A Postprocessed Flux Conserving Finite Element Solution

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We propose a local postprocessing method to get a new finite element solution whose flux is conservative element-wise. First, we use the so-called polynomial preserving recovery (postprocessing) technique to obtain a higher order flux which is continuous across the element boundary. Then, we use special bubble functions, which have a nonzero flux only on one face-edge or face-triangle of each element, to correct the finite element solution element by element, guided by the above super-convergent flux and the element mass. The new finite element solution preserves mass element-wise and retains the quasioptimality in approximation. The method produces a conservative flux, of high-order accuracy, satisfying the constitutive law. Numerical tests in 2D and 3D are presented. © 2017 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 33: 1859–1883, 2017

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I. INTRODUCTION

We solve the Darcy’s equation,

\[
\begin{align*}
-\nabla \cdot (a(x) \nabla p) &= f & \text{in } \Omega, \\
p &= 0 & \text{in } \partial \Omega,
\end{align*}
\]

(1.1)
where $\Omega$ is bounded polygonal domain in 2D or 3D and $a(x) \geq a_0 > 0$ on $\Omega$. Here $p$ is the pressure of fluid and we let the velocity be

$$ u = -a(x) \nabla p. \quad (1.2) $$

The flux of an element-wise mass-conserving solution $u$ of (1.1) satisfies

$$ \int_{\partial K} u \cdot n ds = \int_K f dx, \quad (1.3) $$

where $K$ is a triangle in 2D or a tetrahedron in 3D.

In the mixed finite element method, one seeks a pair of solutions $p_h$ and $u_h$ simultaneously [1, 2], where the flux is conservative element-wise (1.3). But the system of linear equations is not symmetric positive definite. Besides, the constitutive law (1.2) is satisfied weakly, in a lower order of accuracy. The finite volume method also produces a conservative flux solution, but on each dual cell, or control volume, cf. [3–23]. But the finite volume equations are not symmetric either [24].

In addition, due to a loss of flux inside each dual cell, the constitutive law (1.2) holds also weakly and in a low order of accuracy for the finite volume solution. That is, the divergence-free part of the finite volume velocity is of a lower order of accuracy. Conversely, symmetric, high-order finite element equations are naturally defined as an orthogonal projection in a finite dimensional Hilbert space, [25]. Such element equations can be solved effectively by the multigrid method or other fast solvers [25–27]. The goal of this work is to construct a flux-preserving finite element solution which satisfies the constitutive law (1.2) strongly and produces a flux of a high order accuracy.

The idea is to do a post processing on the finite element solution $p_h$, element by element, to get a new solution $\tilde{p}_h$ so that the new flux is conservative element-wise. That is, the solution $\tilde{p}_h$ satisfies the element flux-conservation condition (1.3) with the following consistent condition, and a no-loss of flux condition inside and outside an element,

$$ u_h = -a(x) \nabla \tilde{p}_h, \quad \int_E [u_h \cdot n] ds = 0, \quad (1.4) $$

where $E$ is an edge in 2D or a face triangle in 3D, and $[\cdot]$ denotes the jump of the quantity. To this end, we first use the polynomial preserving recovery technique [28–32] to obtain a superconvergent gradient from the finite element solution $p_h$. We correct the finite element solution $p_h$ by local bubble functions where processed edge/face bubbles are determined by the neighboring flux (1.4), free edge/face bubbles are determined by the super-convergent flux and the last edge/face bubble is determined by the conservation condition (1.3). As a possible accumulation of flux error, the gradient of new solution $\nabla \tilde{p}_h$ may not be superconvergent any more. But numerically, we see $\nabla \tilde{p}_h$ is still superconvergent. This is observed by Carey [33, 34], a super-convergent flux approximation in a large subdomain. But there is no satisfactory proof yet, by our understanding. Nevertheless, we obtain a finite element solution $\tilde{p}_h$ which remains the optimal order approximation in $L^2(\Omega)$, $H^1(\Omega)$ and flux-norms, and satisfying (1.3) and (1.4).

When we use high-order bubble functions to do the correction, we can get a high-order flux conservation, instead of (1.3) and (1.4), for any polynomial $q_k \in P_{k-1}$.

$$ \int_{\partial K} u_h \cdot n q_{k-1} ds = \int_K (f q_{k-1} - u_h \cdot \nabla q_{k-1}) dx, \quad (1.5) $$

and

$$ \int_E [u_h \cdot n] q_{k-1} ds = 0. \quad (1.6) $$

By (1.5) and (1.6), in addition to local mass conservation, we have also a local conservation of first moments and second moments of mass. In physics, they mean local momentum conservation and local energy conservation. None of these can be achieved by the finite volume method, or the mixed finite element (due to the tangential discontinuity), or the discontinuous Galerkin method (directly). The recovered flux \( \mathbf{u}_h \cdot \mathbf{n} = -a(x) \nabla \tilde{p}_h \cdot \mathbf{n} \) is not strictly continuous across interelement boundary, but weakly. So, one can project it onto a \( P_{k-1} \) flux, \( \tilde{\mathbf{u}}_{n,b} \in P_{k-1}(E) \), on each interelement face, by (1.6),

\[
\int_E \tilde{\mathbf{u}}_{n,b} q_{k-1} ds = \int_{E^+} \mathbf{u}_h \cdot \mathbf{n} q_{k-1} ds = \int_{E^-} \mathbf{u}_h \cdot \mathbf{n} q_{k-1} ds \quad \forall q_{k-1} \in P_{k-1}(E).
\]

We present an algorithm for the high-order flux correction (1.5)–(1.6). But we limit our analysis and numerical examples to the zero-th order flux conservation (1.4).

There is a long history of recovering flux from finite element solutions, starting from 1974 by Douglas, Dupont and Wheeler [35]. A detailed review can be found in [2]. In [36], each triangle is subdivided into six to introduce additional degrees of freedom for a flux-recovery locally in \( H(\nabla \cdot) \)-space, for the 2D \( P_1 \) finite element method. Nevertheless, the flux can be lost between macro elements. As our correction of finite element solution is guided by super-convergent point flux values, the idea is used recently by Pouliot et al. [37] who solve a global \( L^2 \) system to recover a high-order convergent flux from supconvergent point values.

Cockburn et al. recover a mass-conserving flux from the finite element solution in two steps, solving a global mass-like system and correcting the flux element by element, cf. [2]. Becker et al. combine the finite element equations with a penalized mixed finite element equations to produce a finite element solution and a mass-conserving flux (the Lagrange multiplier) [38].

Chippada et al. recover a flux-conserving finite element velocity from the flux in the mixed finite element method [39]. Our method recovers a flux-conserving finite element velocity too, by correcting a finite element solution. The difference is that our recovered velocity is consistent with the pressure, that is, they satisfy the Darcy’s law.

On 2D rectangular grids, Chou et al. recovers a conserving Raviart-Thomas mixed element flux locally from the \( C^0 - Q_1 \) finite element [40]. It derives a mixed finite element solution from solving a finite element equation. But it is not known how to recover a consistent (satisfying constitutive law) pressure from the recovered flux and the previous pressure of the finite element solution.

Sun and Liu enrich the usual finite element space by element-wise constants only [41] to get an element-wise flux-conserving finite element solution. It is a special type of discontinuous Galerkin method. Bastian and Riviere proved the flux-conservation property of discontinuous Galerkin method in general [42]. Similar to Sun and Liu [41], Larson [43] solves a global mass system to correct the flux jump of the finite element solution, to get a conservative flux. This leads back to the above work of Cockburn et al. [2]. Some natural conservation property is studied by [44]. Recently Bush et al. proposed another flux recovering method [42, 46]. By postprocessing finite element solutions to \textit{discontinuous} pressures (loss of mass inside an element), they recover a consistent flux, but the consistence holds only at the grid lines of the dual grid.

Our method differs from all the rest in maintaining the consistence condition (1.4). We correct the finite element pressure directly to get a preserving flux \( \mathbf{u}_h \cdot \mathbf{n} \) of high-order accuracy, and a consistent velocity \(-a(x) \nabla \tilde{p}_h = \mathbf{u}_h\). None of the recovered conserving flux above can be used here to correct the finite element solution, satisfying the constitutive law (1.4) and retaining the high order accuracy.

The rest of manuscript is organized as follows. In Section II, we introduce the finite element method and analyze its flux approximation. In Section III, we define a postprocessing algorithm.
We will show the flux-conserving property of the processed solution and its optimal order of convergence. In Section IV, we provide numerical examples.

II. APPROXIMATE LOCAL CONSERVATION PROPERTIES OF POLYNOMIAL PRESERVING RECOVERED GRADIENT

We solve the Darcy’s equation (1.1) by the finite element method. Here, we choose a homogeneous Dirichlet boundary condition for the simplicity in presentation. The analysis is similar for other boundary conditions.

Let $T_h$ be a quasiuniform, and shape-regular simplicial triangulation on the domain $\Omega$:

$$T_h = \{K, \text{ a triangle or tetrahedron} \mid \text{diam}(K) \leq h < 1\}.$$  

With respect to $T_h$, we define an order $k$ finite element subspace

$$V_h := \{v \in C(\overline{\Omega}) : v|_K \in \mathcal{P}_k, \quad \text{for all } K \in T_h, \quad v|_{\partial \Omega} = 0\}.$$ (2.1)

where $\mathcal{P}_k$ is the space of all polynomials of degree equal to or less than $k$. Thus, $V_h \subset H^1_0(\Omega)$. The finite element solution of (1.1) is a function $p_h \in V_h$ such that

$$a(p_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,$$ (2.2)

where the bilinear form $a(\cdot, \cdot)$ is defined by

$$(v, w) = \int_\Omega vw d\mathbf{x}, \quad v, w \in L^2(\Omega).$$

and the inner product $(\cdot, \cdot)$ is defined by

$$a(v, w) = \int_\Omega a(\mathbf{x}) \nabla v \cdot \nabla w d\mathbf{x}, \quad v, w \in H^1_0(\Omega),$$

Let $G_h : V_h \rightarrow V^d_h$ (the vector finite element space of $V_h$, i.e., $V_h \times V_h \times V_h$ in 3D) be a gradient recovery operator of the polynomial preserving recovery technique [28–32]. It satisfies the following super-convergence property.

$$\|G_h p_h - \nabla p\|_{0, \Omega} \lesssim h^{1+k}|p|_{H^{k+2}},$$ (2.3)

and a boundedness property

$$\|G_h p_h\|_{0, \Omega} \lesssim \|\nabla p_h\|_{0, \Omega}.$$ (2.4)

In this article, we adopt this notation $\lesssim$, for “$\leq C$”, where $C$ is a generic, positive constant, independent of the grid size $h$, and some other parameters.

We note that as $G_h p_h \in V^d_h$, $G_h p_h$ is continuous across each interface edge or triangle in $T_h$. However, as the finite element solution, the recovered gradient $G_h p_h$ does not satisfy the following conservation law element-wise

$$\int_K f d\mathbf{x} + \int_{\partial K} a(\mathbf{x}) G_h p_h \cdot \mathbf{n} ds \neq 0 \quad \forall K \in T_h.$$
where \( \mathbf{n} \) is the outward unit normal vector at the boundary \( \partial K \). But the recovered gradient \( G_h p_h \) approximates the true gradient \( \nabla p \) one order higher than that of the finite element gradient \( \nabla p_h \). We will design an element by element correction process on the finite element solution, based on the high accuracy recovered gradient, so that the above element-wise flux-error is zero. We show next that the gradient, recovered from the polynomial preserving recovery technique [28–32], does preserve flux locally in certain degree, one order higher than that of the finite element gradient.

**Theorem 2.1.** Let \( p \in H^{k+2}(\Omega) \) solve (1.1) and \( p_h \) solve (2.2). \( G_h \) is a recovery operator satisfying (2.3) and (2.4). It holds that

\[
\sum_{K \in T_h} \left| \int_K f \, dx + \int_{\partial K} a(x) G_h p_h \cdot \mathbf{n} \, ds \right| \lesssim h^k \| p \|_{H^{k+2}},
\]

(2.5)

where the hidden constant depends only on the shape-regularity of \( T_h \).

**Proof.** For all \( K \in T_h \), by (1.1) and the divergence theorem,

\[
\int_K f \, dx = - \int_{\partial K} a(x) \frac{\partial p}{\partial \mathbf{n}} \, ds.
\]

By (2.2), the finite element solution has a local flux-error,

\[
\int_K f \, dx + \int_{\partial K} a(x) (G_h p_h) \cdot \mathbf{n} \, ds = \int_{\partial K} a(x) (G_h p_h - \nabla p) \cdot \mathbf{n} \, ds.
\]

(2.6)

As \( T_h \) is a shape-regular partition, for all \( K \in T_h \), by the trace theorem and the Schwarz inequality,

\[
\| (G_h p_h - \nabla p) \cdot \mathbf{n} \|_{L^2(\partial K)} \lesssim h_K^{1/2} \| G_h p_h - \nabla p \|_{L^2(K)} + h_K^{1/2} \| G_h p_h - \nabla p \|_{H^{1}(K)},
\]

where \( h_K = \text{diam}(K) \sim |K|^{1/d} \).

Let \( I_h p \) be the Scott-Zhang interpolation [26] in the \( C_0^1 P_{k+1} \) space on the same grid \( T_h \). By the inverse estimate [25, 26] and the regularity of \( p \) [47, Theorem 4.14], as \( K \) is shape-regular, we have

\[
|G_h p_h - \nabla p|_{H^1(K)} \lesssim h_K^{-1} |G_h p_h - \nabla I_h p|_{L^2(K)} + |\nabla I_h p - \nabla p|_{H^1(K)}
\]

\[
\lesssim h_K^{-1} |G_h p_h - \nabla I_h p|_{L^2(K)} + h_K^2 \| p \|_{H^{k+2}(K)},
\]

\[
\lesssim h_K^{-1} |G_h p_h - \nabla p|_{L^2(K)} + |\nabla I_h p - \nabla p|_{L^2(K)} + h_K^2 \| p \|_{H^{k+2}(K)}
\]

\[
\lesssim h_K^{-1} |G_h p_h - \nabla p|_{L^2(K)} + h_K^2 \| p \|_{H^{k+2}(K)}.
\]

Thus, it follows that

\[
|G_h p_h - \nabla p|_{L^2(K)} \lesssim h_K^{-1/2} |\nabla p - G_h p_h|_{L^2(K)} + h_K^{k+1/2} \| p \|_{H^{k+2}(K)}.
\]

Therefore, by (2.6) and the super-convergence property (2.3),

\[
\left| \int_K f \, dx + \int_{\partial K} a(x) G_h p_h \cdot \mathbf{n} \, ds \right| \lesssim h_K^{k+4} \| p \|_{H^{k+2}(K)},
\]

(2.7)
Summing up the above inequalities for all $K \in T$ and using the fact that
\[
\sum_{K \in T_h} h_K^d \| p \|_{H^{k+2}(K)} \lesssim \left( \sum_{K \in T_h} h_K^d \right)^{\frac{1}{2}} \left( \sum_{K \in T_h} \| p \|_{H^{k+2}(K)}^2 \right)^{\frac{1}{2}} \lesssim |\Omega|^{1/2} \| p \|_{H^{k+2}},
\] we obtain the inequality (2.5).

III. ELEMENT BY ELEMENT CONSERVING-FLUX RECOVERY

In this section, we present an algorithm to process the FEM solution $p_h$, guided by $G_h p_h$ so that the new flux is element-wise conserving.

A. One-Sided Bubble Function

For any given triangle or tetrahedron $K \in T_h$, we denote by $\lambda_i = \lambda_{K,i}, i = 1, \ldots, d + 1$ the barycentric coordinates on $K$, that is, $\lambda_i$ is a linear function defined by $\lambda_i(x_j) = \delta_{ij}$ at $(d + 1)$ vertices $\{x_j\}$ of simplex $K$.

For $P_1$ finite elements, we define a $d$-dimensional, one-sided bubble function be
\[
b_{K,i} = s_{K,i} \frac{\lambda_i^2 \cdots \lambda_{d+1}^2}{\lambda_i^2}, \quad i = 1, \ldots, d + 1,
\] where the constant $s_{K,i}$ is defined by (3.2) below. Conventionally, $b_{K,i}$ is defined 0 outside $K$, that is, $b_{K,i}$ is supported on $K$. We note that $b_{K,i}$ has nonzero flux only on one edge/triangle $E_i$ of $K$.

The constant $s_{K,i}$ in $b_{K,i}$ is defined by, via the divergence-theorem,
\[
- \int_K \nabla \cdot (a(x) \nabla b_{K,i}) dx = - \int_{E_i} a(x) \frac{\partial b_{K,i}}{\partial n} ds = 1,
\] noting that $\partial b_{K,i}/\partial n < 0$ on $E_i$, except its boundary vertices/edges. So $s_{K,i} \sim h^{2-d}$. As $T_h$ is shape regular, the scaling argument would give
\[
\|b_{K,i,j}\|_{H^1(K)} \sim h^{-1} \|b_{K,i,j}\|_{L^2(K)} \\
\sim h^{-1} \left( \int_K h^{4-2d} \frac{\lambda_i^4 \cdots \lambda_{d+1}^4}{\lambda_i^2} d\mathbf{x} \right)^{1/2} \\
\sim h^{-1} \left( \int_K h^{4-d} \frac{\lambda_i^4 \cdots \lambda_{d+1}^4}{\lambda_i^2} d\mathbf{x} \right)^{1/2} \\
\sim h^{1-d/2}, \quad \text{in } d - \text{dimension.}
\]

Remark 3.1. In general, for $P_{k-1}$-order moment conservation, we introduce the following bubble functions, for $i = 1, \ldots, d + 1$,
\[
b_{K,i,j} = p_{K,i,j}(\lambda_{i_1}, \ldots, \lambda_{i_d}) \frac{\lambda_{i_1}^2 \cdots \lambda_{i_d+1}^2}{\lambda_i}, \quad p_{K,i,j} \in P_{k-1}^d(K), \quad j = 1, \cdots, \dim(P_{k-1}^{d-1}),
\]
where \( i_1, \ldots, i_{d-1} \neq i \) are the other \((d - 1)\) barycentric coordinate indexes. Note that 
\[ \lambda_1 + \cdots + \lambda_d = 1 - \lambda_i = 1, \] 
on \( E_i \). That is why there are \( \dim P_{d-1}^d(K) \) polynomials in (3.1) but only \( \dim P_{d-1}^{d-1} \) independent bubble functions. Or we can specify \( p_{K,j} \) as homogeneous polynomials of degree \( k - 1 \). On the last face-simplex \( E_i \) of a \( d \)-simplex \( K \), these bubble functions are defined such that

\[
\int_{E_i} p_{K,i,j}(\lambda_{i_1}, \ldots, \lambda_{i_d}) q(\lambda_{i_1}, \ldots, \lambda_{i_d}) [a(x)\lambda_{i_1}^2 \cdots \lambda_{i_d}^2] \, ds
\]

\[
= \int_{E_i} a(x) \nabla(p - \bar{p}_h)q(\lambda_{i_1}, \ldots, \lambda_{i_d}) \, ds,
\]

where \( q(\lambda_{i_1}, \ldots, \lambda_{i_d}) \) are homogeneous polynomials of degree \( k - 1 \) in \( d \)-dimensional space, and \( \bar{p}_h \) is the finite element solution after being corrected on the first \( d \) face edges/triangles.

**B. Element Flux Conservation**

The flux correction, on a \( d \)-dimensional element, is done by the following three steps, using \( d + 1 \) one-sided bubble functions:

\[
\bar{p}_h|_K = p_h + \sum_{i=1}^{d+1} \gamma_{K,i} b_{K,i}, \tag{3.4}
\]

where \( b_{K,i} \) is defined in (3.1) and \( \gamma_{K,i} \) are defined by (3.5-3.7) below. For a face edge/triangle \( E_i \), \( 1 \leq i \leq i_0 \), on the element \( K \) which has been processed on the other side, \( K' \), of element \( K \) and its corrected flux is not the super-convergent \( \bar{G}_h p_h \cdot n \), we preserve the flux on \( K \) side of \( E_i \) too

\[
\gamma_{K,i} = \int_{E_i} a(x) \nabla(\bar{p}_h|_{E_i} - p_h) \cdot n \, ds, \tag{3.5}
\]

where \( \bar{p}_h \) is the processed function defined on the other side of \( E_i, K' \). For previously unprocessed face \( d \)-simplex \( E_i \) or processed face \( E_i \) where the flux on the other side is the super-convergent \( \bar{G}_h p_h \cdot n, i_0 < i \leq d \),

\[
\gamma_{K,i} = \int_{E_i} a(x)(G_h p_h - \nabla p_h) \cdot n \, ds. \tag{3.6}
\]

For the last face edge/triangle, \( E_{d+1} \), we match the total flux on the element \( K \) by letting

\[
\gamma_{K,d+1} = -\int_K f \, ds - \int_{\bigcup_{j=1}^{d+1} E_j} a(x) \frac{\partial \bar{p}_h^+}{\partial n} \, ds
\]

\[
- \int_{\bigcup_{j=1}^{d+1} E_j} a(x)(G_h p_h) \cdot n \, ds + \int_{E_{d+1}} a(x) \frac{\partial p_h}{\partial n} \, ds. \tag{3.7}
\]

**Remark 3.2.** For order \( k - 1 \) moment conservation, using high-order bubbles (3.3), (3.5), (3.6), and (3.7) are replaced by, respectively,

\[
\gamma_{K,i,j} = \int_{E_i} a(x) \nabla(\bar{p}_h^+ - p_h) \cdot n q_j \, ds,
\]
\[
\gamma_{K,i,j'} = \int_{E_i} a(x) (G_h p_h - \nabla p_h) \cdot \mathbf{n} q_{j'} ds,
\]
\[
\gamma_{K,d+1,j'} = -\int_{K} f q_{j'} d\mathbf{x} - \int_{j=1}^{j_0} a(x) \frac{\partial \tilde{p}}{\partial n} q_{j'} ds,
\]
\[
- \int_{j=1}^{j_0} a(x) (G_h p_h) \cdot \mathbf{n} q_{j'} ds + \int_{E_{d+1}} a(x) \frac{\partial p_h}{\partial n} q_{j'} ds.
\]
\[
- \int_{K} a(x) \nabla \tilde{p} \nabla q_{j'} d\mathbf{x}.
\]

Notationally, the corrected finite element solution is
\[
\tilde{p}_h|_K = p_h + \sum_{i=1}^{d} \sum_{j'=1}^{\dim P_{d-1}} \gamma_{K,i,j'} b_{K,i,j'} + \sum_{j'=1}^{\dim P_{d-1}} \gamma_{K,d+1,j'} b_{K,d+1,j'}.
\]

C. A Global Correction Algorithm

Algorithm 3.3. Given the problem (1.1) and a finite element space (2.1).

**Step 1.** Solve the finite element system
\[
a(p_h, v_h) = (f, v_h), \quad \forall v_h \in V_h
\]
to obtain the standard finite element solution \(p_h \in V_h\).

**Step 2.** Recover a \(G_h p_h\) from \(\nabla p_h\) by the polynomial preserving recovery technique [27–32].

**Step 3.** Correct the finite element flux to obtain a conserving flux:

**Step 3.1.** Classify the elements in \(T_h\) to several levels (see Fig. 2 where a triangle marked by a number \(l\) belongs to set \(\mathcal{M}_l\)):

1) Choose an element located roughly at the center of \(\Omega\), denote it as \(K_{1,1}\) and set \(\mathcal{M}_1 = \{K_{1,1}\}\).
2) For \(2 \leq l \leq l_0\), suppose \(\mathcal{M}_{l-1}\) is defined, we set (cf. Fig. 2)
\[
\mathcal{M}_l = \left\{ K_{l,i} \in T_h \setminus \left( \bigcup_{i=1}^{l-1} \mathcal{M}_i \right) \mid \text{meas}_{d-1}(\tilde{K}_{l,i} \cap \bigcup_{K' \in \mathcal{M}_{l-1}} \bar{K}') > 0, \right\}
\]

with two exceptions (on-hold), assuming the domain \(\Omega\) is simply connected for easy stating.

(i) \(K_{l,i}^{*}\) will be dropped from \(\mathcal{M}_l\) if \(\bigcup_{1 \leq j' \leq j} \tilde{K}_{l,j'} \cup K' \in \mathcal{M}_{l-1} \cap \tilde{K}'\) is not simply connected, cf. Fig. 1(A). The drop may not be unique if there are two or more such elements. For example, in Fig. 1(A), \(K_{5,1}\) may be dropped from \(\mathcal{M}_5\) instead of \(K_{5,3}\).

(ii) \(K_{l,i}^{*}\) will be dropped from \(\mathcal{M}_l\) if \(\tilde{K}_{l,i} \cup K' \in \mathcal{M}_{l-1} \cap \tilde{K}'\) is not simply connected, cf. Fig. 1(B).

**Step 3.2.** Process the function \(p_h\) in the order \(K_{l,i} : 1 \leq l \leq l_0, 1 \leq i \leq N_l = \# \mathcal{M}_l\)

1) In the first step, on \(K = K_{1,1}\), by (3.5-3.7), we let the corrected solution be defined by \(p_{h,1} = \tilde{p}_h|_{K_{1,1}}\) in (3.4).
2) For \( l \geq 2 \), we process element \( K_{l, i} \in M_l \), \( i = 1, 2, \ldots, N_l \), sequentially, by (3.5-3.7). Note that the last face edge/triangle \( E_{d+1} \) of \( K_{l, i} \) is not processed before by (3.6), neither (3.7).

With \( K = K_{l, j} \) in (3.5-3.7), we define

\[
    p_{h, i} = \bar{p}_h|_{K_{l, j}} = p_{h, i, j-1} + \sum_{j=1}^{d+1} \gamma_{K_{l, j}, i} b_{K_{l, j}, i}.
\]

Finally, on \( l \)-th level, let \( p_{h, j} = p_{h, j, N_l} \).

Step 3.3. Repeat Step 3.2 until \( M_{l_{0}+1} \) is empty. The local flux preserving finite element solution is defined by

\[
    \bar{p}_h = p_{h, l_{0}} = p_h + \sum_{l=1}^{l_0} \sum_{i=1}^{N_l} \sum_{j=1}^{d+1} \gamma_{K_{l, j}, i} b_{K_{l, j}, i}.
\]

We illustrate the definition of \( M_l \) by a 2D example, shown in Fig. 2. Basically, all the elements are separated into sets of elements such that each element in a set must share at least one edge with some element of a previous level set, and must share at least one edge with some element of a later level set. In this example, all \( M_l \) elements do not share an edge with any other element in \( M_l \). But they do in general.

Lemma 3.4. The Algorithm 3.1 finishes in finite steps. That is, there is an \( l_0 \) such that

\[
    \bigcup_{1 \leq l \leq l_0} \bigcup_{K_{l, i, j} \in M_l} \overline{K_{l, i, j}} = \overline{\Omega}.
\]

Proof. For simplicity, we limit the proof to the case of 2D simply connected domains. We show each set \( M_l \) is nonempty unless the union of them covers \( \Omega \) already. If \( M_{l_{0}+1} = \emptyset \) and (3.10) does not hold, that is,

\[
    S := \bigcup_{1 \leq l \leq l_{0}} \cup_{K_{l, i, j} \in M_l} \overline{K_{l, i, j}} \neq \overline{\Omega},
\]

FIG. 2. Each set $M_l$ contains the triangles marked by $l$, sharing at least one edge with some triangle in $M_{l-1}$ and at least one edge with some triangle in $M_{l+1}$.

FIG. 3. $K_2$ would trap (1 element) less elements than $K_1$ (traps 3 elements).

then there are some elements not in the union. Among these elements, some must share a face or two, or three faces with the elements in $S$. All these element $K$ are dropped from $M_{l_{0+1}}$ (as it is empty) because of Step 3.1.1) (i). That is, the union of any such one $K$ and $S$ is not simply connected.

Let $K_1$ be one of such elements such that the internal region $I$, enclosed by $\overline{K_1}$ and $S$, contains the least number, $n_0$, of elements, see Fig. 3. $n_0 \geq 1$ and $I \neq \emptyset$. So, $I$ has at least one boundary.
edge from $S$ but not of $K_1$. Pick any one element $K_2 \subset I$, which shares one edge (or two) with $S$, cf. Fig. 3. Then $K_2$ is also dropped from $M_{l_0+1}$ in Step 3.1.1) (i). Now the region enclosed by $\bar{K}_2$ and $S$, a sub-region of $I$, contains at most $(n_0 - 1)$ element as $K_2$ is taken off of $I$. The contradiction shows that (3.10) holds when $M_{l_0+1} = \emptyset$.

\textbf{Remark 3.5.} The proof can be easily adapted to the case of nonsimply connected domains in 2D. But the proof would not work for 3D domains as the complements of a nonsimply connected domain may be still connected in 3D. In 3D, we would define a geodesy for each nonsimply connected domain $S \cup \bar{K}_1$, then, we would drop any one element $K_2$ on the geodesy to derive a contradiction.

\textbf{Remark 3.6.} We emphasize the selection of $l$-level elements in $M_l$. These elements of $M_l$ do not form a complete ring. In fact, they form a completely broken ring, that is, they are isolated elements in the sense that their last face edge/triangle is not shared by any elements of this level and earlier levers, $M_l$. Each set $M_l$ contains $\sim 2^d-1\pi r_d-1$ triangles/tetrahedra, that is, a ring of triangles in 2D or a sphere of tetrahedra in 3D, where $r(\sim l)$ is the topological radius (proportional to the number of layers) to the center. This guarantees the length of the path, connecting last face-simplex with the face-simplex(es) of previous last face-simplex, is at most $O(h^{-1})$. That is,

$$l_0 = O(h^{-1}),$$

(3.11)

Every path ends at a boundary face-simplex. So, there are $O(h^{-1})$ paths in 2D, and every element of $T_h$ is crossed by one path. See Fig. 2 for an example of correct numbering, and Fig. 5 for an example of incorrect numbering. Because of Step 3.1.1), the algorithm never produce an incorrect numbering. In general, each path is radial, and more and more new paths start when moving away from the center. Because of the on-hold in Steps 3.1.2.i and 3.1.2.ii, two paths may be combined into one path. For example, on triangle $K_{5,2}^*$ in Fig. 1(B), one path from $K_{5,1}$ and another path from $K_{4,3}$ may be joined. For simplicity and convenience, we assume, for all elements, at Step 3.1.2,

$$i_0 \leq 2.$$  

(3.12)

For a uniform grid shown in Fig. 2, there is none of such two-path merging, that is, $i_0 \leq 1$. But for a bad grid shown in Fig. 4, one path can have eight such mergings (occurs when the path enters a sparse zone from a dense zone). We assume such a merging can happen finite times on each path, independent of $h$, say,

$$\max_{K \cap \Omega \neq \emptyset} \# \{K_{K,i} \mid i_0 = 2 \text{ on } K_{K,i} \} = O(1),$$

(3.13)

where $i_0$ is assumed in (3.12) and $\{K_{K,1}, K_{K,2}, \cdots, K_{K,J_k} = K\}$ are all the elements, sequentially, on one path ending at $K$. The assumption (3.13) does not hold for general quasiuniform grids, for example, for a grid consisting of alternative rings of triangles of size $h$ and size $2h$. But it holds for the family of multilevel grids refined from any initial grid.

\textbf{D. Theory on Local Flux Conservation}

We prove the main theorem.

\textit{Numerical Methods for Partial Differential Equations DOI 10.1002/num}
Without on-hold of Step 3.1.2.i, the paths traveling in the sparse zone will circle around the dense zone before the correction inside is done.

An counter example to Fig. 2. A dot indicates the last face-edge of the element. The path, connecting previous last-face-edge and the last face-edge on this element, goes through all elements, and its length is of $O(h^{-2})$. By the correct numbering in Fig. 2, many new paths start in the middle and each has a length of at most $O(h^{-1})$.

**Theorem 3.7.** The postprocessed finite element solution \( \tilde{\mathbf{p}}_h \) in (3.9) satisfies the local flux-conservation property,

$$
\int_{\partial K} a(\mathbf{x}) \nabla \tilde{\mathbf{p}}_h \cdot \mathbf{n} \, ds = - \int_K f \, d\mathbf{x}
$$

(3.14)

for all \( K \in T_h \). In addition, its flux is weakly continuous such that

$$
\int_E [a(\mathbf{x}) \nabla \tilde{\mathbf{p}}_h \cdot \mathbf{n}] \, ds = 0,
$$

(3.15)

for all internal edge/triangle \( E \) of \( T_h \).

*Numerical Methods for Partial Differential Equations* DOI 10.1002/num
Proof. For the first element $K = K_{1,1}$, by (3.6)–(3.7), we have

\[
\int_{\partial K} a(x) \nabla \bar{p}_h \cdot \mathbf{n} ds = \int_{\partial K} a(x) \nabla p_{h,1} \cdot \mathbf{n} ds \\
= \int_{\partial K} a(x) \nabla p_h \cdot \mathbf{n} ds + \sum_{j=1}^{d+1} \gamma_{K,j} \int_{E_j} a(x) \nabla b_{K,j} \cdot \mathbf{n} ds \\
= \sum_{j=1}^{d+1} \int_{E_j} a(x) \nabla p_h \cdot \mathbf{n} ds + \sum_{j=1}^{d} \int_{E_j} a(x)(G_h p_h - \nabla p_h \cdot \mathbf{n}) ds \\
= - \int_{K} f d\mathbf{x} - \sum_{j=1}^{d} \int_{E_j} a(x)G_h p_h \cdot \mathbf{n} ds - \int_{E_{d+1}} a(x) \nabla p_h \cdot \mathbf{n} ds \\
= - \int_{K} f d\mathbf{x}.
\]

For the rest elements $K_{l,i}$, we prove (3.14) by the mathematical induction. Assume (3.14) holds for all previous elements $K_{l,i'}$, where $l' < l$ or $i' < i$ when $l' = l$. By (3.5–3.7), we have

\[
\int_{\partial K_{l,i}} a(x) \nabla \bar{p}_h \cdot \mathbf{n} ds = \int_{\partial K_{l,i}} a(x) \nabla p_{h,l,i} \cdot \mathbf{n} ds \\
= \int_{\partial K_{l,i}} a(x) \nabla p_h \cdot \mathbf{n} ds + \sum_{j=1}^{d+1} \gamma_{K_{l,i},j} \int_{E_j} a(x) \nabla b_{K_{l,i},j} \cdot \mathbf{n} ds \\
= \sum_{j=1}^{d+1} \int_{E_j} a(x) \nabla p_h \cdot \mathbf{n} ds + \sum_{j=1}^{i-1} \int_{E_j} a(x)(G_h p_h - \nabla p_h \cdot \mathbf{n}) ds \\
+ \sum_{j=i}^{d} \int_{E_j} a(x)(G_h p_h - \nabla p_h) \cdot \mathbf{n} ds \\
= - \int_{K_{l,i}} f d\mathbf{x} - \sum_{j=1}^{i-1} \int_{E_j} a(x) \nabla p_{h,j,i-1}^+ \cdot \mathbf{n} ds \\
- \sum_{j=i}^{d} \int_{E_j} a(x)G_h p_h \cdot \mathbf{n} ds - \int_{E_{d+1}} a(x) \nabla p_h \cdot \mathbf{n} ds \\
= - \int_{K_{l,i}} f d\mathbf{x}.
\]

We prove (3.15) next. For each internal edge/triangle $E$, there are two elements, $K_{l,i}$ and $K_{l',i'}$, where $l' < l$ or $i' < i$ when $l' = l$, sharing this edge/triangle $E$. Let $E$ be the first face edge/triangle of $K_{l,i}$. By (3.5) and (3.8–3.9), it follows that

\[
\int_{E} [a(x) \nabla \bar{p}_h \cdot \mathbf{n}] ds = \int_{E} a(x) \nabla (p_{h,l,i} - p_{h,l',i'}) \cdot \mathbf{n} ds
\]

\[ E = \int_E a(x) \nabla (p_{h,j,i-1} + \gamma_{K_{j,i-1}} b_{K_{j,i-1}} - p_{h,j,i-1}) \cdot \mathbf{n} ds \]

\[ = \int_E a(x) \nabla (p_{h,j,i-1} - p_h) \cdot \mathbf{n} ds = 0. \]

E. Convergence Theory

In addition to having a local flux conservation, the new finite element solution remains an optimal order solution, in approximating the true solution. The processed finite element solution \( \tilde{p}_h \) should be better than the solution \( p_h \) in approximation in \( H^1 \)-norm (accurately, in the energy norm). This can be seen from numerical tests. But in the proof, we estimate it via that of \( p_h \) error and get a large but same order bound for it.

**Theorem 3.8.** The postprocessed finite element solution \( \tilde{p}_h \) in (3.9) remains an optimal order solution to (1.1):

\[ \| p - \tilde{p}_h \|_{0,\Omega} + h \| p - \tilde{p}_h \|_{1,\Omega} \lesssim h^{k+1} \| p \|_{W^{k+2}_\infty(\Omega)}. \] (3.16)

**Proof.** As (3.16) holds if \( \tilde{p}_h \) is replaced by \( p_h \), by (3.9), we only need to estimate the bubble function on each element, \( \sum_{j=1}^{d+1} \gamma_{K_{j,i}} b_{K_{j,i}} \) for all \( 1 \leq l \leq l_0, 1 \leq i \leq N_l \). But each correcting bubble function can be split into two, one matching the super-convergent flux and the other correcting the old flux, for example, in (3.6),

\[ \gamma_{K_{1,1}} = \int_{E_j} a(x) (G_h p_h - \nabla p) \cdot \mathbf{n} ds + \int_{E_j} a(x) (\nabla p - \nabla p_h) \cdot \mathbf{n} ds. \]

The first correction term is to be matched next element and to be accumulated down the path of correction. Fortunately, this term is of a higher order. The second correction term is local, limited to one element and not to be passed down to the next element. The following two \( W^1_\infty \) estimates (cf., [48, 49], [50, Theorem 5.5.2], [30, Theorem 3.1]) will be used in the rest of our proof:

\[ \| \nabla (p - p_h) \|_{L^\infty} \lesssim h^k \| p \|_{W^{k+1}_\infty}, \quad \| \nabla p - G_h p_h \|_{L^\infty} \lesssim h^{k+1} \| p \|_{W^{k+2}_\infty}. \] (3.17)

The recovered super-convergence is originally proved for on an interior domain. We assume it near the boundary.

We estimate the new error on the first element \( K_{1,1} \). By (3.6) and (3.7), we have, on \( K_{1,1} \),

\[ \tilde{p}_h = p_h + \sum_{j=1}^{d+1} \gamma_{K_{1,1,j}} b_{K_{1,1,j}} \]

\[ = p_h + \sum_{j=1}^{d} (\alpha_{K_{1,1,j}} + \beta_{K_{1,1,j}}) b_{K_{1,1,j}} + (\alpha_{K_{1,1,d+1}} + c_{K_{1,1}}) b_{K_{1,1,d+1}}, \]

where

\[ \alpha_{K_{1,1,j}} = \int_{E_j} a(x) (\nabla p - \nabla p_h) \cdot \mathbf{n} ds, \quad j = 1, \ldots, d+1, \] (3.18)
\[
\begin{align*}
\beta_{K,1,j} &= -\int_{E_j} a(x)(\nabla p - G_hp_h) \cdot \mathbf{n} \, ds, \quad j = 1, \ldots, d, \\
c_{K,1} &= -\sum_{j=1}^{d} \beta_{K,1,j} = \int_{\partial K} a(x)(\nabla p - G_hp_h) \cdot \mathbf{n} \, ds.
\end{align*}
\]

They are bounded by, after applying (3.17),
\[
\begin{align*}
|\alpha_{K,1,j}| &\lesssim h^{k+d-1}\|p\|_{W^{k+1}_\infty}, \\
|\beta_{K,1,j}| &\lesssim h^{k+d}\|p\|_{W^{k+1}_\infty}, \\
|c_{K,1}| &\lesssim h^{k+d}\|p\|_{W^{k+2}_\infty}.
\end{align*}
\]

By these bounds, noting \( |b_{K,i,m,j}|^2 \|H_{H^1(K_{K,m})} \lesssim h^{2-d} \)
\[
|p - \tilde{p}_h|^2_{H^1(K_{K,m})} \lesssim |p - p_h|^2_{H^1(K_{K,1})} + \sum_{j=1}^{d} (\alpha_{K,1,j}^2 + \beta_{K,1,j}^2) |b_{K,1,j}|^2_{H^1(K_{K,1})} + (\alpha_{K,1,d+1}^2 + c_{K,1}^2) |b_{K,1,j}|^2_{H^1(K_{K,1})} \lesssim |p - p_h|^2_{H^1(K_{K,1})} + h^{2k+d}\|p\|^2_{W^{k+1}_\infty} + h^{2k+2d}\|p\|^2_{W^{k+2}_\infty}.
\]

Noting \( \|b_{K,i,j}|^2_{L^2(K_{K,1})} \lesssim h|b_{K,i,j}|_{H^1(K_{K,1})} \),
\[
|p - \tilde{p}_h|^2_{L^2(K_{K,1})} \lesssim |p - p_h|^2_{L^2(K_{K,1})} + h^{2k+2d}\|p\|^2_{W^{k+1}_\infty}.
\]

We follow the correction path, that is, the next element sharing the last face-simplex \( E_{d+1} \) of \( K_{1,1} \), to bound the correction bubbles there element by element. Every such a path ends at a boundary face-simplex \( E_{d+1} \) of an element \( K, E_{d+1} \subset \partial \Omega \). We number the elements on a path as in (3.13). All paths ending at \( K \) form a graph tree with its root at the element at the boundary of \( \Omega \). That is, two paths may merge into one. We give a third index to number the path number of paths ending at \( K \):
\[
K_{K,1,1}, \ldots, K_{K,j,j_{K,1}} = K, \\
K_{K,1,2}, \ldots, K_{K,j,j_{K,2}} = K, \\
\vdots = \vdots \\
K_{K,1,j_{K}, \ldots, K_{K,j,j_{K,1} - \ldots, K_{K,j,j_{K}} = K}.
\]

Every element is on at least one path. On the starting element \( K_{K,1,m} \) of any path ending at \( K \) (3.21), the correction is exactly the same as that on \( K_{1,1} \), noting \( G_hp_h \) is continuous across each face-simplex.

On an element \( K_{K,i,m} \) in the middle of path, we have two situations, \( i_0 = 1 \) and \( i_0 = 2 \), cf. (3.12). The correction on the first face-simplex, by (3.5), is
\[
\gamma_{K_{K,i,m}} = \int_{E_{i_1}} a(x)(\nabla \tilde{p}_h - \nabla p) \cdot \mathbf{n} \, ds + \int_{E_{i_1}} a(x)(\nabla p - \nabla p_h) \cdot \mathbf{n} \, ds = -c_{K_{K,j-1,m}} + \alpha_{K_{K,i,m}},
\]
where \( c_{K_{i-1,m}} \) is defined on the last face of previous element similar to (3.20) and \( \alpha_{K_{i,m},j} \) is the local flux correction as in (3.18).

If \( i_0 = 2 \), the neighboring element sharing the face-simplex \( E_2 \) with \( K_{i,m} \) has \( E_2 \) as its last face-simplex. That is, \( K_{i',m'} = K_{i,m} \) for some \( i' \)-th element on another path merged this path at \( K_{i,m} \), cf. (3.21). Then,

\[
\gamma_{K_{i,m}} = \int_{E_2} a(x)(\nabla \tilde{p}_h - \nabla p) \cdot \mathbf{n} ds + \int_{E_2} a(x)(\nabla p - \nabla p_h) \cdot \mathbf{n} ds
\]

\[
= -c_{K_{i',m'}} + \alpha_{K_{i,m}}(x)
\]

where \( c_{K_{i',m'}} \) is defined on the last face of previous element (3.20). On the rest face-simplexes, but not the last one, of \( K_{i,m} \), by (3.6),

\[
\gamma_{K_{i,m}} = \int_{E_j} a(x)(G_h p_h - \nabla p) \cdot \mathbf{n} ds + \int_{E_j} a(x)(\nabla p - \nabla p_h) \cdot \mathbf{n} ds
\]

\[
= \beta_{K_{i,m}} + \alpha_{K_{i,m}}(x)
\]

Together, the error on \( K_{i,m} \) is bounded by, in the case \( i_0 = 2 \), noting \( |b_{K_{i,m}}|^2_{H^1(K_{i,m})} \lesssim h^{2-d} \),

\[
|p - \tilde{p}_h|^2_{H^1(K_{i,m})} \lesssim |p - p_h|^2_{H^1(K_{i,m})} + \sum_{j=0}^{d+1} \alpha^2_{K_{i,m}} h^{2-d} + (c^2_{K_{i-1,m}} + c^2_{K_{i',m'}}) h^{2-d} + \sum_{j=i_0+1}^{d} \beta^2_{K_{i,m}} h^{2-d}
\]

Inductively, by (3.20),

\[
c_{K_{i-1,m}} = -\sum_{j=1}^{i-1} \sum_{\text{unprocessed } E_j} \beta_{K_{i,j'}}
\]

There should be some cancellation, but we do not know theoretically. So, we assume the worst situation that all \( \beta \)'s are of same sign. Therefore,

\[
|p - \tilde{p}_h|^2_{H^1(K_{i,m})} \lesssim |p - p_h|^2_{H^1(K_{i,m})} + h^{2k+1} \|p\|^2_{W^{k+1}_\infty} + (i^2 + i)^2 h^{2k+2+d} \|p\|^2_{W^{k+2}_\infty}.
\]

Continue on the path \( \{K_{1,m}, \ldots, K_{i,m} \} \) to the last element \( K \),

\[
|p - \tilde{p}_h|^2_{H^1(K)} \lesssim |p - p_h|^2_{H^1(K)} + \sum_{j=1}^{d+1} \alpha^2_{K_{j}} h^{2-d} + \sum_{m=1}^{j} \left( c^2_{K_{m,m}} + \sum_{j=i_0+1}^{d} \beta^2_{K_{j,m}} \right) h^{2-d},
\]

where there a branching at $K_{i,j,m+1,m}$. To be precise, the total accumulated correction is

$$c_K = - \sum_{E \subset \partial(j_{K,m}^{K_{i,j,m+1,m}} \cap \partial \Omega)} \beta_E,$$

that is, all $\beta$’s on the boundary simplex of the graph tree rooted at $K$. By (3.11) and (3.13),

$$i_{K,m} \lesssim h^{-1}, \quad j_K \lesssim 1.$$

So, on the last $K = K_{i,j,m}$ or any other $K$,

$$|p - \tilde{p}_h|_{H^1(K)}^2 \lesssim |p - p_h|_{H^1(K)}^2 + h^{2k+d} \|p\|_{W^{k+1}}^2 + (h^{-1})^2 h^{2k+2+d} \|p\|_{W^{k+2}}^2.$$

Summing over all elements,

$$|p - \tilde{p}_h|_{H^1(\Omega)}^2 \lesssim |p - p_h|_{H^1(\Omega)}^2 + h^{2k} \|p\|_{W^{k+1}}^2.$$

Because bubbles have a scaling factor $h$ in $|\cdot|_{H^1}$,

$$\|p - \tilde{p}_h\|_{L^2(\Omega)}^2 \lesssim \|p - p_h\|_{L^2(\Omega)}^2 + h^2 |p_h - \tilde{p}_h|_{H^1}^2.$$

The inequality (3.16) is proved.

**IV. NUMERICAL EXPERIMENTS**

We report the results of four numerical experiments, one by 2D $P_1$ elements, one by 2D $P_2$ elements, one by 3D $P_2$ elements, and the last one by 3D $P_2$ elements on a nonsimply connected domain.

**A. 2D $P_1$ Element**

We solve a Poisson equation on the unit square $\Omega = (0, 1)^2$,

$$-\Delta p = f \quad \text{in } \Omega,$$

$$p = 0 \quad \text{on } \partial \Omega.$$

The exact solution is chosen as

$$p(x, y) = 64(x - x^2)(y - y^3)(1 - 2x)(1 - 3y^2). \quad (4.1)$$

We intentionally choose a nonsymmetric solution (4.1) to avoid the unusual cancellation in postprocessing.

We apply the $P_1$ ($k = 1$ in (2.1)) and $P_2$ conforming finite element method to solve the problem. The first level finite element grid is the unit square cut by a 45° line, $y = x$. Then, each triangle is refined in to four, to form the next level grid. The level four grid is shown in Fig. 2.

In Table I, we list the error of finite element solutions in $L^2$ and in semi-$H^1$ norm, and the order of convergence. This result is standard. Next, we define a norm, cf. (2.5), to measure the flux conservation of the finite element solution:

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TABLE I. The error and order of convergence of $P_1$ finite element solutions $p_h$ for (4.1).

| Grid | $\|p - p_h\|_{L^2}$ | $h^p$ | $|p - p_h|_{H^1}$ | $h^p$ |
|------|-------------------|------|-----------------|------|
| 2    | 0.4737            |      | 4.6408          |      |
| 3    | 0.1764            | 1.4  | 4.0720          | 0.2  |
| 4    | 0.0558            | 1.7  | 2.2808          | 0.8  |
| 5    | 0.0152            | 1.9  | 1.1719          | 1.0  |
| 6    | 0.0039            | 2.0  | 0.5900          | 1.0  |
| 7    | 0.0010            | 2.0  | 0.2955          | 1.0  |
| 8    | 0.0002            | 2.0  | 0.1478          | 1.0  |
| 9    | 0.0001            | 2.0  | 0.0739          | 1.0  |

TABLE II. The local flux conservation error, defined in (4.2), for $P_1$ element solutions.

<table>
<thead>
<tr>
<th>Grid</th>
<th>$F(\nabla p_h)$</th>
<th>$h^p$</th>
<th>$F(G_h p_h)$</th>
<th>$h^p$</th>
<th>$F(\nabla \tilde{p}_h)$</th>
<th>$h^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>48.9607</td>
<td>–</td>
<td>54.70306</td>
<td>–</td>
<td>0.00000</td>
<td>–</td>
</tr>
<tr>
<td>3</td>
<td>50.9868</td>
<td>–</td>
<td>43.02998</td>
<td>0.3</td>
<td>0.00000</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>37.2413</td>
<td>0.4</td>
<td>19.70520</td>
<td>1.1</td>
<td>0.00000</td>
<td>–</td>
</tr>
<tr>
<td>5</td>
<td>29.8715</td>
<td>0.3</td>
<td>8.22280</td>
<td>1.3</td>
<td>0.00000</td>
<td>–</td>
</tr>
<tr>
<td>6</td>
<td>25.6222</td>
<td>0.2</td>
<td>3.73026</td>
<td>1.1</td>
<td>0.00000</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>22.8694</td>
<td>0.2</td>
<td>1.80782</td>
<td>1.0</td>
<td>0.00000</td>
<td>–</td>
</tr>
<tr>
<td>8</td>
<td>21.2707</td>
<td>0.1</td>
<td>0.89599</td>
<td>1.0</td>
<td>0.00000</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>20.4017</td>
<td>0.1</td>
<td>0.44692</td>
<td>1.0</td>
<td>0.00000</td>
<td>–</td>
</tr>
</tbody>
</table>

\[
F(\nabla p_h) = \sum_{K \in T_h} \left( \int_K f \, dx + \int_{\partial K} a(x) \nabla p_h \cdot n \, ds \right). \tag{4.2}
\]

In Table II, in the second and third columns, we can see that the $P_1$ finite element solution almost does not have any local flux conservation. By the polynomial preserving recovery technique [28–32], we have a recovered $P_2$ gradient $G_h p_h$ from the finite element solution $p_h$. In the fourth and fifth columns of Table II, we can find the local flux conservation error and the order of convergence. The result confirms Theorem 2.1. The last two columns of data in Table II verify Theorem 3.8 that the postprocessed finite element solution $\tilde{p}_h$ is locally flux-conservative. Here the error is purely the computer round-off error.

In Table III, we compute the error and the order of convergence of the postprocessed finite element solution $\tilde{p}_h$, defined in (3.9). The error is of optimal order in $L^2$ and in $H^1$ norm, confirming Theorem 3.8. Comparing the errors of $p_h$ in Table I, the $L^2$ error of $\tilde{p}_h$ is slightly larger, but the $H^1$ error is even smaller. Here, $\tilde{p}_h$ is in a larger space, but it is no longer the $H^1$-orthogonal projection. To see which direction the correction is, in Table III, we also list the error and the order of convergence of the bubble correction function:

\[
b_h = \tilde{p}_h - p_h = \sum_{l=1}^{l_0} \sum_{k=1}^{N_l} \sum_{j=1}^{d+1} \lambda_{K_l,k,j} b_{K_l,k,j}.
\]

Finally, we compute the true flux error in $L^2$ norm:

\[
\|\partial_n (p - p_h)\|^2_{h,0} = \sum_{K \in T_h} h \int_{\partial K} (\nabla (p - p_h) \cdot n)^2 \, ds, \tag{4.3}
\]
TABLE III. The error and order of convergence of the postprocessed finite element \( P_1 \) solution \( \tilde{p}_h \), defined in (3.9).

<table>
<thead>
<tr>
<th>Grid</th>
<th>( | p - \tilde{p}<em>h |</em>{L^2} )</th>
<th>( h^n )</th>
<th>( | p - \tilde{p}<em>h |</em>{H^1} )</th>
<th>( h^n )</th>
<th>( | b_h |_{L^2} )</th>
<th>( h^n )</th>
<th>( | b_h |_{H^1} )</th>
<th>( h^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.41042</td>
<td>3.07278</td>
<td>0.36236</td>
<td>4.43220</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.24555</td>
<td>0.7</td>
<td>2.86099</td>
<td>0.1</td>
<td>0.23079</td>
<td>0.7</td>
<td>4.02734</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>0.08161</td>
<td>1.6</td>
<td>1.73287</td>
<td>0.7</td>
<td>0.07712</td>
<td>1.6</td>
<td>2.33875</td>
<td>0.8</td>
</tr>
<tr>
<td>5</td>
<td>0.02209</td>
<td>1.9</td>
<td>0.91234</td>
<td>0.9</td>
<td>0.02086</td>
<td>1.9</td>
<td>1.98800</td>
<td>1.0</td>
</tr>
<tr>
<td>6</td>
<td>0.00564</td>
<td>2.0</td>
<td>0.46228</td>
<td>1.0</td>
<td>0.00532</td>
<td>2.0</td>
<td>0.60092</td>
<td>1.0</td>
</tr>
<tr>
<td>7</td>
<td>0.00142</td>
<td>2.0</td>
<td>0.11606</td>
<td>1.0</td>
<td>0.00033</td>
<td>2.0</td>
<td>0.15011</td>
<td>1.0</td>
</tr>
<tr>
<td>8</td>
<td>0.00035</td>
<td>2.0</td>
<td>0.05804</td>
<td>1.0</td>
<td>0.00008</td>
<td>2.0</td>
<td>0.07504</td>
<td>1.0</td>
</tr>
<tr>
<td>9</td>
<td>0.00009</td>
<td>2.0</td>
<td>0.02572</td>
<td>1.0</td>
<td>0.00009</td>
<td>2.0</td>
<td>0.07504</td>
<td>1.0</td>
</tr>
</tbody>
</table>

TABLE IV. The flux error, defined in (4.3-4.5), and the order of convergence, for the \( P_1 \) element.

<table>
<thead>
<tr>
<th>Grid</th>
<th>( | \partial_n (p - p_h) |_{h,0} )</th>
<th>( h^n )</th>
<th>( | \partial_n (p - G_h p_h \cdot n) |_{h,0} )</th>
<th>( h^n )</th>
<th>( | \partial_n (p - \tilde{p}<em>h) |</em>{h,0} )</th>
<th>( h^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>22.8047</td>
<td>24.40281</td>
<td>25.78607</td>
<td>0.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>14.3138</td>
<td>12.66089</td>
<td>20.29171</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8.4626</td>
<td>4.51281</td>
<td>8.34437</td>
<td>1.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4.4699</td>
<td>1.24440</td>
<td>2.65901</td>
<td>1.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2.2683</td>
<td>0.31578</td>
<td>0.79626</td>
<td>1.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.1384</td>
<td>0.07856</td>
<td>0.24310</td>
<td>1.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.5698</td>
<td>0.01951</td>
<td>0.07751</td>
<td>1.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.2849</td>
<td>0.00486</td>
<td>0.02572</td>
<td>1.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\| \partial_n (p - G_h p_h \cdot n) \|_{h,0}^2 = \sum_{K \in T_h} h \int_{\partial K} ((\nabla p - G_h p_h) \cdot n)^2 ds, \quad (4.4)
\]

\[
\| \partial_n (p - \tilde{p}_h) \|_{h,0}^2 = \sum_{K \in T_h} h \int_{\partial K} (\nabla (p - \tilde{p}_h) \cdot n)^2 ds. \quad (4.5)
\]

By the \( H^1 \) convergence of the standard finite element theory, the flux \( L^2 \) error would converge at \( O(h) \), as shown in the second and third columns of Table IV. By the super-convergence of \( G_h p_h \) (pointwise), its flux \( L^2 \) error would converge at \( O(h^2) \). Nevertheless, we proved only an order \( O(h^{3/2}) \) convergence for the \( \tilde{p}_h \) flux \( L^2 \) error in Theorem 3.8, based on a comparison with \( p_h \). With minor changes, a direct proof would show an order \( O(h) \) convergence for the \( \tilde{p}_h \) flux \( L^2 \) error. This numerical test shows an \( O(h^{3/2}) \) convergence for the \( \tilde{p}_h \) flux error in \( L^2 \)-norm.

B. Two Dimensional \( P_2 \) Element

We compute the \( P_2 \) finite element solutions for (4.1). As we have a second order method, the flux error (4.2) converges at order one, shown in left half of Table V. The same is for the \( P_2 \) interpolation. However, due to super-convergence, the recovered flux is of second order accuracy. Then, by our postprocessing method, we produce an exactly element-wise flux-conservation solution \( \tilde{p}_h \).

In Fig. 6, we plot the error function \( p - p_h \) of the \( P_2 \) finite element solution, and the error \( p - \tilde{p}_h \). We can see, the error is reduced somewhat by the bubble correction. We believe we get not only a flux-conservative and constitutive solution, but also a smaller energy solution (as we have a large approximation space.) This can be seen from Table VI, where the errors and the orders of convergence are listed.

TABLE V. The local flux conservation error, defined in (4.2), for the 2D $P_2$ element.

<table>
<thead>
<tr>
<th>$F(\nabla p_h)$</th>
<th>$h^6$</th>
<th>$F(G_h I_h p)$</th>
<th>$h^6$</th>
<th>$F(G_h p_h)$</th>
<th>$h^6$</th>
<th>$F(\nabla \tilde{p}_h)$</th>
<th>$h^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>55.50000</td>
<td>55.50000</td>
<td>13.73571</td>
<td>0.00000</td>
<td>–</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>24.79438</td>
<td>1.2</td>
<td>23.31868</td>
<td>1.3</td>
<td>8.81381</td>
<td>0.6</td>
<td>0.00000</td>
</tr>
<tr>
<td>4</td>
<td>11.81877</td>
<td>1.1</td>
<td>11.21945</td>
<td>1.1</td>
<td>3.81444</td>
<td>1.2</td>
<td>0.00000</td>
</tr>
<tr>
<td>5</td>
<td>5.67735</td>
<td>1.1</td>
<td>5.57246</td>
<td>1.0</td>
<td>1.22247</td>
<td>1.6</td>
<td>0.00000</td>
</tr>
<tr>
<td>6</td>
<td>2.79467</td>
<td>1.0</td>
<td>2.78162</td>
<td>1.0</td>
<td>0.33632</td>
<td>1.9</td>
<td>0.00000</td>
</tr>
<tr>
<td>7</td>
<td>1.39227</td>
<td>1.0</td>
<td>1.39050</td>
<td>1.0</td>
<td>0.08777</td>
<td>1.9</td>
<td>0.00000</td>
</tr>
<tr>
<td>8</td>
<td>0.69544</td>
<td>1.0</td>
<td>0.69520</td>
<td>1.0</td>
<td>0.02240</td>
<td>2.0</td>
<td>0.00000</td>
</tr>
<tr>
<td>9</td>
<td>0.34763</td>
<td>1.0</td>
<td>0.34760</td>
<td>1.0</td>
<td>0.00565</td>
<td>2.0</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

FIG. 6. The error $p - p_h$ of finite element solution (top) and the error $p - \tilde{p}_h$ (bottom). [Color figure can be viewed at wileyonlinelibrary.com]

TABLE VI. The error and the convergence order of 2D $P_2$ element in $H^1$ and $L^2$ norms, and the processed solution.

| $|p - p_h|_1$ | $h^6$ | $\|p - p_h\|_0$ | $h^6$ | $|p - \tilde{p}_h|_1$ | $h^6$ | $\|p - \tilde{p}_h\|_0$ | $h^6$ |
|-------------|------|---------------|------|----------------|------|----------------|------|
| 2           | 5.2221 | 0.4600309     | 3.2535 | 0.3926263 | 2.8 |
| 3           | 1.9149 | 1.4           | 0.0737763 | 1.5319 | 1.1 | 0.0548444 | 2.8 |
| 4           | 0.5417 | 1.8           | 0.0090624 | 3.0 | 0.4307 | 1.8 | 0.0062695 | 3.1 |
| 5           | 0.1405 | 1.9           | 0.0011065 | 3.0 | 0.1147 | 1.9 | 0.0008222 | 3.0 |
| 6           | 0.0355 | 2.0           | 0.0001373 | 3.0 | 0.0293 | 2.0 | 0.0001012 | 3.0 |
| 7           | 0.0089 | 2.0           | 0.0000171 | 3.0 | 0.0074 | 2.0 | 0.0000127 | 3.0 |
| 8           | 0.0022 | 2.0           | 0.0000021 | 3.0 | 0.0018 | 2.0 | 0.0000016 | 3.0 |
| 9           | 0.0006 | 2.0           | 0.0000003 | 3.0 | 0.0005 | 2.0 | 0.0000002 | 3.0 |

C. Three Dimensional $P_2$ Element on a Simply Connected Domain

We solve the 3D Poisson equation on the unit cube $\Omega = (0, 1)^3$ where the exact solution is

$$u(x, y, z) = 2^6 x(1 - x)y(1 - y)z(1 - z).$$

The first three level grids are displayed in Fig. 7.
FIG. 7. The first three levels of grids, for solving (4.6).

### TABLE VII. The $H^1$ error, the $L^2$ error and the flux conservation error (4.2), for the 3D $P_2$ element solving (4.6).

| Grid | $|p - p_h|_{H^1}$ | $h^n$ | $\|p - p_h\|_{L^2}$ | $h^n$ | $F(\nabla p_h)$ | $h^n$ |
|------|------------------|-------|------------------|-------|----------------|-------|
| 1    | 1.3112897        | 0.2016410 | 8.0000000 |
| 2    | 0.6481522        | 0.0522074 | 5.8260364 | 0.5  |
| 3    | 0.1879966        | 0.0062357 | 2.4363239 | 1.3  |
| 4    | 0.0494076        | 0.0007380 | 1.0722703 | 1.2  |
| 5    | 0.0125422        | 0.000906  | 0.5103404 | 1.1  |
| 6    | 0.0031490        | 0.000113  | 0.2515886 | 1.0  |
| 7    | 0.0007881        | 0.000014  | 0.1253071 | 1.0  |

### TABLE VIII. The nodal error of the polynomial preserving recovered gradient ($g = G_h p_h$), for the 3D $P_2$ element solving (4.6).

| Grid | $|\partial_x p - g_1|_{L^\infty}$ | $h^n$ | $|\partial_y p - g_2|_{L^\infty}$ | $h^n$ | $|\partial_z p - g_3|_{L^\infty}$ | $h^n$ |
|------|----------------------------------|-------|----------------------------------|-------|----------------------------------|-------|
| 1    | 4.0000000                        | 4.0000000 | 4.0000000 |
| 2    | 1.2808497                        | 1.2808497 | 1.2808497 | 1.6  |
| 3    | 0.2805798                        | 0.2805798 | 0.2805798 | 2.2  |
| 4    | 0.0630225                        | 0.0642691 | 0.0642691 | 2.1  |
| 5    | 0.0116278                        | 0.0127516 | 0.0127516 | 2.3  |
| 6    | 0.0019752                        | 0.0020319 | 0.0020319 | 2.6  |
| 7    | 0.0002835                        | 0.0002835 | 0.0002835 | 2.8  |

### TABLE IX. The errors of the postprocessed finite element solution $\tilde{p}_h$, for the 3D $P_2$ element solving (4.6).

| Grid | $|I_h p - \tilde{p}_h|_{H^1}$ | $h^n$ | $|I_h p - \tilde{p}_h|_{L^2}$ | $h^n$ | $F(\nabla \tilde{p}_h)$ | $h^n$ |
|------|-------------------------------|-------|-------------------------------|-------|-------------------------|-------|
| 1    | 2.2143482                     | 0.1740610 | 0.0000000                     |
| 2    | 0.7768436                     | 0.0424808 | 0.0000000                      |
| 3    | 0.2132700                     | 0.0047186 | 0.0000000                      |
| 4    | 0.0527997                     | 0.0004686 | 0.0000000                      |
| 5    | 0.0129907                     | 0.0000505 | 0.0000000                      |
| 6    | 0.0032203                     | 0.0000059 | 0.0000000                      |
| 7    | 0.0008022                     | 0.0000007 | 0.0000000                      |

We list the errors and the order of convergence of $P_2$ finite elements, for solving problem (4.6), in Table VII. We have only a first order of convergence for the local flux conservation. We next list the errors of the polynomial preserving recovered gradient $G_h p_h$ in Table VIII.

Guided by the superconvergent gradient $G_h p_h$, we correct the finite element solution $p_h$ to get a flux conserving solution $\tilde{p}_h$. We list the error and the order of convergence for the new solution in Table IX. By Table IX, $\tilde{p}_h$ is flux conservative.

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The first level grid for the domain with a hole, $\Omega = (0, 1)^3 \setminus \left( \frac{1}{2}, \frac{2}{3} \right)^3$.

### Table X. The errors of the 3D $P_2$ element solving (4.7).

| Grid | $|p - \tilde{p}_h|_{H^1}$ | $h^a$ | $\|p - \tilde{p}_h\|_{L^2}$ | $h^a$ | $F(\nabla p_h)$ | $h^a$ |
|------|-----------------|------|-----------------|------|-----------------|------|
| 1    | 0.0060118       | 0.0  | 0.0001925       | 0.0  | 0.0528415       | 0.0  |
| 2    | 0.0065621       | 0.0  | 0.0000565       | 1.8  | 0.0414112       | 0.4  |
| 3    | 0.0065673       | 0.0  | 0.0000073       | 2.9  | 0.0200744       | 1.0  |
| 4    | 0.0065641       | 0.0  | 0.0000009       | 3.1  | 0.0091254       | 1.1  |
| 5    | 0.0065638       | 0.0  | 0.0000001       | 3.0  | 0.0043410       | 1.1  |

### Table XI. The errors of the postprocessed finite element solution $\tilde{p}_h$, for the 3D $P_2$ element solving (4.7).

| Grid | $|I_h p - \tilde{p}_h|_{H^1}$ | $h^a$ | $\|I_h p - \tilde{p}_h\|_{L^2}$ | $h^a$ | $F(\nabla \tilde{p}_h)$ | $h^a$ |
|------|-----------------|------|-----------------|------|-----------------|------|
| 1    | 0.0074200       | 0.0  | 0.0001698       | 0.0  | 0.0000000       | –    |
| 2    | 0.0024722       | 1.6  | 0.0000447       | 1.9  | 0.0000000       | –    |
| 3    | 0.0008855       | 1.5  | 0.0000062       | 2.9  | 0.0000000       | –    |
| 4    | 0.0002423       | 1.9  | 0.0000007       | 3.2  | 0.0000000       | –    |
| 5    | 0.0000614       | 2.0  | 0.0000001       | 3.1  | 0.0000000       | –    |

### D. Three Dimensional $P_2$ Element on a Nonsimply Connected Domain

In the last experiment, we solve the 3D Poisson equation on a nonsimply connected domain, the unit cube with a cube hole, $\Omega = (0, 1)^3 \setminus \left( \frac{1}{2}, \frac{2}{3} \right)^3$. The exact solution satisfies the homogeneous Dirichlet boundary condition,

$$u(x, y, z) = x(1 - 3x)(2 - 3x)(1 - x)y(1 - 3y)(2 - 3y)(1 - y)z(1 - 3z)(2 - 3z)(1 - z).$$  \hspace{1cm} (4.7)

The first level grid is shown in Fig. 8.

We list the errors and the orders of convergence in Table X. We then postprocess the finite element solutions $p_h$ to get the flux conserving finite element solutions $\tilde{p}_h$. The errors are listed in Table XI. We note that the nonsimply connected domain does not cause any problem to our program, although we assumed a simply connected domain in analysis.

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### References


