A family of optimal Lagrange elements for Maxwell’s equations

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A B S T R A C T

In this paper we propose and study a new Lagrange finite element method for the two-dimensional Maxwell’s equations. Its solution may be singular because of the nonsmooth domain with reentrant corners. The proposed method allows the standard Lagrange elements on barycentric refinements of any order greater than or equal to two. We analyze the proposed method for the eigenvalue problem and the indefinite source problem, obtaining the well-posedness and optimal error estimates. Numerical results are presented for confirming the theoretical results.

1. Introduction

Maxwell’s equations are the fundamental principle in electromagnetism and play an essential role in many modern applications and key technologies. They include applications in the renewable energy science, material science, nanotechnology, life science, magnetic levitation technology, and many others. The Maxwell’s equations are so widely applicable and useful that there have been many finite element methods in the field of computational electromagnetism. The nowadays’ classical method is the edge element method (e.g., [1–3]). Most contributions have been focusing on various discontinuous Galerkin methods, e.g., the nonconforming element method (e.g., [4]), the interior penalty discontinuous Galerkin method (e.g., [5–9]), the locally divergence-free method (e.g., [10]), the nodal discontinuous Galerkin method (e.g., [11]), and the finite volume method (e.g., [12]). In the context of nodal-continuous Lagrange elements, a number of methods have also been available, e.g., the weighted method (e.g., [13,14]), the $H^1$-norm method (e.g., [15]), the $L^2$ projection method (e.g., [16–21]), the mixed method in the classical saddle-point form (e.g., [22]), the least-squares method (e.g., [23]), and the penalty/regularization method (e.g., [24]).

In this paper, we propose a new Lagrange finite element method for solving the Maxwell’s equations in two-dimensions. In this method, a piecewise constant $L^2$ projection operator is applied to the div(divergence) operator, while the double curl is kept in its original variational form. The proposed method is much like the edge element method, except the piecewise constant $L^2$ projection term of the div operator. Such method allows any order greater than or equal to two Lagrange elements.

We mainly study the eigenvalue problem. The proposed method is symmetric, positive definite. The problem is posed in a nonsmooth Lipschitz polygon which may have reentrant corners and is non-convex. As a consequence, there may have
infinite singular eigenfunctions as well as infinite smooth ones. The singular eigenfunction means that it does not belong to the Hilbert space \(H^1(\Omega)^2\), while the smooth eigenfunction is, at least, \((H^1(\Omega))^2\) function. A very important fact about the eigenvalue problem is that it is not likely to priori know which eigenfunctions are singular or smooth. Keeping this fact in mind, we must design the finite element method to well approximate both the singular and smooth eigenfunctions. For this purpose, we propose a family of Lagrange elements on barycentric refinements of any order greater than or equal to two. A key feature for such family is that the Lagrange finite element spaces include the gradients of a family of scalar \(C^1\) finite element spaces as comprehensively studied in [25]. As such, the singular solutions can be well approximated. In other words, aiming at well approximating singular solutions with the strong unbounded singularity in the gradients in the \(L^2\) norm, we require the barycentric refinements. Note that the singularity of the solution of the Maxwell’s equations comes mainly from the gradient of scalar functions (cf., [26,27]), which happen to constitute the infinite kernel of the curl operator. With such inclusion, the singularity and the kernel space of the curl operator could be well approximated. By establishing the coercivity and some type of error estimates which holds uniformly relative to \(L^2(\Omega)^2\), we obtain the collective compactness property which is the key property for the spectral approximation of the compact operator (cf., [28,29]). Consequently, the proposed finite element method is spectrally correct. Moreover, optimal error bounds of the finite element solutions are obtained for singular and smooth solutions.

We would like to remark that the \(C^1\) element is merely for the theoretical purpose for discrete analogs of the Hodge-decomposition and the regular–singular decomposition (cf., [26,27]), having no influence on the finite element discretization in the proposed method which only uses nodal-continuous Lagrange elements. We would also like to remark that the role of the inclusion of the gradients of a \(C^1\) scalar element in the nodal-continuous Lagrange element space is the same as the role of the inclusion of the gradients of a \(C^0\) scalar element in the tangential continuous edge finite element space in the edge element method. Otherwise, the singularity and the kernel space of the curl operator could not be well approximated in either methods, i.e., the incorrectness in both the convergence and the spectral approximation would happen.

Since the proposed method is a type of \(L^2\) projection method, in the spirit of the previous methods in [18,30], we would like to tell the differences here. The previous methods bear an \(L^2\) projection applied to the curl operator, in addition to the \(L^2\) projection applied to the div operator. For higher-order convergence rates, the projection of the curl operator must be higher-order, too. Therein, the linear element or higher-order element projection is used for the div operator. Here, the method only applies a piecewise constant \(L^2\) projection to the div operator while the curl operator keeps its original form with no projections. The previous methods do not have optimal convergence rates for smooth solutions; actually, the convergence rates are the same as the orders of the \(L^2\) projections of the curl operator, but not reaching the orders of the approximation of the solutions (usually, two orders are lost for sufficiently smooth solutions). Here, the method is optimal, with the convergence rate being the same as the order of the approximation of the solution if the solution is sufficiently smooth. The techniques are also different. Since, here, we do not have the \(L^2\) projection for the curl operator, we have to resort to different techniques for establishing a key inf–sup condition which links the curl operator and the div operator. Such technique here is the div inf–sup inequality by [31] and by [32] for the Stokes equations of non-slip boundary condition. In [31], the quadratic and cubic Lagrange elements on barycentric refinements are shown to satisfy the div inf–sup inequality, while in [32], the results therein apply to higher-order Lagrange elements. In addition, here the Lagrange elements are standard except for the barycentric refinements, while, in the previous methods, higher-order element- and face-bubbles need artificially to enrich the Lagrange elements. The element-bubble is some polynomial of higher-order and is zero along the element boundary. The face-bubble is similarly understood.

With the development of the theory for the eigenvalue problem, we generally study the source problem relating to the eigenvalue problem, particularly for the indefinite problem, with the proposed Lagrange finite element method. As a matter of fact, we show a type of inf–sup condition which links the curl operator and the div operator, and then construct the Fortin-type interpolations (cf., [26], as is usually found for the Stokes equations) for the ‘dual variable’ (i.e., curl \(\textbf{u}\) and div \(\textbf{u}\)) and the ‘primal variable’ (i.e., \(\textbf{u}\)), and by the classical Aubin–Nitsche argument (cf., [33]) to show that the proposed method gives optimal convergent approximations for both singular and smooth solutions. In addition, we present numerical results for the Maxwell eigenvalue problem in L-shaped domain which has singular and smooth eigenfunctions to illustrate the proposed finite element method, the theoretical analysis and results.

The rest of this paper is arranged as follows. In Section 2, we review Hilbert spaces and norms, notations, and some results about the curl and div Hilbert spaces. The finite element method is defined in Section 3. We develop the inf–sup condition, and the dual Fortin-type interpolation in Section 4, and we establish the primal Fortin-type interpolation, the coercivity and the error estimates of the finite element method of the source problem in Section 5. Error estimates for the eigenvalue problem are developed in Section 6. Numerical results are presented in Section 7. A concluding remark is made in the last section.

2. Preliminaries

In this section, we introduce and review Sobolev spaces and the curl and div Hilbert spaces and related results.

Given a simply-connected polygon \(\Omega \subset \mathbb{R}^2\), with a connected Lipschitz boundary \(\Gamma\) (a planar curve). Let \(\mathbf{n}\) denote the unit outward normal vector to \(\Gamma\), while \(\mathbf{t}\) the unit tangential vector which orients anticlockwise to \(\Gamma\). Introduce the usual
Hilbert spaces [34]: $H^1(\Omega) = \{ q \in L^2(\Omega) : \nabla q \in (L^2(\Omega))^2 \}$, $H^1_0(\Omega) = \{ q \in H^1(\Omega) : \nabla q \cdot n|_{\Gamma} = 0 \}$, $H^1(\Omega)/\mathbb{R} = \{ q \in H^1(\Omega) : \int_{\Omega} q = 0 \}$. Let $H^1(\Omega)$, $H^1_0(\Omega)$, and $H^1(\Omega)/\mathbb{R}$ be equipped with norm $\| q \|_1^2 = \| q \|^2_0 + \| \nabla q \|^2_0$ and semi-norm $| q |_1 = \| \nabla q \|_0$. We also need Hilbert space $H^1(\Omega)$ with norm $\| q \|$, for $s \in \mathbb{R}$, where $H^s(\Omega) = L^2(\Omega)$. In addition, introduce Hilbert spaces $H(\text{curl}; \Omega) = \{ v \in L^2(\Omega)^2 : \text{curl} v \in L^2(\Omega) \}$, $H_0(\text{curl}; \Omega) = \{ v \in H(\text{curl}; \Omega) : v \cdot t|_{\Gamma} = 0 \}$, $H(\text{div}; \Omega) = \{ w \in L^2(\Omega) : \text{div} w = 0 \}$, $H_0(\text{div}; \Omega) = \{ w \in H(\text{div}; \Omega) : \text{div} w = 0 \}$. The norm for $H(\text{curl}; \Omega)$ is $\| v \|_{H(\text{curl}; \Omega)} = \| \text{curl} v \|_0$ and the norm for $H(\text{div}; \Omega)$ is $\| v \|_{H(\text{div}; \Omega)} = \| \text{div} v \|_0$. The Hilbert space $H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ is equipped with the norm $\| v \|_{H(\text{curl}, \text{div})} = \| \text{curl} v \|_0 + \| \text{div} v \|_0$. As a result of Proposition 2.1, the Hilbert space $H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ can be equipped with the norm $\| v \|_{H(\text{curl}, \text{div})} = \| \text{curl} v \|_0 + \| \text{div} v \|_0$, which is equivalent to the norm $\| v \|_{0, \text{curl}, \text{div}}$. We in addition introduce $L^2_0(\Omega) = \{ w \in L^2(\Omega) : \int_{\Omega} w = 0 \}$. The norm of $L^2_0(\Omega)$ is still denoted by $\| \cdot \|_0$.

Proposition 2.1 ([26,35,36]). For any $v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$, we have $
abla v = \omega^2 u$, $\text{div} u = 0$ in $\Omega$, $u \cdot t = 0$ on $\Gamma$.

The two-dimensional Maxwell eigenvalue problem reads as follows (cf., [37]): Find $\omega^2$ and $u \neq 0$ such that

\[ \text{curl} \text{curl} u = \omega^2 u, \quad \text{div} u = 0 \quad \text{in} \ \Omega, \quad u \cdot t = 0 \quad \text{on} \ \Gamma. \]

(2.1)

With the notation $(\cdot, \cdot)$ for the $L^2$ inner product, the classical variational statement of (2.1) is to find $(\omega^2, u \neq 0) \in \mathbb{R} \times H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ such that

\[ (\text{curl} u, \text{curl} v) = \omega^2 (u, v), \quad \forall v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega). \]

(2.2)

To deal with the divergence-free constraint, a well-known plain regularization method is formulated as follows (cf., [23,24]): Find $(\omega^2, u \neq 0) \in \mathbb{R} \times H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ such that

\[ a(u, v) := (\text{curl} u, \text{curl} v) + (\text{div} u, \text{div} v) = \omega^2 (u, v), \quad \forall v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega). \]

(2.3)

As a second-order elliptic eigenproblem, (2.3) is quite like the eigenproblem of vector Laplacian, the classical nodal-continuous or $H^1$-conforming Lagrange finite element method is most desirable for numerically solving problem (2.3). All the Lagrange element methods which have been studied over the decades are based on (2.3), see [16–21] and references therein. Such regularization is also widely used for the mathematical issues such as singularities (e.g., see [37–39]). However, although the curl operator and the div operator closely relate to the gradient operator $\nabla$ (since $-\Delta v = \text{curl} \text{curl} v - \nabla \text{div} v$, where $\Delta$ is the Laplace operator) the resultant eigenfunctions may have some singular ones, which do not belong to $(H^1(\Omega))^2$ space but to fractional-order Hilbert spaces $(H^s(\Omega))^2$, where $0 \leq r < 1$. This situation with very low regularity eigenfunctions is very commonplace in electromagnetism. A main cause is due to the reentrant corners along the domain boundary $\Gamma$ (e.g., see [26,27,36,38]). For a singular solution, it has been widely recognized that the Lagrange finite element solution which is $H^1$-conforming could not correctly converge. It turns out that the bilinear form in (2.3) accounts for such incorrect convergence. To obtain a correctly convergent Lagrange finite element solution, some modifications on (2.3) are needed, e.g., the $L^2$ projection method (e.g., see [16–21]). The main idea therein is to apply two $L^2$ projections to the curl and the div operators, respectively. In this paper, we shall study a new $L^2$ projection Lagrange finite element method for the eigenvalue problem (2.1) and the related indefinite source problem. Only a piecewise constant $L^2$ projection is applied to the div operators (see Section 3). We propose a family of the standard Lagrange elements on barycentric refinements of any order greater than or equal to two. This family is optimal convergent for both singular and smooth solutions. The new method is much simpler; actually, it is quite like the edge element method.

Proposition 2.2 ([26,36]). We have the following Hodge-decomposition ($L^2$ orthogonal with respect to the $L^2$ inner product $(\cdot, \cdot)$):

\[ (L^2(\Omega))^2 = \text{curl} (H^1(\Omega)/\mathbb{R}) + \nabla H^1_0(\Omega). \]

Proposition 2.3 ([36]). The following continuous embedding holds for some $1/2 < r \leq 1$:

\[ H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \hookrightarrow (H^r(\Omega))^2, \]

where for any $v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$, we have

\[ \| v \|_r \leq C \| v \|_{\text{curl}, \text{div}}. \]

The above continuous embedding is closely related to the Poisson equation of Dirichlet boundary conditions (cf., [36,40]), and the value of $r$ relates to the opening angles at the reentrant corners of $\Gamma$, roughly $r < \pi/k$, for the largest opening angle $\pi \leq k < 2\pi$. Throughout this paper, the regularity index $r$ will always be assumed to be the same as in Proposition 2.3.
3. The finite element method

In this section, we define the finite element method, where two finite element spaces are introduced: one is the Lagrange element space for the solution and the other is the piecewise constant element space for defining the \( L^2 \) projection for the div operator.

Denote by \( \mathcal{T} \) the conforming triangulation of \( \Omega \) into shape-regular triangles (cf., \([33]\)), i.e., \( \mathcal{T} = \{ T \} \), \( h := \max_{T \in \mathcal{T}} h_T \) and \( h_T \) the diameter of \( T \in \mathcal{T} \). Let \( \mathcal{T}_{h/2} \) denote the barycentric refinement of \( \mathcal{T} \) in the following way: for every \( T \in \mathcal{T} \), we divide \( T \) into three sub-triangles by connecting the barycentre to the three vertices. Let \( D \subset \Omega \) be an open subset. Denote by \( P_l(D) \) the space of polynomials on \( D \) of total degree not greater than \( l \), where \( l \geq 0 \) is an integer. Denote either of \( \mathcal{T} \) or \( \mathcal{T}_{h/2} \) by the notation \( \mathcal{N} \). Introduce

\[
V_h^{(\epsilon)}(\mathcal{N}) = \{ v_h \in L^2(\Omega) : v_h|_T \in P_l(T), \forall T \in \mathcal{N} \}. \tag{3.1}
\]

For \( \epsilon \geq 2 \), we define the Lagrange finite element space \( U_h^{(\epsilon)}(\mathcal{T}_{h/2}) \) for the solution

\[
U_h^{(\epsilon)}(\mathcal{T}_{h/2}) = (V_h^{(\epsilon)}(\mathcal{T}_{h/2}) \cap H^1(\Omega))^2 \cap H_0(\text{curl} ; \Omega). \tag{3.2}
\]

In addition, we introduce

\[
Q_h^{(0)}(\mathcal{T}) = V_h^{(0)}(\mathcal{T}), \tag{3.3}
\]

and define the local \( L^2 \) projection \( \Pi_h^{(0)} \) onto \( Q_h^{(0)}(\mathcal{T}) \), i.e., for \( p \in L^2(\Omega) \), \( \Pi_h^{(0)} p \in Q_h^{(0)}(\mathcal{T}) \),

\[
(\Pi_h^{(0)} p, q_h) = (p, q_h), \quad \forall q_h \in Q_h^{(0)}(\mathcal{T}). \tag{3.4}
\]

The finite element method we propose is to find \((\omega_h^2, u_h, v_h) \in \mathbb{R} \times U_h^{(\epsilon)}(\mathcal{T}_{h/2})\) such that

\[
a_h(u_h, v_h) = a_h^2(u_h, v_h), \quad \forall v_h \in U_h^{(\epsilon)}(\mathcal{T}_{h/2}), \tag{3.5}
\]

where

\[
a_h(u, v) = (\text{curl } u, \text{curl } v) + (\Pi_h^{(0)} \text{div } u, \Pi_h^{(0)} \text{div } v). \tag{3.6}
\]

The method is nonconforming, although \( U_h^{(\epsilon)}(\mathcal{T}_{h/2}) \) is conforming to \( H_0(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega) \), because of the \( L^2 \) projection \( U_h^{(0)} \) before the div operator in the bilinear form. The notation \( a_h^2 \) indicates that all the eigenvalues are real and non-negative, since problem (3.5) is symmetric, semi-positive definite.

4. The dual Fortin-type interpolation

In this section we shall establish an inf-sup condition and a dual Fortin interpolation associated with the following trilinear form, which will be the main tools for analyzing the stability and the error bounds. We introduce a trilinear form over \((L^2(\Omega))^2 \times L^2(\Omega) \times L^2(\Omega)\) as follows:

\[
b(v, (\chi, q)) = (\text{curl } v, \chi) + (\text{div } v, q) : ((L^2(\Omega))^2 \times L^2(\Omega) \times L^2(\Omega)) \rightarrow \mathbb{R}. \tag{4.1}
\]

This trilinear form relates to the bilinear form \( a(\cdot, \cdot, \cdot) \) by inserting \( \chi = \text{curl } u \) and \( q = \text{div } u \). In this paper, we call \( v \) and \((\chi, q)\) of \( b(v, (\chi, q)) \) ‘primal’ and ‘dual’ variables, respectively. The main ingredients of the theory for stability and error bounds are the dual Fortin-type interpolation and the inf-sup condition of \( b(\cdot, (\cdot, \cdot)) \) which will be established in this section.

4.1. Auxiliary finite element spaces

We shall introduce three auxiliary finite element spaces for theoretical purpose.

Introduce a third finite element space

\[
W_h^{(\epsilon - 1)}(\mathcal{T}_{h/2}) = V_h^{(\epsilon - 1)}(\mathcal{T}_{h/2}) \cap L^2(\Omega), \tag{4.2}
\]

which is only used for theoretical analysis and will not be involved with the implementation of the finite element method. Let the kernel set of \( b(\cdot, (\cdot, \cdot)) \) be defined by

\[
K_0(b) = \{ v_{0,h} \in U_h^{(\epsilon)}(\mathcal{T}_{h/2}) : b(v_{0,h}, (\chi_h, q_h)) = 0, \forall \chi_h \in Q_h^{(0)}(\mathcal{T}), \forall q_h \in W_h^{(\epsilon - 1)}(\mathcal{T}_{h/2}) \}. \tag{4.3}
\]

**Lemma 4.1.** The kernel set \( K_0(b) \) can also be equivalently defined as follows:

\[
K_0(b) = \{ v_h \in U_h^{(\epsilon)}(\mathcal{T}_{h/2}) : \text{curl } v_h = 0, \quad \Pi_h^{(0)} \text{div } v_h = 0 \} = \{ v_h \in U_h^{(\epsilon)}(\mathcal{T}_{h/2}) : a_h(v_h, v_h) = 0 \}.
\]
Proof. Since, in terms of $\Pi_h^{(0)}$ which is defined by (3.4),
\begin{equation}
\mathbf{b}(\mathbf{v}_h, (\mathbf{X}_h, q_h)) = (\mathbf{v}_h, \mathbf{X}_h) + (\Pi_h^{(0)} \mathbf{div} \mathbf{v}_h, q_h), \quad \forall q_h \in Q_h^{(0)}(\mathcal{T}_h), \forall \mathbf{X}_h \in W_h^{(\ell-1)}(\mathcal{T}_h/2),
\end{equation}
and since
\begin{equation}
\text{curl } U_h^{(\ell)}(\mathcal{T}_h/2) \subset W_h^{(\ell-1)}(\mathcal{T}_h/2),
\end{equation}
from the definition (4.3) of $K_h(b)$, we have
\begin{equation}
K_h(b) = \{ \mathbf{v}_h \in U_h^{(\ell)}(\mathcal{T}_h/2) : \text{curl } \mathbf{v}_h = 0, \quad \Pi_h^{(0)} \mathbf{div} \mathbf{v}_h = 0 \}. \tag{4.3}
\end{equation}
On the other hand, from the definition (3.6) of $a_h(\cdot, \cdot)$, $a_h(\mathbf{v}_h, \mathbf{v}_h) = 0$ if and only if
\begin{equation}
\text{curl } \mathbf{v}_h = 0, \quad \Pi_h^{(0)} \mathbf{div} \mathbf{v}_h = 0.
\end{equation}
The proof is completed. \hfill \square

For the finite dimensional space $U_h^{(\ell)}(\mathcal{T}_h/2)$, we can have the following orthogonal decomposition with respect to the $L^2$ inner product $(\cdot, \cdot)$:
\begin{equation}
U_h^{(\ell)}(\mathcal{T}_h/2) = K_h(b) + K_h(b)^\perp, \tag{4.5}
\end{equation}
where
\begin{equation}
K_h(b)^\perp = \{ \mathbf{v}_{0,h}^{\perp} \in U_h^{(\ell)}(\mathcal{T}_h/2) : (\mathbf{v}_{0,h}, \mathbf{v}_{0,h}^{\perp}) = 0, \quad \forall \mathbf{v}_{0,h} \in K_h(b) \}. \tag{4.6}
\end{equation}
Introduce a bilinear form
\begin{equation}
c_h(\mathbf{u}, \mathbf{v}) := a_h(\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{v}). \tag{4.7}
\end{equation}
It can be seen that (4.5) is also both $a_h(\cdot, \cdot)$ and $c_h(\cdot, \cdot)$ orthogonal, i.e.,
\begin{equation}
a_h(\mathbf{v}_{0,h}, \mathbf{v}_{0,h}^{\perp}) = 0, \quad \forall \mathbf{v}_{0,h} \in K_h(b), \forall \mathbf{v}_{0,h}^{\perp} \in K_h(b)^\perp, \tag{4.8}
\end{equation}
\begin{equation}
c_h(\mathbf{v}_{0,h}, \mathbf{v}_{0,h}^{\perp}) = 0, \quad \forall \mathbf{v}_{0,h} \in K_h(b), \forall \mathbf{v}_{0,h}^{\perp} \in K_h(b)^\perp. \tag{4.9}
\end{equation}
Introduce a fourth finite element space
\begin{equation}
X_h^{(\ell)}(\mathcal{T}_h/2) = U_h^{(\ell)}(\mathcal{T}_h/2) \cap (H^1_0(\Omega))^2, \tag{4.10}
\end{equation}
which, together with $W_h^{(\ell-1)}(\mathcal{T}_h/2)$, is also only used for theoretical analysis for some inf–sup condition relating to the curl operator.

**Lemma 4.2 ([31,32]).** The following surjective relation between $X_h^{(\ell)}(\mathcal{T}_h/2)$ and $W_h^{(\ell-1)}(\mathcal{T}_h/2)$ holds:
\begin{equation}
\text{curl } X_h^{(\ell)}(\mathcal{T}_h/2) \supseteq W_h^{(\ell-1)}(\mathcal{T}_h/2).
\end{equation}
For $\ell \geq 4$, from [41], even if we do not use the barycentric refinements, i.e., $\mathcal{T}_h/2 := \mathcal{T}_h$, Lemma 4.1 still holds, but the meshes must be singular-vertex free, which is a rather restrictive condition. This partly explains why we use the barycentric refinements. Obviously, we also have
\begin{equation}
\text{curl } X_h^{(\ell)}(\mathcal{T}_h/2) \equiv \text{curl } U_h^{(\ell)}(\mathcal{T}_h/2) \equiv W_h^{(\ell-1)}(\mathcal{T}_h/2).
\end{equation}
We finally introduce a fifth finite element space, which is a $C^1$ element,
\begin{equation}
Y_h^{(\ell+1)}(\mathcal{T}_h/2) = V_h^{(\ell+1)}(\mathcal{T}_h/2) \cap H^2_0(\Omega), \tag{4.11}
\end{equation}
which satisfies
\begin{equation}
\nabla Y_h^{(\ell+1)}(\mathcal{T}_h/2) \subset U_h^{(\ell)}(\mathcal{T}_h/2). \tag{4.12}
\end{equation}
Likewise, $Y_h^{(\ell+1)}(\mathcal{T}_h/2)$ is only used for theoretical analysis for discrete analogs of the Hodge-decomposition and the regular–singular decomposition (see Section 5). Such family of $C^1$ element spaces for all $\ell \geq 2$ are available and comprehensively studied in [25]. For a later theoretical use, below we review the local degrees of freedom of $Y_h^{(\ell+1)}(\mathcal{T}_h/2)$ from [25]. Given a triangle $T$ of $\mathcal{T}_h$, whose subdivision consists of three sub-triangles $T_1, T_2, T_3$, all of which belong to $\mathcal{T}_h/2$. The three vertices of $T$ are called interior vertices and the three edges of $T$ interior edges, and the barycentre of $T$ is called interior vertex and the three edges in the interior of $T$ are called interior edges. Thus, $T$ has three exterior vertices, three exterior edges, three interior edges, one interior vertex, and three sub-triangles. The local degrees of freedom are listed below from (1)-(7):
(1) the value and gradient (i.e., $\partial/\partial x_1$, $\partial/\partial x_2$) at the exterior vertices of $T$,
(2) the value at $\ell + 1 - 3$ distinct points in the interior of each exterior edge of $T$,
(3) the normal derivative (i.e., $\partial n/\partial n$) at $\ell + 1 - 2$ distinct points in the interior of each exterior edge of $T$, and if $\ell + 1 \geq 4$,
(4) the value and gradient at the interior vertex of $T$,
(5) the value at $\ell + 1 - 4$ distinct points in the interior of each interior edge of $T$,
(6) the normal derivative at $\ell + 1 - 4$ distinct points in the interior of each interior edge of $T$, and
(7) the value at $(1/2)(\ell + 1 - 4)(\ell + 1 - 5)$ distinct points in the interior of each $K_i$ chosen so that if a polynomial of degree $\ell + 1 - 6$ vanishes at those points, then it vanishes identically.

All the above degrees of freedom who locate on the exterior and interior edges and in the interior of element and sub-elements can be replaced by the average quantities such as edge integrals and volume integrals, e.g., $\int_T \partial q/\partial n$, $\int_T q$ and $\int_T q$ for a given interpolated function $q$.

We emphasize that all these three auxiliary finite element spaces $W_0^{(\ell-1)}(T_h/2)$, $X_0^{(\ell)}(T_h/2)$, $X_0^{(\ell+1)}(T_h/2)$ are merely technicalities of our proofs for establishing an inf-sup condition, dual Fortin-type interpolation, and primal Fortin-type interpolation in the following subsections and in Section 5. They have no influences on the finite element discretization which only uses the Lagrange elements $U_h^{(\ell)}(T_h/2)$ we have defined in Section 3.

### 4.2. Mesh-dependent norms

Corresponding to the bilinear forms $a_h(\cdot, \cdot)$ by (3.6) and $c_h(\cdot, \cdot)$ by (4.7), we introduce mesh-dependent norms.

$$
\|v\|_{a_h}^2 := a_h(v, v) = \|\text{curl } v\|^2_0 + \|I_0^{(0)} \text{div } v\|^2_0,
$$

(4.13)

$$
\|v\|_{c_h}^2 := c_h(v, v) = \|v\|_{0, \text{curl}}^2 + \|I_0^{(0)} \text{div } v\|^2_0.
$$

(4.14)

Moreover, we define

$$
\|v\|_{h}^2 := \|v\|_{c_h}^2 + \mathcal{D}_h(v),
$$

(4.15)

where

$$
\mathcal{D}_h(v) = \sum_{T \in T_h/2} h_T^{-2\ell - 2\ell} \sum_{i=1}^{m_T} \frac{\|v(\psi_i T b_T)\|_0, T(v, \text{curl}(\psi_i T b_T))_0, T}{\sum_{i=1}^{m_T} \|\psi_i T b_T\|_0, T},
$$

(4.16)

and $m_T = \ell(\ell + 1)/2$, and $\{\psi_i T, 1 \leq i \leq m_T\}$ is chosen so that the following local inclusion holds:

$$
\text{div } (v_{h,T}) \in \mathcal{V}_T := \text{span} \{\psi_i T, 1 \leq i \leq m_T\}, \quad \forall v_{h,T} \in U_h^{(\ell)}(T_h/2), \forall T \in T_h/2,
$$

(4.17)

and

$$
b_T = \lambda_1^T \lambda_2^T \lambda_3^T
$$

is the usual element-bubble function, where $\lambda_j^T$, $j = 1, 2, 3$, are the three basis functions of the linear element $P_1(T)$. Note that once $U_h^{(\ell)}(T_h/2)$ is defined, (4.17) can be accordingly easily defined.

**Proposition 4.1.** The $\mathcal{D}_h(\cdot)$ has the following properties:

$$
\mathcal{D}_h(v) \leq C \sum_{T \in T_h/2} h_T^{-2\ell} \|v\|_{0, T}^2, \quad \forall v \in (L^2(\Omega))^2,
$$

(4.18)

$$
\mathcal{D}_h(v) \leq C \sum_{T \in T_h/2} h_T^{-2\ell} \|\text{div } v\|_{0, T}^2, \quad \forall v \in H(\text{div } \Omega),
$$

(4.19)

$$
C \sum_{T \in T_h/2} h_T^{-2\ell} \|\text{div } v_{h,T}\|_{0, T}^2 \leq \mathcal{D}_h(v_h), \quad \forall v_h \in U_h^{(\ell)}(T_h/2).
$$

(4.20)

**Proof.** In fact, by the Cauchy–Schwarz inequality,

$$
\sum_{i=1}^{m_T} \|v(\psi_i T b_T)\|_0, T \leq \|v\|_{0, T}^2 \sum_{i=1}^{m_T} \|\psi_i T b_T\|_0, T^n
$$


and by the definition (4.16), we immediately obtain (4.18). From (4.16), by the formula of integration by parts,

$$(v, \nabla (\psi | T b_T))_{0,T} = - (\text{div} v, \psi | T b_T)_{0,T},$$

and by the Cauchy–Schwarz inequality,

$$| (\text{div} v, \psi | T b_T)_{0,T} | \leq \| \text{div} v \|_{0,T} \| \psi | T b_T \|_{0,T},$$

but, by the Poincaré–Friedrichs’ inequality (cf., Lemma 4.1 in [42]),

$$\| \psi | T b_T \|_{0,T} \leq Ch \| (\psi | T b_T) \|_{0,T},$$

and we have

$$\sum_{i=1}^{m_T} | (v, \nabla (\psi | T b_T))_{0,T} |^2 = \sum_{i=1}^{m_T} | (\text{div} v, \psi | T b_T)_{0,T} |^2 \leq C \| \text{div} v \|_{0,T}^2 \sum_{i=1}^{m_T} \| \nabla (\psi | T b_T) \|_{0,T}^2,$$

from which and the definition (4.16) it follows that (4.19) holds. From the local inclusion (4.17), following the same argument in Lemma 4.3 of [17], we can show (4.20). □

We also emphasize that the mesh-dependent norm $\| \cdot \|_h$ is merely for theoretical purpose for estimating the error bounds of the dual Fortin-type interpolation. We only use the mesh-dependent norms $\| \cdot \|_{h_b}$ and $\| \cdot \|_{h_c}$ in the practical computations of the finite element method which has been defined in Section 3.

4.3. The inf–sup condition

Among $U_h^{(0)}(\mathcal{T}_h/2)$ and $W_h^{(\ell-1)}(\mathcal{T}_h/2)$, $Q_h^{(0)}(\mathcal{T}_h)$, we shall establish an inf–sup condition of $b’(\cdot, (\cdot, \cdot))$.

**Theorem 4.1.** The following inf–sup conditions hold: for all $w_h \in W_h^{(\ell-1)}(\mathcal{T}_h/2)$ and for all $p_h \in Q_h^{(0)}(\mathcal{T}_h)$,

$$\sup_{0 \neq v_h \in U_h^{(0)}(\mathcal{T}_h/2)} \frac{b(v_h, (w_h, p_h))}{\|v_h\|_h} \geq C(\|w_h\|_0 + \|p_h\|_0).$$

**Proof.** For $p_h \in Q_h^{(0)}(\mathcal{T}_h)$, consider the Poisson equation of homogeneous Dirichlet boundary condition: Find $\theta \in H^1_0(\Omega)$ such that

$$- \Delta \theta = p_h \quad \text{in} \quad \Omega, \quad \theta = 0 \quad \text{on} \quad \Gamma.$$

It is known from [40] that, for some $1/2 < r \leq 1$(the same as the one in Proposition 2.3), $\theta \in H^{1+r}(\Omega)$ and $\Delta \theta \in L^2(\Omega)$,

$$\|\Delta \theta\|_0 + \|\theta\|_{1+r} \leq C\|p_h\|_0.$$

Letting $\mathbf{v}^\circ := - \nabla \theta \in H_0(\text{curl } \Omega) \cap H(\text{div } \Omega)$,

we have $\mathbf{v}^\circ \in H_0(\text{curl } \Omega) \cap H(\text{div } \Omega)$,

$$\text{curl } \mathbf{v}^\circ = 0, \quad \text{div } \mathbf{v}^\circ = p_h,$$

$$\mathbf{v}^\circ \in (H^r(\Omega))^2, \quad \|\mathbf{v}^\circ\|_r + \|\mathbf{v}^\circ\|_{0, \text{div}} \leq C\|p_h\|_0.$$

Denote by $\mathcal{F}_h(\mathcal{T}_h)$ the set of all edges of $\mathcal{T}_h$. Define

$\mathbf{v}_h^\circ := - \nabla \theta_h \in U_h^{(0)}(\mathcal{T}_h/2), \quad \theta_h \in Y_h^{(\ell+1)}(\mathcal{T}_h/2),$

where $\theta_h$, which is an averaging-type interpolation of $\theta$, satisfies

$$\int_F \nabla (\theta - \theta_h) \cdot \mathbf{n} = 0, \quad \forall F \in \mathcal{F}_h(\mathcal{T}_h),$$

$$\|\theta - \theta_h\|_0 + h^r|\theta - \theta_h|_1 \leq Ch^{1+r}\|\theta\|_{1+r},$$

$$\left( \sum_{e \in \mathcal{E}_h/2} h_{T_e}^{2-2r} |\theta - \theta_h|_{0,T}^2 + h_{T_e}^{2r} |\theta - \theta_h|_{1,T}^2 \right)^{1/2} \leq C\|\theta\|_{1+r}.$$

We remark that the finite element interpolation $\xi_h \in Y_h^{(\ell+1)}(\mathcal{T}_h/2)$ of $\xi \in H^1_0(\Omega) \cap H^2(\Omega)$ can be found in [25]; in particular, therein $\partial \xi_h / \partial n(a) = \partial \xi / \partial n(a)$ at all the mid-points $a$ of $F \in \mathcal{F}_h(\mathcal{T}_h)$. Such degree of freedom $\partial \xi / \partial n(a)$ can be replaced
by the edge average $|F|^{-1} \int_F \partial \xi \cdot \partial n$, where $|F|$ is the length of $F \in \mathcal{F}_h(\mathcal{T}_h)$, and the regularity of the interpolated function can then be relaxed to $\xi \in H^1_0(\Omega) \cap H^{1+r}(\Omega)$ for $1/2 < r \leq 1$ and $\Delta \xi \in L^2(\Omega)$.

In terms of $\mathbf{v}^e$ and $\mathbf{v}_h^e$,

$$\int_F (\mathbf{v}^e - \mathbf{v}_h^e) \cdot \mathbf{n} = 0, \quad \forall F \in \mathcal{F}_h(\mathcal{T}_h),$$

$$\|\mathbf{v}^e - \mathbf{v}_h^e\|_0 \leq C \|\mathbf{v}^e\|_{r, F},$$

$$\left( \sum_{T \in \mathcal{T}_h} h_T^{-2r} \|\mathbf{v}^e - \mathbf{v}_h^e\|_{0, T}^2 \right)^{1/2} \leq C \|\mathbf{v}^e\|_{r, F}.$$ We also have

$$\|\mathcal{I}_h^{(0)} \text{div} (\mathbf{v}^e - \mathbf{v}_h^e)\|_{0}^2 = \|\mathcal{I}_h^{(0)} \text{div} (\mathbf{v}^e - \mathbf{v}_h^e)\|^2,$$

$$= (\text{div} (\mathbf{v}^e - \mathbf{v}_h^e), \mathcal{I}_h^{(0)} \text{div} (\mathbf{v}^e - \mathbf{v}_h^e)) = \int_{\partial \mathcal{T}_h} (\mathbf{v}^e - \mathbf{v}_h^e) \cdot \mathbf{n} \|\mathcal{I}_h^{(0)} \text{div} (\mathbf{v}^e - \mathbf{v}_h^e)\| = 0.$$ where $\|\cdot\|$ means the jump and $\|\mathcal{I}_h^{(0)} \text{div} (\mathbf{v}^e - \mathbf{v}_h^e)\|$ is constant on $F$. In other words,

$$\mathcal{I}_h^{(0)} \text{div} (\mathbf{v}^e - \mathbf{v}_h^e) = 0.$$

Thus, we have, for all $w_h \in \mathcal{W}_h^{(\ell-1)}(\mathcal{T}_h/2)$,

$$b(\mathbf{v}_h^e, (w_h, p_h)) = (\text{div} \mathbf{v}_h^e, p_h)$$

$$= (\text{div} \mathbf{v}^e, p_h) + (\text{div} \mathbf{v}_h^e - \mathbf{v}^e, p_h)$$

$$= \|p_h\|_0^2 + (\mathcal{I}_h^{(0)} \text{div} (\mathbf{v}_h^e - \mathbf{v}^e), p_h)$$

$$= \|p_h\|_0^2.$$ Actually, we have obtained

$$\mathcal{I}_h^{(0)} \text{div} \mathbf{v}_h^e = \mathcal{I}_h^{(0)} \text{div} \mathbf{v}^e = p_h.$$

$$\|\mathbf{v}_h^e\|_{0, T} \leq \|\mathbf{v}_h^e - \mathbf{v}^e\|_{0, T} + \|\mathbf{v}^e\|_{0, T},$$

$$\|\mathbf{v}_h^e\|_{0}^2 = \|\mathbf{v}_h^e\|_{0, \text{curv}}^2 + \|\mathcal{I}_h^{(0)} \text{div} \mathbf{v}^e\|_{0, T}^2 + \mathcal{D}_h(\mathbf{v}^e)$$

$$= \|\mathbf{v}_h^e\|_{0}^2 + \|\mathcal{I}_h^{(0)} \text{div} \mathbf{v}_h^e\|_{0, T}^2 + \mathcal{D}_h(\mathbf{v}^e)$$

$$\leq C \|\mathbf{v}^e\|_{0, \text{div}}^2 \leq C \|p_h\|_0^2,$$ where we have used $(4.19)$ in Proposition 4.1, i.e.,

$$\mathcal{D}_h(\mathbf{v}^e) \leq \frac{C}{C} \sum_{T \in \mathcal{T}_h/2} h_T^{-2(\ell-2)} \|\text{div} \mathbf{v}^e\|_{0, T}^2 \leq C \|\mathbf{v}^e\|_{0}^2.$$ Consequently, for any given $p_h \in \mathcal{Q}_h^{(0)}(\mathcal{T}_h)$, there exists a $\mathbf{v}_h^e \in \mathcal{U}_h^{(\ell)}(\mathcal{T}_h/2)$, with curl $\mathbf{v}_h^e = 0$, such that, for all $w_h \in \mathcal{W}_h^{(\ell-1)}(\mathcal{T}_h/2)$,

$$b(\mathbf{v}_h^e, (w_h, p_h)) = \|p_h\|_0^2, \quad \|\mathbf{v}_h^e\|_{0, T} \leq C \|p_h\|_0.$$ For any

$$w_h \in \mathcal{W}_h^{(\ell-1)}(\mathcal{T}_h/2),$$

from Lemma 4.2 there exists

$$\mathbf{v}_h^e \in \mathcal{X}_h^{(\ell)}(\mathcal{T}_h/2), \quad (\text{curl} \mathbf{v}_h^e, w_h) = \|w_h\|_{0, T}^2, \quad \|\mathbf{v}_h^e\|_{1} \leq C \|w_h\|_0.$$

(4.21)
From Proposition 4.1, we have
\[ \| v_h^\circ \|_0^2 = \| v_h^\circ \|_{0, W^{2,2}}^2 + \| \mathcal{R}_h^0 \|_{0, W^{2,2}}^2 + \mathcal{P}_h(\mathcal{V}_h^\circ) \leq C \| v_h^\circ \|_1^2 \leq C \| w_h \|_0^2. \] (4.22)

On the other hand, by Cauchy–Schwarz inequality and Young inequality we have
\[ b(v_h^\circ, (w_h, p_h)) = \| w_h \|_0^2 + (\text{div } v_h^\circ, p_h) \geq \| w_h \|_0^2 - \| \text{div } v_h^\circ \|_0 \| p_h \|_0 \]
\[ \geq \| w_h \|_0^2 - C \| v_h^\circ \|_1 \| p_h \|_0 \]
\[ \geq C \| w_h \|_0^2 + \| p_h \|_0^2. \] (4.23)

Taking
\[ v_h = v_h^\circ + \delta v_h^\circ, \quad \delta = \frac{1}{2C_2}, \] (4.24)
from (4.21)–(4.24), we have
\[ \| v_h \|_h \leq C(\| w_h \|_0 + \| p_h \|_0), \]
and hence, it directly follows that the inf–sup condition as stated holds. \( \Box \)

We note that, from the definition (4.3) of \( K_0(b) \), Lemma 4.1, and the decomposition of (4.5)–(4.6), we actually have obtained
\[ \sup_{\theta \neq v_h \in K_0(b)^+} \frac{b(v_h, (w_h, p_h))}{\| v_h \|_h} \geq C(\| w_h \|_0 + \| p_h \|_0). \] (4.25)

for all \( w_h \in W_h^{(\ell-1)}(\mathcal{T}_h/2) \) and for all \( p_h \in Q_h^0(\mathcal{T}_h) \).

4.4. The dual Fortin-type interpolation

In this subsection, with the inf–sup condition, we show that there is a 'dual' Fortin-type interpolation over \( Q_h^0(\mathcal{T}_h) \times W_h^{(\ell-1)}(\mathcal{T}_h/2) \) through the trilinear form \( b(\cdot, \cdot, \cdot) \).

We consider the problem: Given \( p \in L^2(\Omega) \) and \( w \in L^2(\Omega) \), to find \( \tilde{p} \in Q_h^0(\mathcal{T}_h) \) and \( \tilde{w} \in W_h^{(\ell-1)}(\mathcal{T}_h/2) \) such that
\[ b(v_h, (\tilde{w}, \tilde{p})) = b(v_h, (w, p)), \quad \forall v_h \in K_0(b)^+. \] (4.26)

With the choice of \( K_0(b)^+ \), the established inf–sup inequality (4.25), we know that the stated problem has a unique solution. We call \( (\tilde{p}, \tilde{w}) \) as the dual Fortin-type interpolation of \( (p, w) \).

**Theorem 4.2.** Let \( p \in H^1_0(\Omega) \) and \( w \in L^2(\Omega) \), and let \((\tilde{p}, \tilde{w}) \in Q_h^0(\mathcal{T}_h) \times W_h^{(\ell-1)}(\mathcal{T}_h/2) \) be constructed as in problem (4.26). We have
\[ \| \tilde{p} \|_0 + \| \tilde{w} \|_0 \leq C(\| \nabla p \|_0 + \| w \|_0), \] (4.27)
\[ \| p - \tilde{p} \|_0 + \| w - \tilde{w} \|_0 \leq C \inf_{q_h \in Q_h^0(\mathcal{T}_h), \chi_h \in \mathcal{V}_h^\circ(\mathcal{T}_h/2)} \left( \| p - q_h \|_0 + \left( \sum_{T \in \mathcal{T}_h} h_T^{2\ell-2} \| p - q_h \|_{0, T}^2 \right)^{1/2} + \| w - \chi_h \|_0 \right). \] (4.28)

**Proof.** From the inf–sup condition (4.25)
\[ C(\| \tilde{p} \|_0 + \| \tilde{w} \|_0) \leq \sup_{\theta \neq v_h \in K_0(b)^+} \frac{b(v_h, (\tilde{w}, \tilde{p}))}{\| v_h \|_h}, \]
but
\[ b(v_h, (\tilde{w}, \tilde{p})) = b(v_h, (w, p)) = (\text{curl } v_h, w) - (v_h, \nabla p), \]
It follows that (4.27) holds.
Again, from the inf-sup condition (4.25), we have, for any $q_h \in Q_h^{(0)}(\mathcal{T}_h)$ and $\chi_h \in W_h^{(\ell-1)}(\mathcal{T}_h/2)$,

$$C(||p - q_h||_0 + ||\tilde{w} - \chi_h||_0) \leq \sup_{\theta \neq v_h \in \mathcal{X}_h(b)^\perp} \frac{b(v_h, (\tilde{w} - \chi_h, \tilde{p} - q_h))}{\|v_h\|_h},$$

but

$$b(v_h, (\tilde{w} - \chi_h, \tilde{p} - q_h)) = b(v_h, (\tilde{w} - w, \tilde{p} - p)) + b(v_h, (w - \chi_h, p - q_h)) = b(v_h, (w - \chi_h, p - q_h)).$$

For $p$ Corollary 4.1.

Proof. Choosing a $q_h \in Q_h^{(0)}(\mathcal{T}_h)$ as the $L^2$ projection of $p$ such that

$$\|p - q_h\|_{0,T} \leq Ch_\Omega\|p\|_{1,T}, \quad \forall T \in \mathcal{T}_h$$

and $\chi_h \in W_h^{(\ell-1)}(\mathcal{T}_h/2)$ as the $L^2$ projection of $w$ such that

$$\|w - \chi_h\|_{0,T} \leq Ch_{\Omega/2}\|w\|_{1,T}, \quad \forall T \in \mathcal{T}_h/2,$$

we obtain (4.29) from (4.28).

From (4.21) and problem (4.26), if $p = 0$, we infer that $\tilde{p} = 0$, too. Consequently, from (4.28) we have

$$\|w - \tilde{w}\|_0 \leq Ch_{\Omega\Sigma}(\chi_h - w) \leq C \inf_{\chi_h \in \mathcal{X}_h^{(\ell+1)}(\mathcal{T}_h/2)} \|w - \chi_h\|_0,$$

and we have no difficulty in finding a suitable $\chi_h \in \mathcal{X}_h^{(\ell+1)}(\mathcal{T}_h/2)$ such that (4.30) holds. □

5. Coercivity and error estimates of source problem

With the dual Fortin-type interpolations established in the previous section we can study a general indefinite source problem by establishing the coercivity property and analyzing the error estimates.

5.1. Coercivity

Theorem 5.1. We have

$$a_h(v_h, v_h) = \|\text{curl} v_h\|_0^2 + \|\text{div} v_h\|_0^2 \geq C\|v_h\|_0^2, \quad \forall v_h \in K_h(b)^\perp,$$

where $K_h(b)^\perp$ is given by (4.6) and $a_h(\cdot, \cdot, \cdot)$ by (3.6).

Proof. From Proposition 2.2 we write $v_h$ as the following $L^2$ orthogonal Hodge-decomposition:

$$v_h = \text{curl} w - \nabla p, \quad p \in H^1_0(\Omega), \quad w \in H^1(\Omega)/\mathbb{R},$$

$$\|v_h\|_0^2 = \|\nabla p\|_0^2 + \|\text{curl} w\|_0^2,$$

and let $\tilde{p} \in Q_h^{(0)}(\mathcal{T}_h), \tilde{w} \in W_h^{(\ell-1)}(\mathcal{T}_h/2)$ be the dual Fortin-type interpolations of $p$, $w$, respectively, i.e.

$$b(v_h, (\tilde{w}, \tilde{p})) - b(v_h, (w, p)) = 0 \quad \forall v_h \in \mathcal{X}_h(b)^\perp,
such that
\[ \|p\|_0 + \|\vec{w}\|_0 \leq C (\|p\|_1 + \|w\|_0) \leq C \|v_h\|_0. \]
where we have used the Poincaré inequality \( \|w\|_0 \leq C \|w\|_1 \). Let \( \delta > 0 \) be a constant to be given. We have
\[ \|\text{curl } v_h\|_0^2 = \|\text{curl } v_h - \delta \vec{w}\|_0^2 + 2\delta (\text{curl } v_h, \vec{w}) - \delta^2 \|\vec{w}\|_0^2, \]
where
\[ (\text{curl } v_h, \vec{w}) = (\text{curl } v_h, \vec{w} - w) + (v_h, \text{curl } w), \]
\[ (v_h, \text{curl } w) = \|\text{curl } w\|_0^2. \]
and we have
\[ 2\delta (\text{curl } v_h, \vec{w}) = 2\delta (\text{curl } v_h, \vec{w} - w) + 2\delta \|\text{curl } w\|_0^2. \]
On the other hand,
\[ \|I_h^{(0)} \text{div } v_h\|_0^2 = \|I_h^{(0)} \text{div } v_h - \delta \vec{p}\|_0^2 + 2\delta (I_h^{(0)} \text{div } v_h, \vec{p}) - \delta^2 \|\vec{p}\|_0^2, \]
where
\[ (I_h^{(0)} \text{div } v_h, \vec{p}) = (\text{div } v_h, \vec{p}) = (\text{div } v_h, \vec{p} - p) - (v_h, \nabla p), \]
\[ -(v_h, \nabla p) = \|\nabla p\|_0^2. \]
and we have
\[ 2\delta (I_h^{(0)} \text{div } v_h, \vec{p}) = 2\delta (\text{div } v_h, \vec{p} - p) + 2\delta \|\nabla p\|_0^2. \]
We therefore have
\[ \|\text{curl } v_h\|_0^2 + \|I_h^{(0)} \text{div } v_h\|_0^2 = \|\text{curl } v_h - \delta \vec{w}\|_0^2 + \|I_h^{(0)} \text{div } v_h - \delta \vec{p}\|_0^2 + 2\delta (\text{curl } v_h, \vec{w} - w) + 2\delta (\text{div } v_h, \vec{p} - p) + 2\delta \|\text{curl } w\|_0^2 + 2\delta \|\nabla p\|_0^2 - \delta^2 \|\vec{w}\|_0^2 - \delta^2 \|\vec{p}\|_0^2, \]
where
\[ 2\delta (\text{curl } v_h, \vec{w} - w) + 2\delta (\text{div } v_h, \vec{p} - p) = 2\delta b(v_h, (\vec{w} - w, \vec{p} - p)) = 0, \]
\[ 2\delta \|\nabla p\|_0^2 + 2\delta \|\text{curl } w\|_0^2 = 2\delta \|v_h\|_0^2, \]
\[ \delta^2 \|\vec{p}\|_0^2 + \delta^2 \|\vec{w}\|_0 \leq C \delta^2 \|v_h\|_0^2. \]
Hence,
\[ \|\text{curl } v_h\|_0^2 + \|I_h^{(0)} \text{div } v_h\|_0^2 \geq \delta (2 - \delta C) \|v_h\|_0^2, \]
from which the proof is finished, with a suitable \( \delta > 0 \).

5.2. Regularity results

In this subsection we review the regularity results of the source problem.

Given \( f \) and \( \beta \in \mathbb{R} \). Consider the source problem: Find \( z \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) such that, for all \( v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \),
\[ c_\beta(z, v) := (\text{curl } z, \text{curl } v) + (\text{div } z, \text{div } v) - \beta(z, v) = (f, v). \quad (5.1) \]
When \( \beta = -1 \), we simply denote
\[ c(u, v) := c_{-1}(u, v). \quad (5.2) \]
By the well-known Fredholm Alternative Theorem, it is not difficult to show the following proposition, see [43].

**Proposition 5.1.** For any \( f \in (L^2(\Omega))^2 \), for any \( \beta \leq 0 \) or any \( \beta > 0 \) which is not an eigenvalue of the eigenproblem (2.3), the solution \( z \) of the source problem (5.1) exists uniquely and satisfies
\[ \|z\|_{0, \text{curl}, \text{div}} \leq C \|f\|_0. \]
Proposition 5.2. For any \( f \in (L^2(\Omega))^2 \), let \( z \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) be the exact solution of problem (5.1). Then, \( z \in (H^1(\Omega))^2 \) and curl \( z \in H^1(\Omega)/\mathbb{R} \), satisfying

\[
\|z\|_r + \|	ext{curl} \; z\|_1 + \|\text{div} \; z\|_1 + \|\text{curl} \; z\|_0 \leq C\|f\|_0.
\]

Proof. Applying Propositions 2.3 and 5.1 to \( z \), we have the regularity results as stated. \( \square \)

Proposition 5.3. For any \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) with \( \text{curl} \; u \in (L^2(\Omega))^2 \), we have the regular–singular decomposition

\[
u = u^{\text{reg}} + \nabla p^{\text{sing}},
\]

where \( u^{\text{reg}} \in H_0(\text{curl}; \Omega) \cap (H^{1+r}(\Omega))^2 \) denotes the regular part (the superscript ‘reg’ means regular) and \( \nabla p^{\text{sing}} \) denotes the singular part with \( p^{\text{sing}} \in H_0^1(\Omega) \cap H^{1+r}(\Omega) \) (the superscript ‘sing’ means singular), \( r > 1/2 \), satisfying

\[
\|u^{\text{reg}}\|_{1+r} + \|p^{\text{sing}}\|_{1+r} \leq C(\|\text{curl} \; u\|_0 + \|\text{div} \; u\|_0).
\]

Proof. With \( \text{div} \; u \in L^2(\Omega) \), we first find \( \varphi \in H^1_0(\Omega) \) such that \( -\Delta \varphi = \text{div} \; u \) in \( \Omega \), \( \varphi = 0 \) on \( \Gamma \), the Poisson equation of homogeneous Dirichlet boundary condition. From [40], \( \varphi \in H^{1+r}(\Omega) \) for some \( 1 \geq r > 1/2 \), satisfying \( \|\varphi\|_{1+r} \leq C\|\text{div} \; u\|_0 \). Then, let \( u^\perp = u + \nabla \varphi \). We have \( u^\perp \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \), \( u^\perp \in (L^2(\Omega))^2 \), \( \text{div} \; u^\perp = 0 \), and apply [38], Theorem 3.4 on page 243, to get the regular–singular decomposition \( u^\perp = u^{\text{reg}} + \nabla q^{\text{sing}}, \) \( u^{\text{reg}} \in H_0(\text{curl}; \Omega) \cap (H^{1+r}(\Omega))^2 \) and \( q^{\text{sing}} \in H_0^1(\Omega) \cap H^{1+r}(\Omega) \), satisfying \( \|u^{\text{reg}}\|_{1+r} + \|q^{\text{sing}}\|_{1+r} \leq C(\|\text{curl} \; u\|_0 + \|\text{div} \; u\|_0) \). Then, \( u = u^\perp - \nabla \varphi = u^{\text{reg}} + \nabla (q^{\text{sing}} - \varphi) \), letting \( p^{\text{sing}} := q^{\text{sing}} - \varphi \), we obtain the desired regular–singular decomposition. \( \square \)

5.3. The primal Fortin-type interpolation

To study the error estimates of the finite element solution, we need a suitable interpolation for the solution itself, in addition to the dual Fortin-type interpolation in Section 4.

Given \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \). We consider its finite element projection by \( a_0(\cdot, \cdot) \) in the following, called the primal Fortin-type interpolation relative to the dual Fortin-type interpolation defined by \( b(\cdot, \cdot, \cdot) \) in Section 4.

The primal Fortin-type interpolation of \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) is defined as follows.

Find \( \tilde{u} \in \tilde{V}_h(5; \mathcal{T}_h/2) \) such that

\[
a_0(\tilde{u}, v_h) = a_0(u, v_h), \quad \forall v_h \in \tilde{V}_h^{(1)}(\mathcal{T}_h/2).
\]

By the definition (3.6) of \( a_0(\cdot, \cdot) \) and Lemma 4.1 about \( K_0(b) \), the solution \( \tilde{u} \in K_0(b) \) is equivalently determined by solving

\[
a_0(\tilde{u}, v_h) = a_0(u, v_h), \quad \forall v_h \in K_0(b)^\perp.
\]

By the definitions (3.6) and (4.13), we have

\[
\|u - \tilde{u}\|_0 \leq \inf_{v_h \in \tilde{V}_h^{(1)}(\mathcal{T}_h/2)} \|u - v_h\|_0,
\]

and by the coercivity in Theorem 5.1, we also have

\[
\|u - \tilde{u}\|_0 \leq C \inf_{v_h \in \tilde{V}_h^{(1)}(\mathcal{T}_h/2)} \|u - v_h\|_1,
\]

where \( \|\cdot\|_0 \) is given by (4.14).

Lemma 5.1. Assume that \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) and \( \text{curl} \; u \in L^2(\Omega) \).

\[
\|\tilde{u}\|_0 \leq C\|\text{curl} \; u\|_{\text{curl, div}},
\]

\[
\|u - \tilde{u}\|_0 \leq C\|u\|_{(h+c\|\text{div} \; u\|_0)}.
\]

Proof. The first conclusion can be easily obtained. From Proposition 5.3, we have the regular–singular decomposition

\[
u = u^{\text{reg}} + \nabla p^{\text{sing}}, \quad u^{\text{reg}} \in H_0(\text{curl}; \Omega) \cap (H^{1+r}(\Omega))^2 \text{ and } p^{\text{sing}} \in H_0^1(\Omega) \cap H^{1+r}(\Omega) \text{ for some } r > 1/2,
\]

satisfying \( \|u^{\text{reg}}\|_{1+r} + \|p^{\text{sing}}\|_{1+r} \leq C(\|\text{curl} \; u\|_0 + \|\text{div} \; u\|_0) \). We moreover have \( \Delta p^{\text{sing}} \in L^2(\Omega) \) and \( \|\Delta p^{\text{sing}}\|_0 \leq C(\|\text{curl} \; u\|_0 + \|\text{div} \; u\|_0) \).

For such \( p^{\text{sing}} \), similar to the proof of Theorem 4.1, we can construct \( p^{\text{sing}}_h \in V_h^{(\ell+1)}(\mathcal{T}_h/2) \) such that

\[
\|p^{\text{sing}} - p_h^{\text{sing}}\|_0 + h\|p^{\text{sing}} - p_h^{\text{sing}}\|_1 \leq C h^1 + r \|p^{\text{sing}}\|_{1+r},
\]

\[
\int_{\Gamma} (p^{\text{sing}} - p_h^{\text{sing}}) \, \partial n = 0, \quad \forall F \in \mathcal{T}_h(\mathcal{T}_h).
\]
Because $\ell \geq 2$ (e.g., for $\ell = 2$, the pair $(U^2_h(\mathcal{T}_h/2), Q^0_h(\mathcal{T}_h))$ may read as (piecewise)quadratic-constant element), from [26], it is not difficult to find a Fortin-type interpolation $u^{reg}_{h} \in U^2_h(\mathcal{T}_h/2)$ such that
\[
(\text{div} (u^{reg} - u^h_{\text{reg}}), q_h) = 0, \quad \forall q_h \in Q^0_h(\mathcal{T}_h),
\]
\[
\|u^{reg} - u^h_{\text{reg}}\|_0 + h|u^{reg} - u^h_{\text{reg}}|_1 \leq Ch^{1+r} \|u^{reg}\|_{1+r}.
\]
Since $\nabla Y^{(\ell+1)}_h(\mathcal{T}_h/2) \subset U^2_h(\mathcal{T}_h/2)$, we have
\[
\pi_h u := u^{reg}_{\text{reg}} + \nabla p^{reg} \in U^2_h(\mathcal{T}_h/2),
\]
and we find that
\[
\|u - \pi_h u\|_{\text{ch}} \leq Ch^r (\|\text{curl} u\|_{0} + \|\text{div} u\|_{0}),
\]
\[
I^2_h \text{div} (u - \pi_h u) = 0.
\]
The proof is complete. □

5.4. Error estimates of source problem

In this subsection we analyze the error bounds between the exact solution of problem (5.1) and the finite element solution of problem (5.11) below.

The finite element problem of (5.1) is to find $u_{h} \in U^2_h(\mathcal{T}_h/2)$ such that, for all $v_{h} \in U^2_h(\mathcal{T}_h/2)$,
\[
c_{p,h}(z_h, v_h) := (\text{curl} z_h, \text{curl} v_h) + (I^0_h \text{div} z_h, I^0_h \text{div} v_h) - \beta(z_h^{0}, v_h) = (f, v_h).
\]

Lemma 5.2. Let $z \in H_0^{(\text{curl} ; \Omega)} \cap H^{(\text{div} ; \Omega)}$ be the exact solution of problem (5.1) and $z_{h} \in U^2_h(\mathcal{T}_h/2)$ the finite element solution of problem (5.11), respectively. Let $\tilde{p} \in Q^0_h(\mathcal{T}_h)$ and $\tilde{w} \in W^{1,\ell}_h(\mathcal{T}_h/2)$ be the dual Fortin-type interpolations of $p = \text{div} z \in H^1(\Omega)$ and $w = \text{curl} z \in H^1(\Omega) / \mathbb{R}$, respectively. Then,
\[
c_{p,h}(z - z_h, v_h) = (w - \tilde{w}, \text{curl} v_h) + (p - \tilde{p}, I^0_h \text{div} v_h), \quad \forall v_h \in K_h(b)^{1},
\]
\[
|c_{p,h}(z - z_h, v_h)| \leq C(w - \tilde{w}\|_{0} + \|p - \tilde{p}\|_{0})\|v_h\|_{\text{ch}}.
\]

Proof. With $p = \text{div} z \in H^1_0(\Omega)$ and $w = \text{curl} z \in H^1(\Omega) / \mathbb{R}$, for all $v_h \in U^2_h(\mathcal{T}_h/2)$, from (5.11) and (5.1) we have
\[
c_{p,h}(z_h, v_h) = (f, v_h) = (v_h, \text{curl} w - \nabla p) - \beta(z, v_h) = b(v_h, (w, p)) - \beta(z, v_h),
\]
and from (5.11) and (4.4) we have
\[
c_{p,h}(z, v_h) = (\text{curl} z, \text{curl} v_h) + (I^0_h \text{div} z, I^0_h \text{div} v_h) - \beta(z, v_h) = (w, \text{curl} v_h) + (p, I^0_h \text{div} v_h) - \beta(z, v_h) = b(v_h, (\tilde{w}, \tilde{p})) - \beta(z, v_h) + (w - \tilde{w}, \text{curl} v_h) + (p - \tilde{p}, I^0_h \text{div} v_h),
\]
from the definition of $\tilde{p}$ and $\tilde{w}$ by (4.26), we thus obtain (5.12), and (5.13) follows from (5.12) and the definition of $\| \cdot \|_{0}$ by (4.13). □

Corollary 5.1. Let $z \in H_0^{(\text{curl} ; \Omega)} \cap H^{(\text{div} ; \Omega)}$ be the exact solution of problem (5.1) and $z_{h} \in U^2_h(\mathcal{T}_h/2)$ the finite element solution to problem (5.11), respectively. Let $\tilde{z} \in U^2_h(\mathcal{T}_h/2)$ be the primal Fortin-type interpolation of $z$ as defined in the previous subsection. For $v_{h}^{0} := \tilde{z} - z_{h} = v_{0,h} + v_{0,h}^{1}$ with $v_{0,h} \in K_h(b)$ and $v_{0,h}^{1} \in K_h(b)^{1}$, we have
\[
c_{p,h}(v_{h}, v_{h}^{0}) = (w - \tilde{w}, \text{curl} v_{h}^{0}) + (p - \tilde{p}, I^0_h \text{div} v_{h}^{0}) + \beta(z - \tilde{z}, v_{h}^{1}),
\]
\[
|c_{p,h}(v_{h}, v_{h}^{0})| \leq C(w - \tilde{w}\|_{0} + \|p - \tilde{p}\|_{0} + \|z - \tilde{z}\|_{0})\|v_{h}^{0}\|_{\text{ch}},
\]

Proof. By the fact that
\[
c_{p,h}(u, v) = a_{h}(u, v) - \beta(u, v),
\]
and the primal Fortin-type interpolation (5.3), we have
\[
c_{p,h}(\tilde{z} - z, v_{h}^{1}) = \beta(z - \tilde{z}, v_{h}^{1}),
\]
and we have 
\[ c_{ρ,h}(v_h, v_{h}^+) = c_{ρ,h}(\tilde{z} - z_h, v_{h}^+) \]
\[ = c_{ρ,h}(\tilde{z} - z, v_{h}^+) + c_{ρ,h}(z - z_h, v_{h}^+) \]
\[ = c_{ρ,h}(z - z_h, v_{h}^+) + \beta(z - \tilde{z}, v_{h}^+). \]

Hence, it follows from Lemma 5.2 and (4.14) that (5.14) and (5.15) hold. □

To turn to the error estimates, we first analyze the errors between the finite element solution and the finite element interpolation.

Let \( z_h \in U_h^0(\mathcal{T}_h) \) be the finite element solution of (5.11), and \( \tilde{z} \in U_h^0(\mathcal{T}_h) \) be the primal Fortin-type interpolation of the exact solution \( z \in H_0^0(\text{curl } \Omega) \cap H(\text{div } \Omega) \) of problem (5.1). Writing \( z_h = z_{0,h} + z_{h,h}^\dagger \) with \( z_{0,h} \in K_0(b), z_{h,h}^\dagger \in K_0(b)^\perp \), we introduce

\[ v_h := \tilde{z} - z_h = v_{0,h} + v_{h,h}^+ , \quad (5.17) \]

with \( v_{0,h} = -z_{0,h} \in K_0(b), v_{h,h}^\dagger = \tilde{z} - z_{h,h}^\dagger \in K_0(b)^\perp \). Note that \( \tilde{z} \in K_0(b)^\perp \).

We shall estimate \( v_{h,h}^\dagger \) using the classical Aubin–Nitsche duality argument (cf., [33]). With \( v_{h,h}^\dagger \in K_0(b)^\perp \) by (5.17), we consider the following problem: Find \( u^* \in H_0^0(\text{curl } \Omega) \cap H(\text{div } \Omega) \) such that

\[ \text{curl } \text{curl } u^* - \nabla \text{div } u^* - \beta u^* = v_{h,h}^\dagger \quad \text{in } \Omega, \quad u^* \cdot \mathbf{t} = 0, \quad \text{div } u^* = 0 \quad \text{on } \Gamma. \quad (5.18) \]

Let

\[ p^* := \text{div } u^* \in H_0^1(\Omega), \quad w^* := \text{curl } u^* \in H^1(\Omega)/\mathbb{R}, \quad (5.19) \]

we have from Propositions 5.1 and 5.2

\[ ||u^*||_1 + ||w^*||_1 + ||\text{curl } u^*||_1 + ||p^*||_1 + ||\text{curl } w^*||_0 \leq C ||v_{h,h}^\dagger||_0. \quad (5.20) \]

**Theorem 5.2.** Let \( z \in H_0^0(\text{curl } \Omega) \cap H(\text{div } \Omega) \) be the solution of problem (5.1), with \( p = \text{div } z \in H_0^1(\Omega) \) and \( w = \text{curl } z \in H^1(\Omega)/\mathbb{R} \). Let \( \tilde{p} \in Q_0(\mathcal{T}_h) \) and \( \tilde{w} \in W^{0,-1}(\mathcal{T}_h) \) be the dual Fortin-type interpolation of \( p \) and \( w \) defined by (4.26). Let \( v_{h,h}^\dagger \) be defined as in (5.17). For all \( h < h^* \) with \( h^* < 1 \) sufficiently small we have

\[ ||v_{h,h}^\dagger||_0 \leq C(||z - \tilde{z}||_0 + ||w - \tilde{w}||_0 + ||p - \tilde{p}||_0), \quad (5.21) \]

\[ ||v_{h,h}^\dagger||_{h^0} \leq C(||z - \tilde{z}||_0 + ||w - \tilde{w}||_0 + ||p - \tilde{p}||_0). \quad (5.22) \]

**Proof.** From (5.18) we have

\[ ||v_{h,h}^\dagger||^2_0 = (\text{curl } w^*, v_{h,h}^\dagger) - (\nabla p^*, v_{h,h}^\dagger) - \beta(u^*, u_{h,h}^\dagger) \]

\[ = b(v_{0,h}^+, (w^*, p^*)) - \beta(u^*, v_{h,h}^+), \]

\[ = a_0(u^*, v_{h,h}^+) - \beta(u^*, v_{h,h}^+) \]

\[ + b(v_{0,h}^+, (w^*, p^*)) - a_0(u^*, v_{h,h}^+) \]

\[ := E_1 + E_2. \quad (5.23) \]

Where

\[ E_1 = a_0(u^*, v_{h,h}^+) - \beta(u^*, v_{h,h}^+), \]

\[ E_2 = b(v_{0,h}^+, (w^*, p^*)) - a_0(u^*, v_{h,h}^+). \]

Below we estimate \( E_1 \) and \( E_2 \). The estimates are divided into two steps.

**Step 1.** To estimate \( E_1 \).

Let \( u_h^* \in U_h^0(\mathcal{T}_h) \) be the primal Fortin-type interpolation of \( u^* \) as defined by (5.3), and \( u_h^* \in K_0(b)^\perp \), satisfying

\[ a_0(u^* - u_h^*, v_{h,h}^+) = 0. \]

By (4.6), (4.8), (5.17), (5.16), we have

\[ E_1 = a_0(u^*, v_{h,h}^+) - \beta(u^*, v_{h,h}^+) \]

\[ = a_0(u_h^*, v_{h,h}^+) - \beta(u_h^*, v_{h,h}^+) - \beta(u^* - u_h^*, v_{h,h}^+) \]

\[ = a_0(u^*, v_{h,h}^+) - \beta(u^* - u_h^*, v_{h,h}^+) \]

\[ = a_0(u_h^*, Z - z_h) - \beta(u_h^*, Z - z_h) - \beta(u^* - u_h^*, v_{h,h}^+) \]

\[ = a_0(u_h^*, Z - z_h) - \beta(u_h^*, Z - z_h) - \beta(u^* - u_h^*, v_{h,h}^+) \]

\[ + a_0(u_h^*, Z - z_h) - \beta(u_h^*, Z - z_h) \]

\[ = c_{\rho,h}(Z - z_h, u_h^*) - \beta(u_h^*, Z - z_h) - \beta(u^* - u_h^*, v_{h,h}^+). \quad (5.24) \]
where we have used the definite of the primal Fortin-type interpolation $\tilde{z}$ of $z$ as defined by (5.3), i.e.,

$$a_0(u^*_h, \tilde{z} - z) = 0.$$ 

By Lemmas 5.1 and 5.2, (5.20), and Corollary 5.1, we have

$$|\beta(u^*_h, \tilde{z} - z)| \leq C \|u^*_h\|_0 \|z - \tilde{z}\|_0 \leq C \|u^*_h\|_{a_h} |z - \tilde{z}|_0 \leq C \|z - \tilde{z}\|_0 \|u^*_h\|_{\text{curl, div}} \leq C \|z - \tilde{z}\|_0 \|v^+_h\|_{H^1}.$$ 

Combining (5.23), (5.25) and (5.26), we have

$$|\beta(u^* - u^*_h, v^+_{0,h})| \leq C \|u^* - u^*_h\|_0 \|v^+_{0,h}\|_0 \leq Ch' \|v^+_{0,h}\|^2.$$

$|c_{\beta,h}(z - z_h, u^*_h)| \leq C(\|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) \|u^*_h\|_{a_h} \leq C(\|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) \|u^*_h\|_{\text{curl, div}} \leq C(\|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) \|v^+_{0,h}\|_0,$$

and we have

$$|E_i| \leq Ch' \|v^+_{0,h}\|^2 + C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) \|v^+_{0,h}\|_0.$$ 

(5.25)

Step 2. To estimate $E_2.$

Let $\tilde{p}^* \in Q^0(\mathcal{T}_h), \tilde{w}^* \in W^{(e-1)}(\mathcal{T}_{h/2})$ be the dual Fortin-type interpolations of $p^* = \text{div} u^*, w^* = \text{curl} u^*$, respectively, satisfying

$$b(v^+_{0,h}, (\tilde{w}^*, \tilde{p}^*)) = b(v^+_{0,h}, (w^*, p^*)).$$

We have

$$a_0(u^*, v^+_{0,h}) = (\text{curl} u^*, \text{curl} v^+_{0,h}) + (\Pi_h^{(0)} \text{div} u^*, \Pi_h^{(0)} \text{div} v^+_{0,h})$$

$$= (w^*, \text{curl} v^+_{0,h}) + (p^* - \tilde{p}^*, \Pi_h^{(0)} \text{div} v^+_{0,h})$$

$$+ (\tilde{w}^*, \text{curl} v^+_{0,h}) + (\tilde{p}^*, \Pi_h^{(0)} \text{div} v^+_{0,h})$$

$$= (w^* - \tilde{w}^*, \text{curl} v^+_{0,h}) + (p^* - \tilde{p}^*, \Pi_h^{(0)} \text{div} v^+_{0,h}) + b(\Pi_h^{(0)} (\tilde{w}^*), (\tilde{p}^*)),$$

and from Corollary 4.1 and (5.20), we have

$$|E_2| = |b(v^+_{0,h}, (w^*, p^*)) - a_0(u^*, v^+_{0,h})|$$

$$= \left| - (w^* - \tilde{w}^*, \text{curl} v^+_{0,h}) - (p^* - \tilde{p}^*, \Pi_h^{(0)} \text{div} v^+_{0,h}) \right|$$

$$\leq C(\|w^* - \tilde{w}^*\|_0 + \|p^* - \tilde{p}^*\|_0) \|v^+_{0,h}\|_{a_h} \leq Ch' \|v^+_{0,h}\|_0 \|v^+_{0,h}\|_{a_h}.$$ 

(5.26)

Combining (5.23), (5.25) and (5.26), we have

$$(1 - Ch') \|v^+_{0,h}\|^2 \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) \|v^+_{0,h}\|_0 + Ch' \|v^+_{0,h}\|_0 \|v^+_{0,h}\|_{a_h},$$

and for a sufficiently small $h < 1$ so that for all $h < h_*$, $1 - Ch' > 0$, we have

$$\|v^+_{0,h}\|_0 \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) + Ch' \|v^+_{0,h}\|_{a_h},$$

(5.27)

In what follows, we shall estimate $\|v^+_{0,h}\|_{a_h}$. We have, from (4.8) and (5.16),

$$\|v^+_{0,h}\|^2_{a_h} = a_0(v^+_{0,h}, v^+_{0,h}) = a_0(v^+_{0,h}, v^+_{0,h})$$

$$= a_0(h, v^+_{0,h}) - \beta(h, v^+_{0,h}) + \beta(h, v^+_{0,h})$$

$$= c_{\beta,h}(v^+_{0,h}, v^+_{0,h}) + \beta(h, v^+_{0,h}),$$

where, from Corollary 5.1,

$$|c_{\beta,h}(v^+_{0,h}, v^+_{0,h})| \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) \|v^+_{0,h}\|_{a_h},$$

and from Theorem 5.1,

$$\|v^+_{0,h}\|_{a_h} \leq C \|v^+_{0,h}\|_{a_h},$$

and from (5.27),

$$|\beta(h, v^+_{0,h})| = \|\beta(v^+_{0,h}, v^+_{0,h})\|^2 \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0)^2 + Ch^2 \|v^+_{0,h}\|^2_{a_h},$$

and we have

$$\|v^+_{0,h}\|^2_{a_h} \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0) \|v^+_{0,h}\|_{a_h}$$

$$+ C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0)^2 + Ch^2 \|v^+_{0,h}\|^2_{a_h},$$

(5.28)
where, for an $\epsilon > 0$,

$$C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0)\|v_{0,h}\|_{a_0} \leq \epsilon \|v_{0,h}\|_{a_0}^2 + C_\epsilon^{-1}(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0)^2.$$

and we have

$$(1 - \epsilon - C\epsilon^2)\|v_{0,h}\|_{a_0}^2 \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0)^2,$$

for $\epsilon = 1/2$ and a second sufficiently small $h^* < h_*$ so that for all $h < h^*$, $1 - \epsilon - C\epsilon^2 > 0$, we have, from (5.28),

$$\|v_{0,h}\|_{a_0} \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0 + \|p - \tilde{p}\|_0).$$

Hence, (5.27) and (5.29) lead to the conclusions. \(\square\)

**Corollary 5.2.** Under the same assumptions as in Theorem 5.2,

$$\|v_{0,h}\|_0 \leq Ch'(\|z\|_{\text{curl} \cdot \text{div}} + \|\text{div } z\|_1 + \|\text{curl } z\|_0),$$

and

$$\|v_{0,h}\|_{a_0} \leq Ch' \|z\|_{\text{curl} \cdot \text{div}} + \|\text{div } z\|_1 + \|\text{curl } z\|_0.$$  \(5.30\)

**Proof.** From Theorem 5.2, Lemma 5.1, and Corollary 4.1, it follows that both (5.30) and (5.31) hold. \(\square\)

From Corollary 5.2, Proposition 5.2, and Lemma 5.1, we can have the following corollary.

**Corollary 5.3.** Under the same assumptions as in Theorem 5.2, we have

$$\|v_{0,h}\|_0 + \|v_{0,h}\|_{a_0} \leq C h' \|f\|_0.$$  \(5.32\)

As a result, since $z - z_{0,h} = z - \tilde{z} + v_{0,h}$, the error estimates hold uniformly relative to $(L^2(\Omega))^2$:

$$\|z - z_{0,h}\|_0 + \|z - z_{0,h}\|_{a_0} \leq C h' \|f\|_0.$$  \(5.33\)

**Corollary 5.4.** The finite element problem (5.11) has a unique solution.

**Proof.** We follow the argument in [44]. Note that uniqueness and existence are equivalent for a finite dimensional square system. We need only consider the uniqueness. We use the argument by contradiction. If the solution of (5.11) is not unique, then for $f = 0$, there should have a solution $z_h \neq 0$. For such $z_h = z_{0,h} + z_{0,h}$ with $z_{0,h} \in K_0(b)$, $z_{0,h} \in K_0(b)^{\perp}$, from the error estimates (5.33) we know that $z_{0,h} = 0$, since $f = 0$ leads to $z = 0$. Meanwhile, noting that $z_{0,h}$ is determined by

$$-\beta(z_{0,h}, v_h) = (f, v_h) \quad \forall v_h \in K_0(b),$$

but $f = 0$, we have $z_{0,h} = 0$. Hence $z_h = 0$, this contradicts $z_h \neq 0$. \(\square\)

### 5.5 Application to indefinite Maxwell’s equations

Given $f$ with

$$\text{div } f = 0$$  \(5.34\)

and a number $\beta \in \mathbb{R}$ which is not any eigenvalue of the Maxwell eigenvalue problem (2.1). We consider the indefinite Maxwell’s equations as follows:

$$\text{curl } \text{curl } z - \beta z = f \quad \text{in } \Omega, \quad \text{div } z = 0 \quad \text{in } \Omega, \quad z \cdot t = 0 \quad \text{on } \Gamma.$$  \(5.35\)

The variational problem is to find $z \in H_0(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega)$ such that

$$c_{\beta}(z, v) = (f, v), \quad v \in H_0(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega),$$

and the finite element problem is to find $z_h \in U_h^{(\ell)}(\mathcal{R}_{1/2})$ such that

$$c_{\beta,h}(z_h, v_h) = (f, v_h), \quad \forall v_h \in U_h^{(\ell)}(\mathcal{R}_{1/2}).$$  \(5.37\)

Because of (5.34), the finite element solution $z_h$ of problem (5.37) belongs to $K_0(b)^{\perp}$ (i.e., $z_h$ does not have the component in $K_0(b)$). With the regularity results in Proposition 5.2, from Corollary 5.3, we have

$$\|z - z_h\|_{a_0} \leq Ch' \|f\|_0.$$  \(5.38\)
For the source problem (5.35) in a non-convex polygon, usually, the solution \( z \) has no more regularity than that is stated in Proposition 5.2. For the eigenvalue problem in a non-convex polygon, the situation is complex. There may have infinite singular eigenfunctions in the following sense:

\[
\begin{align*}
u \in (H^2(\Omega))^2, \quad \text{curl} \, \nu \in H^1(\Omega), \quad \text{div} \, \nu = 0.
\end{align*}
\]

In other words, Proposition 5.2 applies. At the same time, there may still have infinite smooth and even analytical eigenfunctions, although there seem no theory about these smooth eigenfunctions who live in a non-convex polygon.

Therefore, if assuming that \( z \) is sufficiently smooth, we need to study the error estimates so that the error bounds the same as \( \ell \) can be obtained. Since the order of approximation in \( U^f_h(\mathcal{T}_h/2) \) is \( \ell \geq 2 \), we assume that

\[
\begin{align*}z \in (H^{1+\ell}(\Omega))^2.\tag{5.40}\end{align*}
\]

Moreover, for the eigenvalue problem, since \( \text{curl} \, z = \omega^2 z \), \( \text{curl} \, z \) is smoother, i.e.,

\[
\begin{align*}\text{curl} \, z \in H^{2+\ell}(\Omega).\tag{5.41}\end{align*}
\]

Since \( p := \text{div} \, z = 0, \bar{p} = 0 \) from Corollary 4.1. Thus, from Theorem 5.2, by the fact that \( v_h = \tilde{z} + z_h = \nu_0^1 \) because both \( \tilde{z} \) and \( z_h \) belong to \( K_h(b)^\perp \) where \( \tilde{z} \) is the primal Fortin-type interpolation of \( z \), we have

\[
\begin{align*}\|z - z_h\|_{ch} \leq \|\tilde{z} - z_h\|_{ch} + \|\tilde{z} - z_h\|_{ch} \leq C(\|z - \tilde{z}\|_0 + \|w - \tilde{w}\|_0),\tag{5.42}\end{align*}
\]

where \( \tilde{w} \in W^{(l-1)}(\mathcal{T}_{h/2}) \) is the dual Fortin-type interpolation of \( w = \text{curl} \, z \in H^{l}(\Omega) \), and from Corollary 4.1,

\[
\begin{align*}\|w - \tilde{w}\|_0 \leq C h^l \|w\|_\epsilon,\tag{5.43}\end{align*}
\]

while, from (5.5), since \( z \) is smooth, with the interpolation like (5.6)–(5.7) but for \( z \) itself, it is easy to obtain

\[
\begin{align*}\|z - \tilde{z}\|_{ch} \leq C h^\ell \|z\|_{1+\epsilon},\tag{5.44}\end{align*}
\]

and we obtain

\[
\begin{align*}\|z - z_h\|_{ch} \leq C h^\ell \|z\|_{1+\epsilon}.\tag{5.45}\end{align*}
\]

We summarize the above as the following theorem.

**Theorem 5.3.** Let \( z \) be the solution of problem (5.34)–(5.35) and \( z_h \in U^f_h(\mathcal{T}_{h/2}) \) the solution of problem (5.37). Assume that (5.40) holds. Then,

\[
\begin{align*}\|z - z_h\|_{ch} \leq C h^\ell \|z\|_{1+\epsilon}.\end{align*}
\]

### 6. Error estimates for eigenproblem

In this section, we shall analyze the error estimates of the eigenproblem within the spectral theory for compact operators in [28,29].

We first investigate the well-posedness of the finite element eigenproblem (3.5). For that goal, we restate problem (3.5) as follows: Find \( (\lambda_h, u_h \neq 0) \in \mathbb{R} \times U^f_h(\mathcal{T}_{h/2}) \) such that

\[
\begin{align*}c_h(u_h, v_h) = c_{-1,h}(u_h, v_h) = a_h(u_h, v_h) + (u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in U^f_h(\mathcal{T}_{h/2}),\tag{6.1}\end{align*}
\]

where

\[
\begin{align*}\lambda_h := \omega^2_h + 1.\tag{6.2}\end{align*}
\]

**Theorem 6.1.** The eigenproblem (6.1) is well-posed, and \( \lambda_h = 1 \) is the only one whose eigenfunction space is \( K_h(b) \). Any other eigenvalues \( \lambda_h \neq 1 \) it satisfies

\[
\begin{align*}\lambda_h \geq 1 + C > 0,\tag{6.3}\end{align*}
\]

and its eigenfunction space belongs to \( K_h(b)^\perp \), where the constant \( C \) comes from Theorem 5.1. Moreover, for different eigenvalues, their eigenfunctions are orthogonal relative to both the \( a_h \)- and \( c_h \)-induced inner products \( \langle \cdot , \cdot \rangle_h = a_h(\cdot , \cdot ) \) and \( \langle \cdot , \cdot \rangle_h = c_h(\cdot , \cdot ) \) and to the \( L^2 \) inner product \( \langle \cdot , \cdot \rangle \).

**Proof.** The eigenproblem (6.1) is well-posed, thanks to the coercivity of \( c_h(\cdot , \cdot ) \) on \( U^f_h(\mathcal{T}_{h/2}) \), i.e., for all \( v_h = v_{0,h} + \nu_0^1 \)

\[
\begin{align*}c_h(v_h, v_h) = \|v_h\|_{a_h}^2 + \|v_h\|_{\nu_0}^2 
\begin{align*}&= \|\nu_0^1\|_{a_h}^2 + \|v_{0,h}\|_{\nu_0}^2 + \|\nu_0^1\|_{\nu_0}^2 
\begin{align*}&\geq (1 + C)\|v_{0,h}\|_{\nu_0}^2 + \|v_{0,h}\|_{\nu_0}^2 \geq \|v_h\|_{\nu_0}^2.
\end{align*}
\end{align*}
\]

With \( v_{0,h} \in K_h(b) \), \( v_{0,h}^\perp \in K_h(b)^\perp \).
We first note that all the eigenvalues are real numbers, because of the symmetry property of (6.1). It can be trivially verified that the orthogonality of eigenfunctions corresponding to different eigenvalues with respect to $(\cdot, \cdot)_{\Omega_2}$ and $(\cdot, \cdot)$. In addition, we can see that $\lambda_h = 1$ is an eigenvalue and its eigenspace is $K_0(b)$. For any other eigenvalue $\lambda_h \neq 1$, with eigenfunction $u_h \in \mathcal{U}_h^c(\mathcal{T}_h/2)$ of (6.1), we must have

$$u_h \in K_0(b)^\perp.$$  

(6.5)

In fact, write

$$u_h = u_{0, h} + u_{\perp, h}$$ where $u_{0, h} \in K_0(b)$ and $u_{\perp, h} \in K_0(b)^\perp$.

If $u_{0, h} = 0$, then (6.5) holds. Otherwise, from $c_0(u_h, v_h) = (u_h, v_h) = (u_{0, h}, v_h)$ for all $v_h \in K_0(b)$, from (6.1) we have

$$(u_{0, h}, v_h) = \lambda_h (u_{0, h}, v_h) \ \forall v_h \in K_0(b),$$

and both $(u_{0, h}, u_{0, h}) = \lambda_h (u_{0, h}, u_{0, h})$ and $\lambda_h \neq 1$ lead to $\|u_{0, h}\|_0 = 0$, i.e., $u_{0, h} = 0$. Hence, $u_h = u_{\perp, h} \in K_0(b)^\perp$, i.e., (6.5) holds. Since $u_h \in K_0(b)^\perp$, $u_h \neq 0$, $\lambda_h \neq 1$, we have from (6.1) and Theorem 5.1

$$\lambda_h \|u_h\|^2_0 = \|u_h\|^2_{\Omega_2} = \|u_h\|^2_{\Omega_2} + \|u_h\|^2_0 \geq (1 + C)\|u_h\|^2_0,$$

and therefore, (6.3) holds. □

According to Theorem 6.1, the only priori-known eigenvalue $\lambda_h = 1$ should be abandoned, whose eigenfunction space is $K_0(b)$. Such a situation also exists in other methods such as the edge element method. All the other eigenvalues solve problem (3.5), with eigenfunctions belong to $K_0(b)^\perp$, which are well separated from the eigenvalue $\lambda = 1$ according to (6.3), while the eigenvalue $\lambda_h = 1$ is already priori identified.

To study the error bounds within the general spectral theory [28,29], we need to introduce the continuous solution operator which is compact and the discrete solution operator.

Corresponding to problem (6.1), we consider the source problem: Given $f \in (L^2(\Omega))^2$, Find $z \in H_0(curl : \Omega) \cap H(div ; \Omega)$ such that, for all $v \in H_0(curl ; \Omega) \cap H(div ; \Omega)$,

$$c(z, v) = c_{\cdot, 1}(z, v) = (curl z, curl v) + (div z, div v) + (z, v) = (f, v).$$

(6.6)

The boundary value problem reads as follows: Given $f \in (L^2(\Omega))^2$, to find $z$ such that

$$c(curl z - \nabla div z + z = f \ in \ \Omega, \ \ z \cdot t = 0, \ \ div z = 0 \ on \ \Gamma'.$$

(6.7)

With (6.6), a linear operator $\mathcal{T} : (L^2(\Omega))^2 \to H_0(curl ; \Omega) \cap H(div ; \Omega)$ is defined as follows: for any given $f \in (L^2(\Omega))^2$,

$$z = \mathcal{T}f \in H_0(curl ; \Omega) \cap H(div ; \Omega)$$

satisfies

$$c(\mathcal{T}f, v) = (f, v), \quad \forall v \in H_0(curl ; \Omega) \cap H(div ; \Omega).$$

(6.8)

It can be easily verified that the linear operator $\mathcal{T}$ is bounded from $(L^2(\Omega))^2$ to $H_0(curl ; \Omega) \cap H(div ; \Omega)$. Moreover, $\mathcal{T}$ is self-adjoint, in fact, symmetric positive definite with respect to both the $L^2$ inner product $(\cdot, \cdot)$ and the $a(\cdot, \cdot)$-induced inner product $(\cdot, \cdot)_a := a(\cdot, \cdot)$ and the $c(\cdot, \cdot)$-induced inner product $(\cdot, \cdot)_c := c(\cdot, \cdot)$. Moreover, $\mathcal{T} : (L^2(\Omega))^2 \to (L^2(\Omega))^2$ is compact (due to the compact embedding of $H_0(curl ; \Omega) \cap H(div ; \Omega) \hookrightarrow (L^2(\Omega))^2$). Also, $\mathcal{T} : H_0(curl ; \Omega) \cap H(div ; \Omega) \to H_0(curl ; \Omega) \cap H(div ; \Omega)$ is compact. With

$$\lambda := \omega^2 + 1,$$

the continuous eigenvalue problem (2.3) may also read as follows: Find $\lambda, u \in \mathbb{R} \times H_0(curl ; \Omega) \cap H(div ; \Omega)$ such that

$$c(u, v) = \lambda (u, v), \quad \forall v \in H_0(curl ; \Omega) \cap H(div ; \Omega),$$

(6.10)

and the corresponding boundary value problem is as follows: Find $\lambda, u \neq 0$ such that

$$c(curl u - \nabla div u + u = \lambda u \ in \ \Omega, \ \ u \cdot t = 0, \ \ div u = 0 \ on \ \Gamma'.$$

(6.11)

Since $H_0(curl ; \Omega) \cap H(div ; \Omega)$ is compactly embedded into $(L^2(\Omega))^2$ (see [35,36]), the eigenproblem (6.10) has an infinite sequence of eigenvalues

$$0 < 1 + C \leq \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty,$$

(6.12)

where the constant $C$ only depends on $\Omega$ and comes from Proposition 2.1. Clearly, an eigenpair $(\lambda, u \neq 0)$ of (6.10) if and only if $(\nu = 1/\lambda, u \neq 0)$ is an eigenpair of $\mathcal{T}$, i.e.,

$$\mathcal{T}u = v u \quad with \ v = 1/\lambda,$$

(6.13)

and the sequence of eigenvalues of $\mathcal{T}$ satisfies

$$0 \to \cdots \to v_2 \leq v_1 \leq 1/(1 + C).$$

(6.14)
The finite element problem corresponding to (6.6) is to find $z_h \in U_h^{(2)}(\mathcal{B}_{\mathcal{H}/2})$ such that

$$c_h(z_h, v_h) = c_{-1,h}(z_h, v_h) = a_h(z_h, v_h) + \langle z_h, v_h \rangle = (f, v_h), \quad \forall v_h \in U_h^{(2)}(\mathcal{B}_{\mathcal{H}/2}).$$  \hspace{1cm} (6.15)

From (6.4) it follows that problem (6.15) is well-posed, and it induces a linear bounded operator $T_h : (L^2(\Omega))^2 \to U_h^{(2)}(\mathcal{B}_{\mathcal{H}/2})$, i.e., for any $f \in (L^2(\Omega))^2$, $T_h f \in U_h^{(2)}(\mathcal{B}_{\mathcal{H}/2})$ solves

$$c_h(T_h f, v_h) = (f, v_h), \quad \forall v_h \in U_h^{(2)}(\mathcal{B}_{\mathcal{H}/2}).$$  \hspace{1cm} (6.16)

By decomposing $T_h f$ as follows:

$$T_h f = T_{0,h} f + T_{0,h}^{\perp} f \quad \text{where} \quad T_{0,h} f \in K_h(b) \text{ and } T_{0,h}^{\perp} f \in K_h(b)^\perp,$$

we know that any eigenpair $(\lambda, u_h \neq 0)$ satisfies (6.1) if and only if

$$T_{0,h} u_h = \nu_h u_h \quad \text{with} \quad \nu_h = 1/\lambda_h \neq 1 \text{ and } u_h \in K_h(b)^\perp.$$  \hspace{1cm} (6.18)

In the sequel, we will analyze the errors between the eigenvalues of (6.13) and (6.18). Recall Corollary 5.3 in terms of $T$ and $T_{0,h}^{\perp}$ in the following.

$$\|Tf - T_{0,h}^{\perp}f\|_0 \leq C \|f\|_0, \quad \forall f \in (L^2(\Omega))^2.$$  \hspace{1cm} (6.19)

which is a uniform error bound, i.e.,

$$\lim_{h \to 0} \sup_{h \in (L^2(\Omega))^2} \frac{\|Tf - T_{0,h}^{\perp}f\|_0}{\|f\|_0} = 0.$$  \hspace{1cm} (6.20)

Moreover, for all $0 \leq s \leq r$, from (6.19) and Proposition 5.2 and the inverse estimates of finite element functions, we can easily obtain

$$\|Tf - T_{0,h}^{\perp}f\|_s \leq C \|f\|_0, \quad \forall f \in (L^2(\Omega))^2.$$  \hspace{1cm} (6.20)

In fact, since $Tf \in (H^r(\Omega))^2$ for some $r > 1/2$, let $I_h$ denote some averaging-type interpolation operator onto $U_h^{(2)}(\mathcal{B}_{\mathcal{H}/2})$ such as Scott–Zhang interpolation [45], and let $I_h T f \in U_h^{(2)}(\mathcal{B}_{\mathcal{H}/2})$, satisfying

$$\|Tf - I_h T f\|_s \leq C \|f\|_0, \quad \|Tf - I_h T f\|_0 \leq C \|Tf\|_r \leq C \|f\|_0.$$  \hspace{1cm} (6.21)

where from Proposition 5.2, $\|Tf\|_r \leq C \|f\|_0$. Then,

$$\|Tf - T_{0,h}^{\perp}f\|_s \leq \|Tf - I_h T f\|_s + \|I_h T f - T_{0,h}^{\perp}f\|_s \leq C \|f\|_0 + C \|f - T_{0,h}^{\perp}f\|_s \leq C \|f\|_0 + C \|f - T_{0,h}^{\perp}f\|_0 \leq C \|f\|_0.$$  \hspace{1cm} (6.20)

We recall the following theorem on spectral correctness for compact operators, due to [28,29].

**Theorem 6.2.** Let $X$ be a Hilbert space with inner product $(\cdot, \cdot)_X$ and norm $\| \cdot \|_X$ and $A : X \to X$ a self-adjoint and compact operator. Let $\Theta = \{h \in \mathbb{R} : 1 \leq n < \infty\}$ be a discrete subset such that $h_n \to 0$ as $n \to \infty$. Let $A_h$ which defined with respect to $h \in \Theta$ denote a family of linear self-adjoint operators. Assume that $A_h$ converges pointwise to $A$ and that the set $A = \{A_h : X \to X, h \in \Theta\}$ is collectively compact. Let $\nu$ be an eigenvalue of $A$ of multiplicity $m$ and let $\phi_i$, $1 \leq i \leq m$, be the associated orthonormal eigenvectors. Then, (a) for any $\epsilon > 0$ such that the disk $B(\nu, \epsilon)$ which centers at $\nu$ with radius $\epsilon$ contains no other eigenvalues of $A$, there exists $h_\epsilon$ which depends on $\epsilon$ such that, for all $h < h_\epsilon$, $A_h$ has exactly $m$ eigenvalues (repeated according to their multiplicity) in the disk $B(\nu, \epsilon)$ (b) For $h < h_\epsilon$, denoting the set of the eigenvalues of $A_h$ in the disk $B(\nu, \epsilon)$ as $h_{\nu,i}$, $1 \leq i \leq m$, we have, for all $1 \leq i \leq m$,

$$|\nu - h_{\nu,i}| \leq \sum_{k=1}^m |(A - A_h)\phi_k, \phi_k|_X + \sum_{k=1}^m \|A - A_h\phi_k\|_X^2. \quad \Box$$  \hspace{1cm} (6.21)

Put $X := (L^2(\Omega))^2$, $(\cdot, \cdot)_X := (\cdot, \cdot)$ and $\| \cdot \|_X := \| \cdot \|_0$. $A := \mathcal{T}, \Theta := \{h : h = \max_{T \in \mathcal{T}_{\mathcal{H}/2}} h_T\}$, and $A_h := T_{0,h}^{\perp}$. As mentioned earlier, $T$ is self-adjoint and compact from $X$ to $X$. It can be easily verified that $T_{0,h}^{\perp} : X \to X$ is also self-adjoint. The error estimates in (6.19) ensure the pointwise convergence of $T_{0,h}^{\perp}$ to $T$. Therefore, Theorem 6.2 holds, once the following Lemma 6.1 is shown for the collective compactness of the set $A_h = \{A_h : X \to X, h \in \Theta\}$. Recall that the so-called collectively compact property of the set $A$ says if, for any bounded set $M \subset X$, the image set of $A M := \{A_h f, f \in M, A_h \in A\}$ is relatively compact in $X$.

**Lemma 6.1.** The sequence $\{T_{0,h}^{\perp}\}_{h>0}$ is collectively compact.
Proof. We first note that, for all $s > 0$, $(H^s(\Omega))^2$ is compactly embedded into $(L^2(\Omega))^2$. Thus, we can conclude from the following stability result:

$$\|\tau_{0,h}^+ f\|_s \leq C \|f\|_0 \quad \text{for all} \ f \in (L^2(\Omega))^2, \ \text{for some} \ s > 0.$$ 

This result holds, indeed, from Proposition 5.2 and (6.20).

From Theorem 6.2 and (6.19), we immediately have the following error bounds for eigenvalues.

Corollary 6.1. The following error estimates between the continuous eigenvalue $\nu$ and its discrete counterpart $\nu_h$ hold:

$$|\nu - \nu_h| \leq C h^r.$$  

(6.22)

Now, we shall study the optimal error bound for the eigenvalues of the Maxwell eigenvalue problem (2.1). Note that $\omega^2 = \lambda - 1$. Here, we first assume that the continuous eigenfunctions corresponding to $\lambda$ are singular in the following sense:

$$u \in (H^r(\Omega))^2, \quad \text{curl} \ u \in H^r(\Omega).$$

By (2.1), we actually have more regularity on curl $u$:

$$\text{curl} \ u \in H^{1+r}(\Omega).$$

We also have

$$\text{div} \ u = 0.$$ 

The second term in (6.21) will be $O(h^{2r})$ from (6.19), but the first term in (6.21) is not so obvious. In the following, we shall estimate the first term in (6.21). Let $\lambda = \nu^{-1} = \omega^2 + 1$, and any orthonormal eigenfunction $\nu \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ with $\|\nu\|_0 = 1$ corresponding to $\lambda$, is determined by (2.1):

$$\text{curl} \ \text{curl} \ \nu = \omega^2 \nu, \quad \text{div} \ \nu = 0 \quad \text{in} \ \Omega, \quad \nu \cdot t = 0, \quad \text{on} \ \Gamma.$$ 

(6.23)

It also solves

$$\text{curl} \ \text{curl} \ \nu - \nabla \text{div} \ \nu + \nu = \lambda \nu, \quad \text{in} \ \Omega, \quad \nu \cdot t = 0, \quad \text{div} \ \nu = 0 \quad \text{on} \ \Gamma.$$ 

(6.24)

and

$$c(\nu, z) = (\lambda \nu, z), \quad \forall z \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega).$$ 

(6.25)

We know from Propositions 5.1 and 5.2 that $\nu \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ has the following regularity,

$$\|\nu\|_{l, r} + \|\nu\|_1 + \|\text{curl} \ \nu\|_1 + \|\text{curl} \ \text{curl} \ \nu\|_0 \leq C |\lambda| \|\nu\|_0 = C,$$ 

(6.26)

$$\text{div} \ \nu = 0.$$ 

(6.27)

Corresponding to the same $\lambda$, for another eigenfunction $u$ with $\|u\|_0 = 1$, $\text{div} \ u = 0$, we still have (6.23)–(6.27). Set

$$u^* = T u^*, \quad u_h^* = T_{0,h}^+ u^*,$$ 

(6.28)

and let $\tilde{u}^* \in K_0(b)^\perp$ be the primal Fortin-type interpolation of $u^*$ as defined in Section 5.3. Let $\nu^* \in K_0(b)^\perp$ be the primal Fortin-type interpolation of $\nu^*$. By (6.7) and (6.8), since $\text{div} \ u^* = 0$, we have

$$\text{div} \ u^* = 0,$$ 

(6.29)

and by Propositions 5.1 and 5.2, we have

$$\|u^*\|_{l, r} + \|u^*\|_1 + \|\text{curl} \ u^*\|_1 + \|\text{curl} \ \text{curl} \ u^*\|_0 \leq C \|u^*\|_0 = C.$$ 

(6.30)

Note that $\text{div} \ u^* = 0$, i.e., $u^* = \text{curl} \ A$ for a scalar $A$, and $(u^*, z_h) = (\text{curl} \ z_h) = 0$ for all $z_h \in K_0(b)$, and $T_{0,h}^+ u^* \equiv 0$, i.e.,

$$u_h^* = T_{0,h}^+ u^* = T_h u^* \in K_0(b)^\perp.$$ 

(6.31)

By (6.16), (6.8) and (6.29), for all $z_h \in K_0(b)^\perp$,

$$c_h(u^*_h, z_h) = (u^*_h, z_h) = (u^*, \text{curl} \ z_h) = (\text{curl} \ u^*, \text{curl} \ z_h) + (u^*, z_h)$$

$$= (\text{curl} \ u^*, \text{curl} \ z_h) + (\Pi_h^{(0)} \text{div} \ u^*, \Pi_h^{(0)} \text{div} \ z_h) + (u^*, z_h) = c_h(u^*, z_h),$$

that is

$$c_h(u^* - u_h^*, z_h) = 0, \quad \forall z_h \in K_0(b)^\perp.$$ 

(6.32)
Lemma 6.2. We have
\[ |\langle (\mathcal{T} - \mathbb{T}_{0,h}^{\perp})u^h, v^h \rangle| \leq Ch^{2\ell}. \]

Proof. From (6.25), (6.27), (6.32), and the fact that \( \tilde{v}^i \in K_0(\mathcal{B})^i \), Corollary 5.3, Lemma 5.1, we have
\[ (u^* - u^*_h, \lambda v^i) = c(u - u^*_h, v^i) \]
\[ = (\text{curl} (u^* - u^*_h), \text{curl} v^i) + (\text{div} (u^* - u^*_h), \text{div} v^i) + (u^* - u^*_h, v^i) \]
\[ = (\text{curl} (u^* - u^*_h), \text{curl} v^i) + (\mathcal{I}^0_h \text{div} (u^* - u^*_h), \mathcal{I}^0_h \text{div} v^i) + (u^* - u^*_h, v^i) \]
\[ = c_h (u^* - u^*_h, v^i) = \lambda c_h (u^* - u^*_h, v^i) \]
\[ \leq \|u^* - u^*_h\|_{C_0} \|v^i - \tilde{v}_h^i\|_{C_0} \]
\[ \leq Ch^{2\ell} \|u^i\|_{1,C_0} \|v^i\|_{1,C_0}. \] (6.33)

Consequently, we have
\[ |\langle (\mathcal{T} - \mathbb{T}_{0,h}^{\perp})u^h, v^i \rangle| = |(u^* - u^*_h, v^i)| \leq Ch^{2\ell}, \]
and the proof is completed. \(\square\)

From Lemma 6.2 and Theorem 6.2 and (6.19), we have obtained the following error estimates for the Maxwell eigenvalue problem.

Theorem 6.3. Let \( v^{-1} = \lambda = \omega^2 + 1 \), \( \omega^2 \) denotes the eigenvalue of the Maxwell eigenvalue problem (2.1) and \( \nu^{-1} = \lambda = \omega^2 \), the eigenvalue of the finite element eigenproblem (3.5). Then, under the regularity results (6.26)–(6.27) for the corresponding orthonormal eigenfunctions,
\[ |v - \nu_h| \leq Ch^{2\ell}. \] (6.34)

When the eigenfunctions are smooth, say \( v^i \in (H^{1+\ell}(\Omega))^2 \), from (5.44), Theorem 5.3 and (6.33), we have
\[ |(u^* - u^*_h, v) \rangle| \leq \|u^* - u^*_h\|_{C_0} \|v^i - \tilde{v}_h\|_{C_0} \leq C^{2\ell} \|u^*\|_{1+\ell} \|v^i\|_{1+\ell}, \] (6.35)

but, from (6.24) we have \( u^* = \mathcal{T} u^* = \nu \mathcal{T} (\lambda u^* - v \nu u^*) \) (see also (6.13)), i.e.,
\[ u^* \in (H^{1+\ell}(\Omega))^2, \] (6.36)

and from (6.35), we have
\[ |\langle (\mathcal{T} - \mathbb{T}_{0,h}^{\perp})u^h, v^i \rangle| = |(u^* - u^*_h, v^i)| \leq Ch^{2\ell} \|u^*\|_{1+\ell} \|v^i\|_{1+\ell}. \] (6.37)

On the other hand, from Theorem 5.3 and (6.36), we have
\[ \|\langle (\mathcal{T} - \mathbb{T}_{0,h}^{\perp})u^h, v^i \rangle\|_0^2 \leq Ch^{2\ell} \|u^*\|_{1+\ell} \|v^i\|_{1+\ell}. \] (6.38)

Hence, from (6.21) in Theorem 6.2, (6.37) and (6.38), we have obtained the following theorem.

Theorem 6.4. Let \( v^{-1} = \lambda = \omega^2 + 1 \), \( \omega^2 \) denotes the eigenvalue of the Maxwell eigenvalue problem (2.1) and \( \nu^{-1} = \lambda = \omega^2 \), the eigenvalue of the finite element eigenproblem (3.5). Then, under the regularity results (6.34) for the corresponding orthonormal eigenfunctions, the optimum in the error bound relative to the order of approximation holds:
\[ |v - \nu_h| \leq Ch^{2\ell} \sum_{k=1}^{m} \|u^i_k\|_{1+\ell}^2, \]
where \( u^i_k, 1 \leq k \leq m \), are the orthonormal eigenfunctions corresponding to the eigenvalue \( v \).

7. Numerical results

In this section, we use the finite element method (3.5), with \( (P_i)^2 \) elements \( \ell = 2, 3, 4 \), to solve the Maxwell eigenvalue problem (2.1) in L-shaped domain. Because there are singular eigenfunctions as well as smooth eigenfunctions in this eigenproblem, the numerical results suffice to illustrate the proposed finite element method and the theoretical results. For comparisons, we choose the benchmark computed results (https://perso.univ-rennes1.fr/monique.dauge/benchmax.html) as the exact solutions. We use the uniform triangle-element meshes with barycentric refinements.
The regularity for the eigenfunctions of the above five eigenvalues is as follows: for any \( \epsilon > 0 \), the 1st Maxwell eigenfunction has the strong unbounded singularity, belonging to \((H^{2/3-\epsilon}(\Omega))^2\), the 2nd one belongs to \((H^{4/3-\epsilon}(\Omega))^2\), the 3rd and 4th ones are analytic (exact value of the eigenvalue \( \pi^2 = 9.86960440108936 \)), and the 5th one seems belonging to \((H^{4/3-\epsilon}(\Omega))^2\). The computed results are reported in Tables 1–3, corresponding to \((P_2)^2\) elements, \( \ell = 2, 3, 4 \).

From Table 1, we see that corresponding to the five eigenvalues, respectively, the convergence rate is about \( 4/3 \approx 2r \) for \( r = 2/3 - \epsilon, 8/3 \approx 2r \) for \( r = 4/3 - \epsilon, 4 \approx 2\ell \) for \( \ell = 2 \) for analytical eigenfunction and \( 4 \approx 2\ell \) for \( \ell = 2 \) for analytical eigenfunction, \( 8/3 \approx 2r \) for \( r = 4/3 - \epsilon \). These are consistent with the theoretical results.

From Table 2, we see that corresponding to the five eigenvalues, respectively, the convergence rate is about \( 4/3 \approx 2r \) for \( r = 2/3 - \epsilon, 8/3 \approx 2r \) for \( r = 4/3 - \epsilon, 6 \approx 2\ell \) for \( \ell = 3 \) for analytical eigenfunction and \( 6 \approx 2\ell \) for \( \ell = 3 \) for analytical eigenfunction, \( 8/3 \approx 2r \) for \( r = 4/3 - \epsilon \). These are consistent with the theoretical results. For the 3rd and 4th eigenvalues, when \( h = 1/32 \), the computed are already very accurate close to the exact solutions with relative errors about \( 1.0E - 12 \) so that the computed convergence rates for successive meshes do not reflect the predicted.

From Table 3, we see that corresponding to the five eigenvalues, respectively, the convergence rate is about \( 4/3 \approx 2r \) for \( r = 2/3 - \epsilon, 8/3 \approx 2r \) for \( r = 4/3 - \epsilon, 8 \approx 2\ell \) for \( \ell = 4 \) for analytical eigenfunction and \( 8 \approx 2\ell \) for \( \ell = 4 \) for analytical eigenfunction, \( 8/3 \approx 2r \) for \( r = 4/3 - \epsilon \). These are consistent with the theoretical results. For the 3rd and 4th eigenvalues, when \( h = 1/8 \), the computed are already very accurate close to the exact solutions with relative errors about \( 3.0E - 12 \) so that the computed convergence rates for successive meshes do not reflect the predicted.

8. Concluding remarks

In this paper, we have proposed a new Lagrange finite element method for the Maxwell’s equations, where a piecewise constant \( L^2 \) projection is applied to the \( \text{div} \) operators. A family of Lagrange elements of order greater than or equal two
Table 2
Finite element eigenvalues in L-shaped domain using $P_3$ element.

| $\omega^2$ | $1/h$ | $\omega_h^2$ | $|\omega^2 - \omega_h^2|/|\omega^2|$ | Rate |
|------------|-------|--------------|-----------------------------------|------|
| 2          | 1.38301499473508 | 6.2758E−02  | –                                 | –    |
| 4          | 1.43791471088196  | 2.553E−02   | 1.30                              |      |
| 8          | 1.46049840451247  | 1.0249E−02  | 1.32                              |      |
| 16         | 1.46959419404285  | 4.0848E−03  | 1.33                              |      |
| 32         | 1.47322556215085  | 1.6239E−03  | 1.33                              |      |
| 64         | 1.47467018771464  | 6.4491E−04  | 1.33                              |      |
| 3.53403136678 | 2 | 3.52886865764843 | 1.4609E−03 | –                           |
| 4          | 3.53322804446830  | 2.2731E−04  | 2.68                              |      |
| 8          | 3.53390533135496  | 3.5663E−05  | 2.67                              |      |
| 16         | 3.53401154225098  | 5.6096E−06  | 2.67                              |      |
| 32         | 3.53402824587770  | 1.0249E−02  | 1.32                              |      |
| 64         | 3.53403087532610  | 1.3906E−07  | 2.67                              |      |

Table 3
Finite element eigenvalues in L-shaped domain using $P_4$ element.

| $\omega^2$ | $1/h$ | $\omega_h^2$ | $|\omega^2 - \omega_h^2|/|\omega^2|$ | Rate |
|------------|-------|--------------|-----------------------------------|------|
| 2          | 1.42444047269485 | 3.4684E−02  | –                                 | –    |
| 4          | 1.4549826116324  | 1.3987E−02  | 1.31                              |      |
| 8          | 1.46737531773555  | 5.5885E−03  | 1.32                              |      |
| 16         | 1.47234004926175  | 1.7110E−09  | 5.98                              |      |
| 32         | 1.4743179581944  | 1.1748E−10  | 6.00                              |      |
| 64         | 1.47510414223819  | 1.2952E−12  | 6.50                              |      |
| 3.53403136678 | 2 | 3.53246157895511 | 4.4419E−04 | –                           |
| 4          | 3.53738556730148  | 6.9522E−05  | 2.68                              |      |
| 8          | 3.53390274140685  | 1.0030E−05  | 2.67                              |      |
| 16         | 3.53402582782150  | 1.7203E−06  | 2.67                              |      |
| 32         | 3.53403040951117  | 2.7087E−07  | 2.67                              |      |
| 64         | 3.53403121603966  | 4.2654E−08  | 2.67                              |      |
| 11.38947939790 | 2 | 11.382504048089 | 6.124E−04 | –                           |
| 4          | 11.3833429972479  | 9.9776E−05  | 2.62                              |      |
| 8          | 11.3892996908835  | 1.5778E−05  | 2.66                              |      |
| 16         | 11.38964040200891  | 2.4893E−06  | 2.66                              |      |
| 32         | 11.3896404019635  | 6.4308E−13  | 7.18                              |      |
| 64         | 11.38964040107525  | 1.4948E−12  | – 1.22                           |      |
on barycentric refinements are shown to be optimally convergent, for singular solutions as well as smooth solutions. The finite element method is applicable to the indefinite Maxwell’s equations and the eigenvalue problem. When applied to the Maxwell eigenvalue problem, the spectrally correct property of the proposed finite element method is also shown. The numerical results for the Maxwell eigenvalue problem with singular and smooth eigenfunctions in L-shaped domain have confirmed the theoretical results.

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