tr>
<td>$v_{S,0}$</td>
<td>boundary part of $v_{S,h}$</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>$L^2$ projection operator to $[P_{k_0}(T_S)]^d$</td>
</tr>
<tr>
<td>$Q_b$</td>
<td>$L^2$ projection operator to $[P_{k_b}(e_S)]^d$</td>
</tr>
<tr>
<td>$Q_h$</td>
<td>$Q_h := { Q_0, Q_b }$</td>
</tr>
<tr>
<td>$Q_{h,b}$</td>
<td>$L^2$ projection operator to $W_{D,h}$</td>
</tr>
<tr>
<td>$Q_s$</td>
<td>$L^2$ projection operator to $[P_{k_0-1}(T_S)]^{d \times d}$</td>
</tr>
<tr>
<td>$Q_{h,b}$</td>
<td>$L^2$ projection operator to $P_{k_{b-1}}(T_S)$</td>
</tr>
<tr>
<td>$\pi_h$</td>
<td>$L^2$ projection operator to $P_{k_s}(T_S)$</td>
</tr>
<tr>
<td>$e_{S,h}$</td>
<td>$Q_{h,b}u_s - u_{S,h}$</td>
</tr>
<tr>
<td>$e_{D,h}$</td>
<td>$Q_{h,b}p_s - p_{S,h}$</td>
</tr>
<tr>
<td>$\epsilon_{D,h}$</td>
<td>$Q_{h}p_D - p_{D,h}$</td>
</tr>
</tbody>
</table>

**Acknowledgments**

The work was supported in part by China Natural Science Foundation(11971198, 91630201, 11871245, 11771179, 11826101), and by the Program for Cheung Kong Scholars(Q2016067), Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of E-
ducation, Jilin University, Changchun, 130012, P.R.China.

References


(2019), 998-1020.


