Coefficient Jump-Independent Approximation of the Conforming and Nonconforming Finite Element Solutions

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Abstract. A counterexample is constructed. It confirms that the error of conforming finite element solution is proportional to the coefficient jump, when solving interface elliptic equations. The Scott-Zhang operator is applied to a nonconforming finite element. It is shown that the nonconforming finite element provides the optimal order approximation in interpolation, in $L^2$-projection, and in solving elliptic differential equation, independent of the coefficient jump in the elliptic differential equation. Numerical tests confirm the theoretical finding.

AMS subject classifications: 65N30, 65D05

Key words: Jump coefficient, finite element, $L^2$ projection, weighted projection, Scott-Zhang operator.

1 Introduction

When studying the finite element solutions of elliptic boundary value problems with discontinuous coefficients, i.e., interface problems, a useful tool is the weighted $L^2$ projection operator, cf. [5, 17, 19]. To be specific, we consider a domain $\Omega$ which is subdivided into finitely many, bounded, polygonal subdomains $\{\Omega_i, \ i = 1, \cdots, J\}$, in $d = 2$ or 3 space-dimension. On each subdomain $\Omega_i$, we are given a positive constant $\omega_i$, and we have a quasi-uniform triangulation $T_h(\Omega_i)$ of size $h$ on $\Omega_i$, cf. [7], shown by Fig. 1 as an example. Thus each $\Omega_i$ and $\Omega$ is Lipschitz. We further assume the subdomain grids are matching at the interface so that we can define conforming and nonconforming linear finite
element spaces on the combined grid $T_h$ over the domain $\Omega$, [7]:

$$V_h = \{ v \in H_0^1(\Omega) \cap C(\Omega) | v|_K \in P_1, \forall K \in T_h \},$$ (1.1a)

$$V_{h,c} = \{ v|_K \in P_1, \forall K \in T_h, v \text{ is continuous at all mid face } \partial K \text{ and 0 at mid face } \partial K \cap \partial \Omega \}. \quad (1.1b)$$

The weighed $L^2$ and semi-$H^1$ inner products are defined by

$$\langle u, v \rangle_{L^2(\Omega)} = \sum_{i=1}^I \omega_i \int_{\Omega_i} uv dx,$$ (1.2a)

$$\langle u, v \rangle_{H^1(\Omega)} = \sum_{i=1}^I \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v dx.$$ (1.2b)

The induced norms are denoted by $\| \cdot \|_{L^2(\Omega)}$ and $| \cdot |_{H^1(\Omega)}$, respectively. The full $H^1$ weighted norm is $\| \cdot \|_{H^1_w} = | \cdot |_{H^1_w} + \| \cdot \|_{L^2(\Omega)}$. The weighted $L^2$ projection $Q^w_h : L^2(\Omega) \mapsto V_h$ is defined by

$$\langle Q^w_h u, v \rangle_{L^2(\Omega)} = \langle u, v \rangle_{L^2(\Omega)}, \quad \forall v \in V_h.$$ (1.3)

The following important theorem is proved by Bramble and Xu in 1991.

**Theorem 1.1** (see [5]). If for all $i$, the $(d-1)$-dimensional Lebesgue measure of $\partial \Omega_i \cap \partial \Omega$ is positive, then for all $u \in H_0^1(\Omega)$,

$$\| u - Q^w_h u \|_{L^2(\Omega)} + h |Q^w_h u|_{H^1(\Omega)} \leq C h |\log h|^{1/2} |u|_{H^1(\Omega)},$$ (1.4)

where $C$ is independent of $\{ \omega_i \}$.

Trying to show the necessity that all subdomains have a part of boundary $\partial \Omega$, and of the log term in the bound, several examples are constructed by Xu in [18]. However, these examples are constructed by some limit argument where no specific function $u$ can be used in computation to show the sharpness of (1.4). Thus, some people are still...
skeptic about the sharpness and the condition for (1.4). In this work, we construct a counterexample, where \( \partial \Omega \cap \partial \bar{\Omega} \) is a set of isolated points, i.e., has a Lebesgue measure zero, violating the condition in Theorem 1.1. Basically, we construct a discontinuous, but \( H^1 \) function. With this example we show that Bramble-Xu’s Theorem 1.1 can never have the constant \( C \) in the bound independent of coefficient jump, if the boundary assumption is taken off. In other words, by this example, we show the following standard error bound is sharp:

\[
\| u - Q^w_h u \|_{L^2(\Omega)} + h |Q^w_h u|_{H^1(\Omega)} \leq Ch \frac{\max \{ \sqrt{\omega_i} \}}{\min \{ \sqrt{\omega_i} \}} \| u \|_{H^1(\Omega)}.
\]

By this explicit counterexample, we put a solid period to the problem on weighted projection of conforming finite elements. This example points out that it is inappropriate to use conforming finite element methods for large jump-coefficient problems. What about the nonconforming finite element? Yes. With the help of Scott-Zhang interpolation operator, we show the following theorem that the approximation of weighted projection is independent of weights for nonconforming finite elements. In addition, we show that the nonconforming finite element solves the jump-coefficient elliptic problems optimally, independent of the jump.

**Theorem 1.2.** For all \( u \in H^1_0(\Omega) \),

\[
\| u - Q^w_{h,nc} u \|_{L^2(\Omega)} + h |Q^w_{h,nc} u|_{H^1(\Omega)} \leq Ch \| u \|_{H^1(\Omega)},
\]

where \( C \) is independent of \( \{ \omega_i \} \), and the weighted \( L^2 \)-projection operator \( Q^w_{h,nc} \) is defined by

\[
(Q^w_{h,nc} u, v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega)}, \quad \forall v \in V^{nc}_h.
\]

Though we prove the jump-independent convergence for the non-conforming finite element method only, it would be straightforward to prove the same property for many other exotic finite elements, in particular, for the mixed finite elements [8, 14], for the discontinuous Galerkin methods [1], for the discontinuous Petrov-Galerkin methods [11], for the hybridizable discontinuous Galerkin methods [10], for the virtual element methods [4], and for the weak Galerkin methods [13, 16].

The rest manuscript is organized as follows. In Section 2, we define a counterexample showing the weighted \( L^2 \) projection does depend on the jump coefficient. In Section 3, we introduce the Scott-Zhang operator and show its jump-independent approximation. Consequently we prove the main theorem. Numerical results are given in Section 4, supporting the new theory.

## 2 A counterexample

By the standard finite element theory, we show first a weight-dependent approximation of the weighted \( L^2 \) projection. Then we construct a counterexample, showing this standard approximation property cannot be improved, i.e., the approximation property of weighted \( L^2 \) projection in conforming finite element spaces is weight-dependent.
**Theorem 2.1.** For any \( u \in H^1(\Omega) \),

\[
|u - Q_h^\omega u|_{L^2(\Omega)} + h|Q_h^\omega u|_{H^1(\Omega)} \leq Ch \max_i \frac{\sqrt{\omega_i}}{\min \{ \omega_i \}} ||u||_{H^1(\Omega)},
\]

where \( Q_h^\omega \) is the weighted \( L^2 \) projection into the conforming finite element space \( V_h \) (1.1a), defined in (1.3).

**Proof.** Let \( I_h \) be the Scott-Zhang interpolation operator (cf. [15] and next section):

\[
(I_h : H^0_0(\Omega) \cap H^{e+d/2}(\Omega) \rightarrow V_h, \quad I_h u(x_i) = \int_{K_i} \psi_i(x)u(x)d\chi \quad \text{at a vertex } x_i \text{ of } \mathcal{T}_h. \quad (2.2a)
\]

Here \( K_i \) is a \((d-1)\) simplex (an edge in 2D, and a triangle in 3D) with \( x_i \) as one of its vertices. For \( x_i \in \partial \Omega \), \( K_i \) is chosen so that \( K_i \subset \partial \Omega \). The \( \psi_i \) above is the dual basis function on \( K_i \) with respect to the finite element nodal basis \( \phi_i \) at \( x_i \) (restricted on \( K_i \)). By the standard theory [15], without boundary value requirement,

\[
||u - I_h u||_{L^2(\Omega)} \leq Ch \|u\|_{H^1(\Omega)},
\]

\[
|I_h u|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)}.
\]

Thus,

\[
||u - Q_h^\omega u||_{L^2(\Omega)} \leq ||u - I_h u||_{L^2(\Omega)} \leq \max_i \{ \sqrt{\omega_i} \} \|u - I_h u||_{L^2(\Omega)}
\]

\[
\leq C \max_i \{ \sqrt{\omega_i} \} \|u\|_{H^1(\Omega)} \leq Ch \max_i \{ \sqrt{\omega_i} \} \|u\|_{H^1(\Omega)}.
\]

For the weighted semi-\( H^1 \) norm, we need an inverse inequality [7].

\[
|Q_h^\omega u|_{H^1(\Omega)}^2 = \sum_i \omega_i |Q_h^\omega u|_{H^1(\Omega_i)}^2 \leq \sum_i \omega_i 2 \left( |I_h u - Q_h^\omega u|_{H^1(\Omega_i)}^2 + |I_h u|_{H^1(\Omega_i)}^2 \right)
\]

\[
\leq C \sum_i \omega_i \left( h^{-2} \|I_h u - u||_{L^2(\Omega_i)}^2 + \|u - Q_h^\omega u||_{L^2(\Omega_i)}^2 + |u|_{H^1(\Omega_i)}^2 \right)
\]

\[
= C \left( h^{-2} \|I_h u - u||_{L^2(\Omega)}^2 + h^{-2} \|u - Q_h^\omega u||_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2 \right)
\]

\[
\leq C \max_i \{ \omega_i \} \|u\|_{H^1(\Omega)}^2.
\]

Thus, we complete the proof. \( \square \)
We show next a counterexample. We define a "reference" function on the reference tetrahedron \( \hat{K} = \{x+y+z \leq 1|x,y,z \geq 0\} \):

\[
f_1(x,y,z) = \begin{cases} 
0, & \text{if } x = y = z = 0, \\
\frac{z}{x+y+z}, & \text{elsewhere on } \hat{K}.
\end{cases}
\] (2.3)

The function \( f_1(x,y,z) \) is continuous everywhere except at \((0,0,0)\). At point \((0,0,0)\), the directional limit of \( f_1(x,y,z) \) is between 0 and 1, depending on the direction. In particular, \( f_1(x,y,z) = 1 \) on the vertical axes, but it jumps to zero at the origin. Actually, the function \( f_1 \) is very smooth in terms of Sobolev spaces. It is in \( H^{3/2-\epsilon}(\hat{K}) \). It can be shown under the spherical coordinates that \( |f_1|_{H^{3/2}(\hat{K})} = \infty \). For our need, we verify it is in \( H^1(\hat{K}) \), via spherical coordinates:

\[
|f_1|^2_{H^1(\hat{K})} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{1/(\cos \theta + \sin \theta (\cos a + \sin a))} |\nabla f_1|^2 p^2 \sin \theta \rho d\rho d\theta d\alpha = \frac{5}{12}.
\] (2.4)

Next, we glue two \( f_1 \) functions and a constant function on \( 1/6 \) of unit cube, i.e., on a tetrahedron \( K_{0126} = \{x \geq 0, z \geq x, x \geq y, z \leq 1\} \). \( K_{0126} \) is shown in Fig. 2 as tetrahedron 0126. For convenience, we let \( F_{ijkl} \) be the affine mapping (standard finite element reference mapping) from \( K \) to \( K_{ijkl} \): \( F(0,0,0) = x_i, F(1,0,0) = x_j, F(0,1,0) = x_k, F(0,0,1) = x_l \). Note that \( f_1 \) has a constant value 0 on one face triangle \((z = 0)\) while having a value 1 on one edge \((x = 0, y = 0)\), on the reference tetrahedron \( \hat{K} \). Here we need one such function, but also another function which has a value 1 on one face triangle but 0 on an edge, opposite to \( f_1 \). We define \( f_2 \) on \( K_{0126} \), cf. Fig. 2,

\[
f_2(x,y,z) = \begin{cases} 
1, & \text{on } K_{0135}, \\
1 - f_1(F_{1356}^{-1}(x,y,z)), & \text{on } K_{1356}, \\
f_1(F_{1623}^{-1}(x,y,z)), & \text{on } K_{1623}.
\end{cases}
\] (2.5)
Finally, we define $u$ shown in Fig. 2:

Thus, $1 - [f_2|_{K_{1623}}]_{x_1 x_2 x_3}$ would have a constant value too, $|x_3 x_6|/|x_3 x_6|$, on each radial line connecting $x_1$ and $x$ for some $x$ on the line segment $x_3 x_6$. This is the key point of the counterexample. Later, we have another face-value matching, but that is a matching of two linear functions. $f_2$ is continuous everywhere except at one point, $(0,0,1)$. Its semi-$H^1$ norm can be computed by affine mapping and (2.4),

$$\int_{K_{1623} \cup K_{1635}} |\nabla f_2|^2 dx = \frac{2}{3} + \frac{13}{72}.$$  

(2.6)

By two reflections, we glue three copies of function $f_2$, to get a function on $1/2$ of a cube:

$$f_3(x,y,z) = \begin{cases} 
  f_2(x,y,z) & \text{on } K_{0126}, \\
  f_2(F_{0126}(F_{0426}^{-1}(x,y,z))) & \text{on } K_{0426}, \\
  f_2(F_{0126}(F_{0486}^{-1}(x,y,z))) & \text{on } K_{0486}.
\end{cases}$$  

(2.7)

Here in (2.7), the affine mapping $F_{ijkl}$ is no longer the finite element reference mapping as we allow “negative” volume (negative Jacobian). This is to form a mirror image for the piecewise function. We compute its norm, cf. Fig. 2,

$$|f_3|^2_{H^1(K_{1623} \cup K_{1635} \cup K_{2463} \cup K_{4536} \cup K_{4103,6} \cup K_{48,4,10})} = \frac{2}{3} + \frac{13}{72} + \frac{3}{4} + \frac{13}{72} + \frac{3}{4} = \frac{65}{24}.$$  

We will reflect $f_3$ once to define our counterexample $u$ on one cube $C_{04871269} = [0,1]^3$, shown in Fig. 2:

$$f_4 = \begin{cases} 
  f_3(x,y,z) & \text{on } x_0 x_1 x_2 x_4 x_6 x_8, \\
  f_3(y,x,z) & \text{on } x_0 x_1 x_9 x_7 x_8 x_8.
\end{cases}$$

Finally, we define $u$ on a big cube $\Omega = [-1,1]^3$:

$$u(x,y,z) = \begin{cases} 
  f_4(x,y,z) & \text{on } 0 \leq x,y,z \leq 1, \\
  f_4(-x,y,z) & \text{on } 0 \leq -x,y,z \leq 1, \\
  f_4(-x,-y,z) & \text{on } 0 \leq -x,-y,z \leq 1, \\
  f_4(x,-y,z) & \text{on } 0 \leq x,-y,z \leq 1, \\
  f_4(x,y,-z) & \text{on } 0 \leq x,y,-z \leq 1, \\
  f_4(-x,y,-z) & \text{on } 0 \leq -x,y,-z \leq 1, \\
  f_4(-x,-y,-z) & \text{on } 0 \leq -x,-y,-z \leq 1, \\
  f_4(x,-y,-z) & \text{on } 0 \leq x,-y,-z \leq 1.
\end{cases}$$  

(2.8)

The constructed function $u(x,y,z)$ is continuous everywhere except at six mid-face points, $(\pm 1,0,0)$, $(0,\pm 1,0)$ and $(0,0,\pm 1)$. The function is depicted in Fig. 3, where $u=1$ at purple
colored region, and \( u = 0 \) at red colored region (only the boundary of the cube). By (2.4), via affine mappings, it follows that

\[
u(x,y,z) \in H^1_0((-1,1)^3),
\]

with

\[
|\nu|_{H^1_0}(\Omega) = \sqrt{16|\nu|_{H^1(X_0X_1X_2X_3X_4X_5X_6X_7X_8)}^2} = \frac{1}{3}\sqrt{390} \approx 6.6833.
\]

This value is also verified by our numerical integration, see the data in column 4 of Table 1, in column 4 of Table 2, and in column 6 of Table 4. In fact, the weighted semi-\( H^1 \) norm of \( u \) is independent of the jump of coefficients, see (2.10) below, as we purposely made the semi-\( H^1 \) norm zero in the region of large weight.

We define a weight function of two constant weights on the domain:

\[
\omega_{\epsilon_0}(x,y,z) = \begin{cases} 
\omega_1 = \epsilon_0^{-1}, & \text{if } |x| + |y| + |z| \leq 1, \\
\omega_2 = 1, & \text{elsewhere on } [-1,1]^3.
\end{cases}
\]  

(2.9)

Note that, \( u \equiv 1 \) in the diamond region \( \{ |x| + |y| + |z| \leq 1 \} \) where the weight could be big, \( \epsilon_0^{-1} \). Thus, \( |\nu|_{H^1_0(|x|+|y|+|z|\leq1)} = 0 \) and

\[
|\nu|_{H^1_0(\Omega)} = |\nu|_{H^1(\Omega)} = \frac{1}{3}\sqrt{390}, \quad \forall \epsilon_0 > 0 \text{ in (2.9).}
\]  

(2.10)
Table 1: The error and order of convergence, for conforming elements.

<table>
<thead>
<tr>
<th>ε₀ = 1 in (2.9)</th>
<th>ε₀ = 10⁻¹² in (2.9)</th>
</tr>
</thead>
<tbody>
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<td></td>
<td># CG</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>0.2017</td>
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<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>0.0258</td>
</tr>
<tr>
<td>7</td>
<td>0.0092</td>
</tr>
</tbody>
</table>

The first level triangulation on Ω is shown in Fig. 4. Then the standard multigrid refinement [20] is applied recursively to define the higher level grids. The second level grid is also shown in Fig. 4. We note that there is no interior node on the first two level grids for the conforming finite element functions. So the meaningful computation starts from the level 3 grid. To verify Theorem 2.1, we select two ε₀, 1 and 10⁻¹², in the weight ω(x,y,z) (2.9). When ε₀ = 1, the weight ω(x,y,z) is identically 1 on the whole domain Ω. The approximation is standard, shown in Table 1. Note that u ∈ H³/₂−ε(Ω), the L² convergence order is 1.5. More details on the computation will be explained in Section 4, comparing to that of nonconforming finite element. To confirm the ratio, max{√ω_i}/min{√ω_i} = √e₀⁻¹ = 10⁶, in Theorem 2.1, we list the ratio of errors below,

\[
\frac{\|u - Q^h_{ω_i}u\|_{L^2}}{\|u - Q^h_{ω_i^0}u\|_{L^2}} \leq \frac{\min\{\sqrt{ω_i}\}}{\max\{\sqrt{ω_i}\}} = 0.428, 0.424, 0.420, 0.414, 0.411,
\]

when h = 1/4, 1/8, 1/16, 1/32, 1/64, respectively. That is, the constant C in Theorem 2.1 is roughly 0.4 (C depends also on shape regularity of grids).

In this counterexample, the weighted semi-\(H^1\) norm is independent of the coefficient jump. Thus, on a fixed grid, as the conforming finite element is continuous that its semi-\(H^1\) norm on the inner diamond region is nonzero, bounded below by some constant depending on the grid. Thus the conforming finite element projection, interpolation, or PDE solution would have its weighted semi-\(H^1\) norm proportional to the \(1/\sqrt{ε₀}\), as shown in Theorem 2.1.

3 The Scott-Zhang operator on nonconforming elements

We define first the Scott-Zhang interpolation for the nonconforming finite element. The key is that the interpolation is subdomain independent, i.e., jump-independent. Then,
with the standard theory on the Scott-Zhang interpolation, we will show the $L^2$ weighted projection is of optimal order, independent of jump coefficients. Also, we show that the nonconforming finite element solution is of optimal order, jump-independent, when solving elliptic partial differential equations with large jump coefficients.

Because the nodal degree of freedom of $P_1$ nonconforming finite elements (1.1b) in at the barycenter of a $(d-1)$ dimensional simplex, the Scott-Zhang operator is defined uniquely without any face-selection

$$I^{nc}_h : H^1_0(\Omega) \rightarrow V^{nc}_h,$$

$$(I^{nc}_h u)(\chi_i) = \frac{1}{|K_i|} \int_{K_i} u(x) d\chi$$ at a barycenter $\chi_i$ of $K_i \subset K \in \mathcal{T}_h$.

The Scott-Zhang operator preserves the image function that

$$I^{nc}_h v_h = v_h, \quad \forall v_h \in V^{nc}_h.$$

Because the value of $I^{nc}_h u(\chi_i)$ is defined by the value of $u$ on the common interface on the two sides of $K_j$, it follows that

$$I^{nc}_h = I^{nc}_h |_{\Omega_1} \oplus \cdots \oplus I^{nc}_h |_{\Omega_J}.$$

This cannot be done for continuous finite element functions. In this section, to avoid introducing discrete norms and inner products, we simply extend the conventional notations to nonconforming finite elements, for example,

$$|v_h|_{H^1(\Omega)}^2 = \sum_{K \in \mathcal{T}_h} |v_h|_{H^1(K)}^2, \quad \forall v_h \in V^{nc}_h.$$

**Lemma 3.1.** For any $u \in H^1_0(\Omega)$, it holds that

$$|I^{nc}_h u|_{H^1_0(\Omega)} \leq C \|u\|_{H^1_0(\Omega)},$$

$$\|I^{nc}_h u - u\|_{L^2(\Omega)} \leq Ch \|u\|_{H^1_0(\Omega)},$$

where $C$ are independent of the jump $\{\omega_i\}$.

**Proof.** On one subdomain, by trace theorem, as in [15], we have

$$|I^{nc}_h u|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)},$$

$$\|I^{nc}_h u - u\|_{L^2(\Omega)} \leq Ch \|u\|_{H^1(\Omega)}.$$  

Thus, summing up all subdomains,

$$|I^{nc}_h u|_{H^1(\Omega)}^2 = \sum_i \omega_i |I^{nc}_h u|_{H^1(\Omega)}^2 \leq C \sum_i \omega_i \|u\|_{H^1(\Omega)}^2 = C \|u\|_{H^1(\Omega)}^2.$$

The $L^2$ estimate is proved similarly. \qed
We are ready to prove the main theorem, Theorem 1.2.

**Proof of Theorem 1.2.** The proof is standard (cf. [15]) except that it involves weights. Let $u \in H^1_0(\Omega)$. Let $Q^\omega_{h,nc}u$ be defined in (1.6). By Lemma 3.1,

$$||u - Q^\omega_{h,nc}u||^2_{L^2(\Omega)} \leq ||u - I^nc_hu||^2_{L^2(\Omega)} \leq Ch^2 \sum \omega_i ||u||^2_{H^1(\Omega)} = Ch^2 ||u||^2_{H^1(\Omega)}.$$

Next, for semi-$H^1_ω$ norm, we have, by the inverse inequality,

$$||Q^\omega_{h,nc}u||^2_{H^1_ω(\Omega)} = \sum \omega_i ||Q^\omega_{h,nc}u||^2_{H^1(\Omega)}$$

$$\leq C \sum \omega_i ||Q^\omega_{h,nc}u - I^nc_hu||^2_{H^1(\Omega)} + ||I^nc_hu||^2_{H^1(\Omega)}$$

$$\leq C \sum \omega_i h^{-2} ||Q^\omega_{h,nc}u - I^nc_hu||^2_{L^2(\Omega)} + ||I^nc_hu||^2_{H^1(\Omega)}$$

$$\leq Ch^{-2} (||Q^\omega_{h,nc}u - u||^2_{L^2(\Omega)} + ||u - I^nc_hu||^2_{L^2(\Omega)}) + C ||I^nc_hu||^2_{H^1_ω(\Omega)}$$

So, we complete the proof. □

Finally, we study the nonconforming finite element solution for jump-coefficient elliptic problems. The weighted Laplace equation is:

$$- \nabla \cdot (\omega(\xi) \nabla u) = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega.$$

The weak variational form is: Find $u \in H^1_0(\Omega)$ such that

$$(u,v)_{H^1_0(\Omega)} = (f,v)_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega).$$

We seek the nonconforming finite element solution: Find $u_h \in V^nc_h$ such that

$$(u_h,v_h)_{H^1_0(\Omega)} = (f,v_h)_{L^2(\Omega)}, \quad \forall v_h \in V^nc_h. \quad (3.1)$$

**Theorem 3.1.** The nonconforming finite element solution in (3.1) approximates the true solution in the optimal order, independent of the jump coefficients $\{\omega_i\}$, i.e.,

$$|u - u_h|_{H^1_0(\Omega)} \leq Ch^{\beta - 1} ||u||_{H^p_0(\Omega)},$$

where $\beta > 1$ is determined by the regularity of the true solution $u$.

**Proof.** By Lemma 3.1, the proof is standard, cf. [6], for example. First, by the second Strang lemma, cf. [9],

$$|u - u_h|_{H^1_0(\Omega)} \leq C \inf_{w_h \in V^nc_h} |u - w_h|_{H^1_0(\Omega)} + C \sup_{w_h \in V^nc_h} \frac{(u,w_h)_{H^1_0(\Omega)} - (f,w_h)_{L^2(\Omega)}}{|w_h|_{H^1_0(\Omega)}}.$$
The first term is the approximation error, bounded by the interpolation error, by Lemma 3.1,

\[ |u - w_h|_{H^1_0(\Omega)}^2 \leq |u - I_h^{nc} u|_{H^1_0(\Omega)}^2 \]

\[ = \sum_i \omega_i \sum_{K \in \mathcal{T}_h \cap \Omega_i} |u - I_h^{nc}|_{\Omega_i} u |_{H^1(K)}^2 \]

\[ \leq \sum_i \omega_i \sum_{K \in \mathcal{T}_h \cap \Omega_i} C \inf_{p_1 \in P_1(K)} |I_h^{nc}|_{\Omega_i} (u - p_1) |_{H^1(K)}^2 \]

\[ \leq \sum_i \omega_i \sum_{K \in \mathcal{T}_h \cap \Omega_i} C h^{2\beta - 2} |u |_{H^{1}(K)}^2 \]

\[ = Ch^{2\beta - 2} |u |_{H^1(\Omega)}^2 \]

where \( \mathcal{S}_K \) is the union of triangles touching triangle \( K \). We note that \( \mathcal{S}_K \) can cross several regions \( \Omega_i \). But as \( u \in H^1(\Omega) \), the existence of \( p_1 \) is guaranteed.

The second term is the consistent error, cf. [6].

\[ (u, w_h)_{H^1_0(\Omega)} - (f, w_h)_{L^2(\Omega)} = \sum_i \omega_i \sum_{K \in \mathcal{T}_h \cap \Omega_i} \int_{\partial K} \frac{\partial u}{\partial n} w_h ds = \sum_{e \in \mathcal{T}_h} \int_e \omega_i \frac{\partial u}{\partial n_e} [w_h] ds, \]

where \( \mathcal{T}_h \) is the set of edges in the triangulation \( \mathcal{T}_h \), \( [w_h] \) is the jump of \( w_h \) on the edge of \( e \), and \( \omega_i \frac{\partial u}{\partial n_e} \) is continuous on the two sides of \( e \). We introduce two notations,

\[ \overline{f} = \int_e \frac{f ds}{|e|} \quad \text{and} \quad \hat{f} = \int_K \frac{f dx}{|K|}. \]

Then, we continue to estimate the second term,

\[ \left| \sum_{e \in \mathcal{T}_h^+} \int_e \omega_i \frac{\partial u}{\partial n_e} [w_h] ds \right| \]

\[ = \left| \sum_{e \in \mathcal{T}_h^+} \int_e \left( \omega_i \frac{\partial u}{\partial n_e} - \omega_j \frac{\partial u}{\partial n_e} \right) (w_h|_e^+ - w_h|_{K_e^+} - w_h|_e^- + w_h|_{K_e^-}) ds \right| \]

\[ \leq \left( \sum_i \sum_{K \in \mathcal{T}_h \cap \Omega_i} \sum_{e \subseteq K} \right) \left( \omega_i \frac{\partial u}{\partial n_e} - \omega_j \frac{\partial u}{\partial n_e} \right)^2 \]

\[ \leq \left( \sum_i \sum_{K \in \mathcal{T}_h \cap \Omega_i} \sum_{e \subseteq K} \omega_i \int_e 2(w_h - \overline{w}_h)^2 ds \right)^{1/2}, \]

noticing that on edge \( e, w_h|_{K_e^+} \) and \( w_h|_{K_e^-} \) are two constants, averaging \( w_h \) on the plus and the minus side triangle of edge \( e \), respectively,
\[ \frac{1}{2} \sum_{e \in T} \sum_{i \in T \cap \Omega_i, e \in \partial K} \left( \left\| \frac{\partial u}{\partial n_e} - \frac{\partial I_h u}{\partial n_e} \right\|^2_{L^2(e)} \right)^{1/2} \cdot \left( \sum_{i} \omega_i \sum_{K \in T_h \cap \Omega_i} \left( 1 \right) \sum_{e \subset \partial K} \left( \left\| \nabla (u - I_h u) \right\|^2_{L^2(K)} + h^2 \beta^{-3} \left\| \nabla (u - I_h u) \right\|^2_{H^{\beta-1}(K)} \right) \right)^{1/2} \]
\[ \leq C \left( \sum_{i} \omega_i h^2 \beta^{-3} \left\| u \right\|_{H^\beta(\Omega_i)}^2 \right)^{1/2} \cdot h^{1/2} \left| w_h \right|_{H^\beta(\Omega)} \]
\[ = C h^{\beta-1} \left| u \right|_{H^\beta(\Omega_i)} \left| w_h \right|_{H^\beta(\Omega)}. \]

Combining the two estimates, the theorem is proved.

\[ \square \]

4 Numerical tests

We numerically solve the weighted \( L^2 \) projection problems (1.3) and (1.6), for \( u \) defined in (2.8) and \( \omega(x,y,z) \) defined in (2.9). The initial grid is shown in Fig. 4. The standard multigrid refinement is performed to generate high level grids, [20].

In the first test, we use the conforming element \( V_h \) in (1.1a). The error and the order of convergence is listed in Table 1. The resulting linear system is solved by the conjugate gradient method. It is interesting to notice that the number of CG iterations for the rough coefficient is about 60 times of that for constant 1 coefficient. It is apparent the weighted
Table 2: The error and order of convergence, for nonconforming elements.

| $\varepsilon_0 = 1$ in (2.9) | $h^n$ | $|Q_{h,nc}^\omega u|_{H^1_\omega}$ | # CG | # dof |
|-----------------------------|-------|---------------------------------|------|------|
| 2 | 0.4827 | 1.2 | 4.8000 | 27 | 104 |
| 3 | 0.1695 | 1.5 | 5.9324 | 34 | 736 |
| 4 | 0.0599 | 1.5 | 6.1224 | 35 | 5504 |
| 5 | 0.0215 | 1.5 | 6.3954 | 33 | 42496 |
| 6 | 0.0077 | 1.5 | 6.5347 | 29 | 333824 |
| 7 | 0.0028 | 1.5 | 6.6064 | 26 | 2646016 |

$\varepsilon_0 = 10^{-12}$ in (2.9)

| $h^n$ | $|Q_{h,nc}^\omega u|_{H^1_\omega}$ | # CG | # dof |
|-------|---------------------------------|------|------|
| 2 | 0.4929 | 18.7 | 4.8575 | 120 | 104 |
| 3 | 0.1772 | 1.1 | 5.5887 | 562 | 736 |
| 4 | 0.0789 | 1.3 | 6.1207 | 844 | 5504 |
| 5 | 0.0556 | 0.5 | 6.3944 | 879 | 42496 |
| 6 | 0.0519 | 0.1 | 6.5342 | 860 | 333824 |
| 7 | 0.0513 | 0.0 | 6.6061 | 826 | 2646016 |

Table 3: The error for nonconforming elements.

| $h^n$ | $|Q_{h,nc}^\omega u|_{H^1_\omega}$ | $|Q_{h,nc}^\omega u|_{L^2_\omega}$ | $|Q_{h,nc}^\omega u|_{H^1_\omega}$ | # CG | # dof |
|-------|---------------------------------|---------------------------------|---------------------------------|------|------|
| 2 | 0.6271 | 1.0 | 0.4903 | 8.8 | 2.7376 | 0.6 |
| 3 | 0.2937 | 1.1 | 0.1696 | 1.5 | 2.2915 | 0.3 |
| 4 | 0.1159 | 1.3 | 0.0600 | 1.5 | 1.6410 | 0.5 |
| 5 | 0.0443 | 1.4 | 0.0215 | 1.5 | 1.1974 | 0.5 |
| 6 | 0.0167 | 1.4 | 0.0077 | 1.5 | 0.8812 | 0.4 |
| 7 | 0.0063 | 1.4 | 0.0028 | 1.5 | 0.6494 | 0.4 |

$L^2$ error deteriorates when $\varepsilon_0 \to 0$, at order $1/\sqrt{\varepsilon_0}$.

In the second test, we use the nonconforming element $V_{h,nc}^{nc}$ defined in (1.1b). On grids shown in Fig. 4, we have internal degree of freedom for $V_{h,nc}^{nc}$ functions on level 2 and higher. We note that the arithmetic steps are about the same for both conforming and nonconforming methods on a same grid, though the $\text{dim} V_{h,nc}^{nc} \approx 7 \text{dim} V_h$. In fact, the condition number for the nonconforming element is even better, as seen from the number of CG iterations that the number for nonconforming element is about 1/4 of that for conforming element. We listed the weighted $L^2$ error and the order of convergence in Table 2. Also in the fourth column, we list the $H^1_\omega$ norm of $u_h$. Table 2 confirms the estimate in Theorem 1.2. In particular, we can see in the 4th column of Table 2 that the $|Q_{h,nc}^\omega u|_{H^1_\omega}$ is truly independent of $\omega_i$, as the two sets of data are nearly identical. However, it seems that the computation is beyond computer accuracy that the $|u - Q_{h,nc}^\omega u|_{L^2_\omega}$ could not assume the convergence order, 1.5 when $\varepsilon_0 = 10^{-12}$. We can compare the data in the second column of Table 2 with that 4, supposedly $|u - Q_{h,nc}^\omega u|_{L^2_\omega} \leq |u - I_h^\omega u|_{L^2_\omega}$. For a computer accuracy checking, we let $\varepsilon_0 = 10^{-6}$ in Table 3. We also increased the stop accuracy for the CG iteration in Table 3. Now, in Table 3, the $|u - Q_{h,nc}^\omega u|_{L^2_\omega}$ converges at the correct order.
is less than that of interpolation error, and is identical to that error when $\epsilon_0 = 1$.

Next, we verify Lemma 3.1, checking the Scott-Zhang operator for nonconforming finite elements. The weighted $L^2$ error and $H^1_\omega$ error are listed in Table 4. It turns out the numbers are all identical for the two cases $\epsilon_0 = 1$ and $\epsilon_0 = 10^{-12}$ in (2.9). Thus, the interpolation for nonconforming finite elements is absolutely independent of the jump-coefficients, proved in Lemma 3.1.

Why the conforming finite element functions do not approximate our counterexample $u(x,y,z)$ in (2.8)? The problem is at the 6 discontinuous points of $u$. When $u_h$ is forced to be zero at $\partial \Omega$, it destroys the constant 1 property of $u$ in the large coefficient $\omega_1 = \epsilon_0^{-1}$ region. We can see the plots of conforming and nonconforming finite element solutions in Fig. 5. Nonconforming finite element solutions can match exactly $u = 1$ in $\omega_1$ region.

Table 4: The interpolation error for the nonconforming element.

<table>
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<tr>
<th>$\epsilon_0 = 1$ and $\epsilon_0 = 10^{-12}$ in (2.9)</th>
<th>$|u - I_h^n u|_{L^2}$</th>
<th>$H^1$</th>
<th>$|u - I_h^n u|_{H^1}$</th>
<th>$H^1$</th>
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References


