The simplest nonconforming mixed finite element method for linear elasticity in the symmetric formulation on \( n \)-rectangular grids

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\textbf{A B S T R A C T}

A family of mixed finite elements is proposed for solving the first order system of linear elasticity equations in any space dimension, where the stress field is approximated by symmetric finite element tensors. This family of elements has a perfect matching between the stress and the displacement. The discrete spaces for the normal stress \( \tau_{ii} \), the shear stress \( \tau_{ij} \) and the displacement \( u_i \) are span\{1, \( x_i \)\}, span\{1, \( x_i, x_j \)\} and span\{1\}, respectively, on rectangular grids. In particular, the definition remains the same for all space dimensions. As a result of these choices, the theoretical analysis is independent of the spatial dimension as well. In 1D, the element is nothing else but the 1D Raviart–Thomas element, which is the only conforming element in this family. In 2D and higher dimensions, they are new elements but of the minimal degrees of freedom. The total degrees of freedom per element are 2 plus 1 in 1D, 7 plus 2 in 2D, and 15 plus 3 in 3D. These elements are the simplest element for any space dimension.

The well-posedness condition and the optimal a priori error estimate of the family of finite elements are proved. Numerical tests in 2D and 3D are presented to show a superiority of the new elements over others, as a superconvergence is exhibited and proved.

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1. Introduction

The first order system of equations, for the symmetric stress field \( \sigma \in \Sigma := H(\text{div}, \Omega; \mathbb{S}) \) and the displacement field \( u \in V := L^2(\Omega; \mathbb{R}^n) \), reads: Find \((\sigma, u) \in \Sigma \times V \) such that

\[
\begin{align*}
    (A\sigma, \tau) + (\text{div}\tau, u) &= 0 \quad \forall \tau \in \Sigma, \\
    (\text{div}\sigma, v) &= (f, v) \quad \forall v \in V.
\end{align*}
\]  

(1.1)
Here the operator $A$ is symmetric and positive definite, the symmetric tensor-valued stress space $\Sigma$ and the vector-valued displacement space $V$ are, respectively,

$$
H(\text{div}, \Omega; \mathbb{S}) = \left\{ (\tau_{ij})_{i,j=1}^n \in H(\text{div}, \Omega; \mathbb{R}^{n \times n}) \mid \tau_{ij} = \tau_{ji} \right\},
$$

(1.2)

$$
L^2(\Omega; \mathbb{R}^n) = \left\{ (u_1, \ldots, u_n)^T \mid u_i \in L^2(\Omega; \mathbb{R}) \right\}.
$$

(1.3)

In 1D, one example of the problem (1.1) is the mixed formulation of the 1D Poisson equation; In 2D and 3D, the stress–displacement formulation based on the Hellinger–Reissner principle for the linear elasticity can be regarded as a celebrated example of (1.1).

Because of the symmetry constraint on the stress tensor, $\tau_{ij} = \tau_{ji}$, it is extremely difficult to construct stable conforming finite elements of (1.1) even if for 2D and 3D, as stated in the plenary presentation to the 2002 International Congress of Mathematicians by D. N. Arnold. Hence compromised works use composite elements [1,2], or enforce the symmetry condition weakly [3–10]. In Arnold and Winther [11] and Arnold, Awanou, and Winther [12], a sufficient condition of the discrete stable method was proposed, which states that a discrete exact sequence guarantees the stability of the mixed methods. Based on such a condition, conforming mixed finite elements on the simplicial and rectangular triangulations were developed for both 2D and 3D [13,14,12,11]. In order to keep conformity the vertex degrees of freedom are in particular employed in those methods. To avoid the complexity of conforming mixed elements and also vertex degrees of freedom, new weak–symmetry finite elements [15–18], non-conforming finite elements [19–24] were constructed. See also [25,26] for the enrichment of nonconforming elements of [21,22] to conforming elements. However, most of these elements are difficult to implement; numerical implementation can only be found in [27,28,24] so far, all in 2D. See [29] for the DPG method for the linear elasticity problem. Recently, based on a crucial structure of symmetric matrix-valued piecewise polynomial $H(\text{div})$ space and two basic algebraic results, the first and fourth authors developed a new framework to design and analyze the mixed finite elements of the linear elasticity problem. As a result, on both simplicial and product grids, the first families of both symmetric and optimal mixed elements with polynomial shape functions in any space dimension were achieved; see more details in [30–34].

Superconvergence is one of the most active research fields for finite element methods. A lot of fundamental results can be found for conforming, nonconforming and mixed finite elements of model problems in literature, see for instance, [35–39]. However, no results can be found for the mixed finite element methods under consideration in literature so far. A very recent paper [40] analyzed superconvergence of a family of conforming rectangular mixed finite element methods for the two dimensional linear elasticity problem of [25]. However, in the conclusion, it was pointed out that the technique therein cannot be applied to the mixed elements under consideration.

In this paper, a new family of minimal, any space-dimensional, symmetric, nonconforming mixed finite elements for the problem (1.1) is constructed. It is motivated by a simple fact that, by (1.2), the derivative on a normal stress component $\tau_{ni}$ is only in $x_i$ direction; while those on $\tau_{ij}$ are only in $x_i$ and $x_j$ directions. Thus, the minimal finite element space for $\tau_{ni}$ would be span$\{1, x_i\}$; the minimal finite element space for $\tau_{ij}$ would be span$\{1, x_i, x_j\}$. For the displacement (1.3), there is no derivative and the minimal finite element space would be the constant space span$\{1\}$. Surprisingly, it is shown that these minimal finite element spaces can actually form a family of stable and convergent methods for (1.1). However, the analysis herein has to overcome the difficulty to prove the discrete inf–sup condition, one key ingredient for the stability analysis of the mixed finite element methods [41], and the difficulty related to nonconformity of the discrete spaces for the stress. To overcome the first difficulty, an explicit constructive proof is given for the discrete inf–sup condition. In order to deal with the second difficulty, a superconvergence property of the consistency error is proved. The mathematical elegance and beauty of this family of minimal elements is gestated within, besides the perfect matching, the independence of the spatial dimension $n$. In $n$ dimension, the constructive proof of the discrete inf–sup condition can be divided into $n$ steps of that for the 1D Raviart–Thomas element, and the consistency error can be decomposed as $n$ two-dimensional consistency errors (For 1D, there is no consistency error), In [42], we extended this family of elements to conforming mixed elements.

Meanwhile, superconvergence of the nonconforming mixed element in 2D, which was demonstrated by numerical examples, is proved as well. One difficulty is that the canonical interpolation operator for the discrete stress space has no commuting properties, which are indispensable ingredients for superconvergence analysis for mixed finite elements for the Poisson equation, see for instance, [43,44,37,45,46], also for the linear elasticity problem [40]. Another difficulty is that the normal stresses are coupled and consequently the superclose analysis used in [37] for the mixed finite element of the Poisson equation cannot be extended to the present case. To overcome these difficulties, we follow [47] to adopt a new expansion for the canonical interpolation operator. In addition, we develop a new technique to derive a new and sharp superconvergence of the consistency error, which settles down a third difficulty caused by nonconformity of the discrete stress space.

The rest of the paper is organized as follows. The minimal element in 2D is introduced in Section 2. The well-posedness of the finite element problem, i.e. the discrete coerciveness and discrete inf–sup condition, is proved in Section 3. The optimal order convergence is shown in Section 4, which is followed by Section 5 where superconvergence analysis is presented. The element is extended to any space-dimension in Section 6. Numerical results in 2D and 3D, are provided in Section 7, which show a superconvergence of the minimal elements herein.
2. A minimal element in 2D

The 2D element is presented separately in this section for fixing the main idea while the whole family will be developed in Section 6. Consider the following linear elasticity problem:
\[
\begin{align*}
\text{div}(A^{-1} \varepsilon(u)) &= f & \text{in } \Omega, \\
\mathbf{u} &= 0 & \text{on } \Gamma = \partial \Omega.
\end{align*}
\]

The domain is assumed to be a rectangle (it is straightforward that results can be extended to domains which can be covered by rectangles), which is subdivided by a family of rectangular grids \(T_h\) (with grid size \(h\)).

The set of all edges in \(T_h\) is denoted by \(\mathcal{E}_h\), which is divided into two sets, the set \(\mathcal{E}_h^v\) of horizontal edges and the set \(\mathcal{E}_h^v\) of vertical edges. Let \(\mathcal{E}_h^v\) and \(\mathcal{E}_h^v\) denote the sets of all the interior vertical and horizontal edges of \(T_h\), respectively, and \(\mathcal{E}^v\) be the set of all the internal vertices of \(T_h\). Given vertex \(A \in \mathcal{E}^v\), let \(\mathcal{E}(A)\) be the set of edges that take \(A\) as one of their endpoints.

Given any edge \(e \in \mathcal{E}_h\), one fixed unit normal vector \(n\) with components \((n_1, n_2)^T\) is assigned. For each \(K \in T_h\), define the affine invertible transformation
\[
F_K : \hat{K} \rightarrow K,
\]
with the center \((x_0, y_0)\) of \(K\), the horizontal length \(h_{x,K}\), and the vertical length \(h_{y,K}\), and the reference element \(\hat{K} = [-1, 1]^2\). For convenience, suppose that the domain \(\Omega\) is a unit square \([0, 1]^2\) which is triangulated evenly into \(N^2\) elements, \(\{K\}\). This implies that \(h = h_{x,K} = h_{y,K} = \frac{1}{N}\) for any element \(K\).

On each element \(K \in T_h\), a constant finite element space for the displacement is defined by
\[
V_K = \mathcal{P}_0(K; \mathbb{R}^2) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1, v_2 \in \mathcal{P}_0(K) \right\},
\]
while the symmetric linear finite element space for the stress is defined by
\[
\Sigma_K = \left\{ \tau \in \begin{pmatrix} P_1(x) & \mathcal{P}_1(x, y) \\ \mathcal{P}_1(x, y) & P_1(y) \end{pmatrix} \mid \tau \in \mathbb{S}_{1,2} \right\},
\]
where subscript \(\mathbb{S}\) indicates a symmetric matrix stress, and
\[
\begin{align*}
P_1(x) &= \text{span}\{1, x\}, \\
\mathcal{P}_1(x, y) &= \text{span}\{1, x, y\}, \\
P_1(y) &= \text{span}\{1, y\}.
\end{align*}
\]

The dimension of the space \(V_K\) is 2, and that of \(\Sigma_K\) is 7. The degrees of freedom for \((v_1, v_2)^T, \tau_{11}, \text{ and } \tau_{22}\), are

- The moment of degree 0 on \(K\) for \(v_1\) and \(v_2\);
- The moments of degree 0 on two vertical edges of \(K\) for \(\tau_{11}\);
- The moments of degree 0 on two horizontal edges of \(K\) for \(\tau_{22}\).

The degrees of freedom for \(\tau_{12}\) will be studied as follows. Locally \(\mathcal{P}_1(x, y)\) is the space of linear polynomials. Globally, let \(W_h\) be the \(P_1\)-nonconforming space on \(T_h\), which was first introduced in [48] as a nonconforming approximation space to \(H^1(\Omega)\) on the quadrilateral mesh; see also [49]. To be exact, \(W_h\) is the space of piecewise linear polynomials, which are continuous at all mid-edge points of triangulation \(T_h\). \(W_h\) is the finite element space approximating function \(\tau_{12}\).

The global spaces \(\Sigma_h\) and \(V_h\) are defined by
\[
\Sigma_h := \left\{ \tau_h = \begin{pmatrix} \tau_{11,h} \\ \tau_{12,h} \end{pmatrix} \in L^2(\Omega; \mathbb{S}) \mid \tau_h|_K \in \Sigma_K \text{ for all } K \in T_h, \right\},
\]
\[
\begin{align*}
\tau_{11,h} &\text{ is continuous on all vertical interior edges,} \\
\tau_{12,h} &\text{ is continuous on all horizontal interior edges,} \\
\tau_{12,h} &\text{ is continuous at all mid-points of interior edges}
\end{align*}
\]
\[
V_h := \{ v_h \in L^2(\Omega; \mathbb{R}^2) \mid v_h|_K \in V_K \text{ for all } K \in T_h \}.
\]

Since \(\tau_{11,h}\) is continuous on all vertical interior edges, the derivative \(\partial_y \tau_{11,h}\) is well-defined in \(L^2(\Omega)\). However, \(\tau_{12,h}\) is not continuous on \(\Omega\) so that \(\partial_y \tau_{12,h}\) and \(\partial_x \tau_{12,h}\) are not in \(L^2(\Omega)\). Therefore the discrete stress space \(\Sigma_h\) is a nonconforming
approximation to $H(\text{div}, \Omega; \mathbb{S})$. So the discrete divergence operator $\text{div}_h$ is defined elementwise with respect to $\mathcal{T}_h$.

$$\text{div}_h \tau_h|_K = \text{div}(\tau_h|_K) \quad \forall \tau_h \in \Sigma_h.$$  

The mixed variational form for (2.1a) is (1.1). The mixed finite element approximation of Problem (1.1) reads: Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$
\begin{align*}
\begin{cases}
(A \sigma_h, \tau_h) + (\text{div}_h \tau_h, u_h) & = 0 \quad \forall \tau_h \in \Sigma_h, \\
(\text{div}_h \sigma_h, v_h) & = (f, v_h) \quad \forall v_h \in V_h.
\end{cases}
\end{align*}
$$

(2.6)

It follows from the definition of $\Sigma_h$ that $\text{div}_h \tau_h$ are piecewise constant for any $\tau_h \in \Sigma_h$, which leads to

$$\text{div}_h \Sigma_h \subset V_h.$$  

This, in turn, leads to a strong discrete divergence-free space:

$$
\begin{align*}
Z_h := \{ \tau_h \in \Sigma_h & \mid (\text{div}_h \tau_h, v_h) = 0 \quad \forall v_h \in V_h \} \\
= \{ \tau_h \in \Sigma_h & \mid \text{div}_h \tau_h = 0 \text{ pointwise} \}.
\end{align*}
$$

(2.7)

For the analysis, define the following broken norm:

$$\| \tau_h \|^2_{H(\text{div}, \Omega)} = \left( \| \tau_h \|_0^2 + \| \text{div}_h \tau_h \|_0^2 \right)^{1/2} \quad \forall \tau_h \in \Sigma_h.$$  

(2.8)

The rest of this section is devoted to an alternative definition to $W_h$, the space for $\tau_{12,h}$ in $\Sigma_h$. The dimension of the space $\mathcal{P}_1(x, y)$ is three, less than the number of edges or vertices of element $K$. Here the idea from [49] of a frame for $\mathcal{P}_1(x, y)$ will be used. To this end, define the frame for the space $\mathcal{P}_1(\hat{x}, \hat{y}) = \text{span}\{1, \hat{x}, \hat{y}\}$ by

$$
\hat{\phi}^{(0)} = \frac{1 - \hat{x} - \hat{y}}{4}, \quad \hat{\phi}^{(1)} = \frac{1 + \hat{x} - \hat{y}}{4}, \\
\hat{\phi}^{(2)} = \frac{1 + \hat{x} + \hat{y}}{4}, \quad \hat{\phi}^{(3)} = \frac{1 - \hat{x} + \hat{y}}{4}.
$$

(2.9)

This frame is depicted in Fig. 1.

An interpolation operator $\Pi_{12}$, from $H^2(\Omega)$ (i.e., some continuous functions) to $W_h$ is needed. The interpolation on $\hat{K}$ is defined as

$$\hat{\Pi}_{12} \tau_{12} = \tau_{12}(\hat{x}_{1,\hat{K}}, \hat{y}_{1,\hat{K}}) \hat{\phi}^{(0)} + \tau_{12}(\hat{x}_{2,\hat{K}}, \hat{y}_{2,\hat{K}}) \hat{\phi}^{(1)} + \tau_{12}(\hat{x}_{3,\hat{K}}, \hat{y}_{3,\hat{K}}) \hat{\phi}^{(2)} + \tau_{12}(\hat{x}_{4,\hat{K}}, \hat{y}_{4,\hat{K}}) \hat{\phi}^{(3)},$$

where the four vertices are numbered counterclockwise,

$$(\hat{x}_{1,\hat{K}}, \hat{y}_{1,\hat{K}}) = (-1, -1), \\
(\hat{x}_{2,\hat{K}}, \hat{y}_{2,\hat{K}}) = (1, -1), \\
(\hat{x}_{3,\hat{K}}, \hat{y}_{3,\hat{K}}) = (1, 1), \\
(\hat{x}_{4,\hat{K}}, \hat{y}_{4,\hat{K}}) = (-1, 1).$$

In the same fashion, the interpolation $\Pi_{12}$ is defined on all $K \in \mathcal{T}_h$ by

$$\Pi_{12} \tau_{12}(x, y) = \tau_{12}(x_{1,K}, y_{1,K}) \hat{\phi}^{(0)}(F^{-1}_K(x, y)) + \tau_{12}(x_{2,K}, y_{2,K}) \hat{\phi}^{(1)}(F^{-1}_K(x, y)) + \tau_{12}(x_{3,K}, y_{3,K}) \hat{\phi}^{(2)}(F^{-1}_K(x, y)) + \tau_{12}(x_{4,K}, y_{4,K}) \hat{\phi}^{(3)}(F^{-1}_K(x, y)),$$

(2.10)

where $(x, y) \in K$, and $(x_{i,K}, y_{i,K})$ are the four vertices of $K$. As $\hat{\phi}^{(0)}(0, -1) = \hat{\phi}^{(1)}(0, -1) = 1/2$, it follows that

$$\Pi_{12} \tau_{12}|_{K_a}(e_m) = \frac{1}{2} \left( \tau_{12}(e(A_1)) + \tau_{12}(e(A_2)) \right).$$
where \( K_+ \) and \( K_- \) are two elements that share the edge \( e \in E_h, e_m \) is the mid-point of \( e \), and \( e(A_1) \) and \( e(A_2) \) are two endpoints of \( e \). That is, \( \Pi_{12} \tau_{12} \) is continuous at all mid-points of edges. For a vertex in \( T_h \),

\[
c_{ij} = (ih, jh), \quad 0 \leq i, j \leq N, \quad N = 1/h,
\]

it may be shared by one, or two, or four elements \( K \in T_h \). The combination of the frame functions at the vertex \( c_{ij} \) forms one global frame function \( \phi_{i,j} \). For example, at vertex \( c_{0,1} \), as it is shared by two elements, \( K_{1,1} = [0, h] \times [0, h] \) and \( K_{1,2} = [0, h] \times [h, 2h] \),

\[
\psi_{0,1} = \begin{cases} 
\phi^{(3)} \left( \frac{2}{h} (x - \frac{h}{2}), \frac{2}{h} (y - \frac{h}{2}) \right) & (x, y) \in K_{1,1}, \\
\phi^{(0)} \left( \frac{2}{h} (x - \frac{h}{2}), \frac{2}{h} (y - \frac{3h}{2}) \right) & (x, y) \in K_{1,2}, \\
0 & \text{elsewhere on } \Omega.
\end{cases}
\]

Note that \( \psi_{i,j} \) is not continuous at \( c_{i,j} \). Thus, the finite element space for \( \tau_{12,h} \) in (2.4) is

\[
W_h := \left\{ s \in L^2(\Omega) \mid s = \sum_{i,j=0}^N p_{ij} \psi_{i,j} \right\}.
\]

(3.11)

3. Well-posedness of the discrete problem in 2D

This section considers the well-posedness of the discrete problem (2.6), which needs the following two conditions.

(1) \( K \)-ellipticity. There exists a constant \( C > 0 \), independent of the meshsize \( h \) such that

\[
(A\tau_h, \tau_h) \geq C \| \tau_h \|_{H(\text{div},\Sigma_h)}^2 \quad \forall \tau_h \in \Sigma_h,
\]

where \( \Sigma_h \) is the divergence-free space defined in (2.7).

(2) Discrete B–B condition. There exists a positive constant \( C > 0 \) independent of the meshsize \( h \), such that

\[
\inf_{0 \neq v_h \in V_h} \sup_{0 \neq \tau_h \in \Sigma_h} \frac{(\text{div} \tau_h, v_h)}{\| \tau_h \|_{H(\text{div},\Sigma_h)} \| v_h \|_0} \geq C.
\]

Theorem 3.1. For the discrete problem (2.6), the \( K \)-ellipticity (3.1) and the discrete B–B condition (3.2) hold uniformly. Consequently, the discrete mixed problem (2.6) has a unique solution \( (\tau_h, \upsilon_h) \in \Sigma_h \times V_h \).

Proof. It follows from (2.7) that for all \( \tau_h \in \Sigma_h, \text{div} \tau_h = 0 \). Thus \( \| \text{div} \tau_h \|_0 = 0 \) and \( \| \tau_h \|_{H(\text{div},\Sigma_h)} = \| \tau_h \|_0 \). Since the operator \( A \) is symmetric and positive definite, the \( K \)-ellipticity of the bilinear form \( (A\tau_h, \tau_h) \) follows.

It remains to show the discrete B–B condition (3.2). Since the usual technique based on canonical interpolations operators for discrete stress spaces \([12,11]\) is inapplicable here, a constructive proof is adopted. For any \( v \in V_h \), it can be decomposed as a sum,

\[
v_h = \sum_{i=1}^N \sum_{j=1}^N V_{ij} \varphi_{ij}(x, y),
\]

where \( \varphi_{ij}(x) \) is the characteristic function on the element \( K_{ij} \), and \( V_{ij} = (V_{1,ij}, V_{2,ij})^T = (\upsilon_h | K_{ij}) \). A discrete stress function \( \tau_h \in \Sigma_h \) will be constructed with

\[
\text{div} \upsilon_h \tau_h = \upsilon_h \quad \text{and} \quad \| \tau_h \|_{H(\text{div},\Sigma_h)} \leq C \| \upsilon_h \|_0.
\]

The construction of \( \tau_h \) is motivated by a simple proof of the inf–sup condition of the 1D Raviart–Thomas element for the 1D Poisson problem. The shear stress \( \tau_{12,h} \) can be taken zero, i.e., \( \tau_{12,h} = 0 \); the normal stress \( \tau_{11,h} \) (resp. \( \tau_{22,h} \)) can be constructed so that it is independent of the second (resp. first) component of \( \upsilon_h \). In addition, \( \tau_{11,h} \) (resp. \( \tau_{22,h} \)) can be a continuous piecewise linear function of the variable \( x \) (resp. \( y \)) and a piecewise constant function of \( y \) (resp. \( x \)). Therefore, they are of form

\[
\tau_{11,h}(x, y) = h \sum_{m=1}^{i-1} V_{1,mj} + V_{1,ij}(x - x_{i-1}),
\]

\[
\tau_{22,h}(x, y) = h \sum_{k=1}^{j-1} V_{2,ik} + V_{2,ij}(y - y_{j-1}),
\]

(3.4)
for $x_{i-1} \leq x < x_i$ and $y_{j-1} \leq y < y_j$. $(x_i, y_j)$ is the upper-right corner vertex of square $K_{ij}$.) Thus, define
\[ \tau_h = \begin{pmatrix} \tau_{11,h} & 0 \\ 0 & \tau_{22,h} \end{pmatrix} \in \Sigma_h. \]

By this construction, $\partial_x \tau_{11,h} = (v_h)_1$ and $\partial_y \tau_{22,h} = (v_h)_2$. This gives
\[ \text{div}_h \tau_h = v_h. \] (3.6)

An elementary calculation gives
\[ \|v_h\|_0^2 = \sum_{i,j=1}^N \|V_{ij}\|_{0,K_{ij}}^2 = \sum_{i,j=1}^N \int_{K_{ij}} |V_{ij}|^2 \, dx \, dy \]
\[ = \sum_{i,j=1}^N ((V_{1,ij})^2 + (V_{2,ij})^2)h^2. \]
By the Schwarz inequality,
\[ \|\tau_{11,h}\|_0^2 = \sum_{i,j=1}^N \int_{K_{ij}} \left( h \sum_{m=1}^{i-1} V_{1,mj} + V_{1,ij}(x - x_{i-1}) \right)^2 \, dx \, dy \]
\[ \leq \sum_{i,j=1}^N \int_{K_{ij}} \left( h^2 \sum_{m=1}^{i-1} (V_{1,mj})^2 + (V_{1,ij})^2(x - x_{i-1})^2 \right) \, dx \, dy. \]
Further, since $N = 1/h$ and $\int_{K_{ij}} = h^2$,
\[ \|\tau_{11,h}\|_0^2 \leq \sum_{i,j=1}^N \left( h^2 \sum_{m=1}^{i-1} (V_{1,mj})^2 \right) \cdot Nh^2 \leq \sum_{j=1}^N \left( h^2 \sum_{m=1}^N (V_{1,mj})^2 \right) \cdot N^2 h^2 \]
\[ = h^2 \sum_{i,j=1}^N (V_{1,ij})^2. \]
A similar argument leads to
\[ \|\tau_{22,h}\|_0^2 \leq h^2 \sum_{i,j=1}^N (V_{2,ij})^2. \]

The combination of the aforementioned two identities and two inequalities yields
\[ \|\tau_h\|_{H(\text{div},h)}^2 = \|\tau_h\|_0^2 + \|\text{div}_h \tau_h\|_0^2 \]
\[ = \|\tau_{11,h}\|_0^2 + \|\tau_{22,h}\|_0^2 + \|v_h\|_0^2 \leq 2\|v_h\|_0^2. \]
Hence, for any $v_h \in V_h$, the B–B condition (3.2) holds with $C = 1/\sqrt{2}$:
\[ \inf_{0 \neq v_h \in V_h} \sup_{0 \neq \tau_h \in \Sigma_h} \frac{\langle \text{div}_h \tau_h, v_h \rangle}{\|\tau_h\|_{H(\text{div},h)} \|v_h\|_0} \geq \inf_{v_h \in V_h} \frac{\|v_h\|_0^2}{\sqrt{2} \|v_h\|_0} = \frac{1}{\sqrt{2}}. \]
This completes the proof. \hfill \blacksquare

4. Error analysis in 2D

The section is devoted to the error estimate stated in Theorem 4.3, which is based on the approximation error estimate of Theorem 4.1 and the consistency error estimate of Remark 4.1.

In order to analyze the approximation error, for any $\tau \in H(\text{div}; \Omega) \cap H^2(\Omega; S)$, define an interpolation
\[ \Pi_h \tau = \begin{pmatrix} \Pi_{11} \tau_{11} & \Pi_{12} \tau_{12} \\ \Pi_{12} \tau_{12} & \Pi_{22} \tau_{22} \end{pmatrix} \in \Sigma_h. \] (4.1)
where $\Pi_{11}$ and $\Pi_{22}$ are standard, satisfying, respectively,
\[\int_e \Pi_{11} \tau_{11} ds = \int_e \tau_{11} ds \quad \text{for any vertical edge } e \in \mathcal{E}_h, \quad (4.2)\]
\[\int_e \Pi_{22} \tau_{22} ds = \int_e \tau_{22} ds \quad \text{for any horizontal edge } e \in \mathcal{E}_h. \quad (4.3)\]

$\Pi_{12}$ is the interpolation operator defined in (2.10), from the space $H^2(\Omega; \mathbb{R})$ to $W_h$. It was shown by Hu and Shi [49], Park and Sheen [48] that
\[|w - \Pi_{12} w|_{m, K} \leq Ch^{2-m} |w|_{2, K}, \quad m = 0, 1, K \in \mathcal{T}_h, \quad \text{for any } w \in H^2(K; \mathbb{R}). \quad (4.4)\]

**Theorem 4.1.** For any $\tau \in H^2(\Omega; \mathbb{S})$, it holds that
\[\|\tau - \Pi_h \tau\|_0 \leq Ch\|\tau\|_2, \quad \|\text{div}_h(\tau - \Pi_h \tau)\|_0 \leq Ch\|\tau\|_2. \]

**Proof.** By the scaling argument and the standard approximation theory, the following two estimates will be proved
\[|\tau_{11} - \Pi_{111} \tau_{11}|_{0, K} \leq Ch |\tau_{11}|_{1, K} \quad \forall K \in \mathcal{T}_h, \quad (4.5)\]
\[\left| \frac{\partial}{\partial x} (\tau_{11} - \Pi_{111} \tau_{11}) \right|_{0, K} \leq Ch \left| \frac{\partial \tau_{11}}{\partial x} \right|_{1, K} \quad \forall K \in \mathcal{T}_h. \quad (4.6)\]

For any element $K \in \mathcal{T}_h$, by (4.2) (i.e., the interpolation (4.2) is equivalent to a mid-point interpolation),
\[\|\tau_{11} - \Pi_{111} \tau_{11}\|_{0, K}^2 = \frac{h^2}{4} \int_K |\hat{\tau}_{11} - \hat{\Pi}_{111} \hat{\tau}_{11}|^2 d\hat{x} d\hat{y} \leq Ch^2 |\hat{\tau}_{11}|_{1, K}^2 \leq Ch^2 |\tau_{11}|_{1, K}^2. \]

This is (4.5). By the reference mapping,
\[\left| \frac{\partial}{\partial x} (\tau_{11} - \Pi_{111} \tau_{11}) \right|_{0, K}^2 = \int_K \left| \frac{\partial}{\partial x} (\hat{\tau}_{11} - \hat{\Pi}_{111} \hat{\tau}_{11}) \right|^2 d\hat{x} d\hat{y}. \quad (4.7)\]

Now
\[\int_K \frac{\partial}{\partial x} \hat{\Pi}_{111} \hat{\tau}_{11} d\hat{x} d\hat{y} = \int_{-1}^1 \left( (\hat{\Pi}_{11} \hat{\tau}_{11}(1, \hat{y}) - (\hat{\Pi}_{11} \hat{\tau}_{11})(-1, \hat{y}) \right) d\hat{y} \]
\[= \int_{-1}^1 \left( \hat{\tau}_{11}(1, \hat{y}) - \hat{\tau}_{11}(-1, \hat{y}) \right) d\hat{y} \]
\[= \int_K \frac{\partial \hat{\tau}_{11}}{\partial x} d\hat{x} d\hat{y}. \]

This means $\frac{\partial}{\partial x} (\hat{\Pi}_{11} \hat{\tau}_{11}) = \Pi_0^0 (\hat{\tau}_{11})_x$, where $\Pi_0^0_k$ is the projection operator onto the constant space on element $\hat{K}$. A substitution of it into (4.7) leads to
\[\left| \frac{\partial}{\partial x} (\tau_{11} - \Pi_{111} \tau_{11}) \right|_{0, K}^2 \leq C \left| \frac{\partial \hat{\tau}_{11}}{\partial x} - \Pi_0^0_k \left( \frac{\partial \hat{\tau}_{11}}{\partial x} \right) \right|_{0, K}^2 \]
\[\leq C \inf_{c \in \mathbb{R}} \left| \left( \frac{\partial \hat{\tau}_{11}}{\partial x} - c \right) \right|_{0, K}^2. \]

By the Bramble–Hilbert Lemma,
\[\left| \frac{\partial}{\partial x} (\tau_{11} - \Pi_{111} \tau_{11}) \right|_{0, K}^2 \leq C \left| \frac{\partial \tau_{11}}{\partial x} \right|_{1, K}^2 \leq Ch^2 \left| \frac{\partial \tau_{11}}{\partial x} \right|_{1, K}^2. \]

This is (4.6).

A similar argument yields
\[\|\tau_{22} - \Pi_{22} \tau_{22}\|_{0, K} \leq Ch |\tau_{22}|_{1, K} \quad \forall K \in \mathcal{T}_h, \quad (4.8)\]
\[\left| \frac{\partial}{\partial y} (\tau_{22} - \Pi_{22} \tau_{22}) \right|_{0, K} \leq Ch \left| \frac{\partial \tau_{22}}{\partial y} \right|_{1, K} \quad \forall K \in \mathcal{T}_h. \quad (4.9)\]
Noting that the $L^2$ norm on $\Sigma$ is
$$
\|\tau\|_{0, h}^2 = \|\tau_{11}\|_{0, K}^2 + 2\|\tau_{12}\|_{0, K}^2 + \|\tau_{22}\|_{0, K}^2.
$$
A combination of the estimates (4.5), (4.6), (4.8), (4.9) and (4.4) completes the proof. ■

In the sequel, we will need some results on Sobolev spaces. They are formulated in the following lemma. First of all, define $\partial\Omega_h$ as the subset of points having (Euclidean) distance less than $h$ from the boundary:
$$
\partial\Omega_h := \{ \mathbf{x} \in \Omega : \exists y \in \partial\Omega : \text{dist}(x, y) \leq h \}.
$$

**Lemma 4.1** ([43,50]). For $v \in H^s(\Omega)$ with $0 \leq s \leq 1/2$, it holds
$$
\|v\|_{0, \partial\Omega_h} \leq Ch^s\|v\|_s. \tag{4.10}
$$

**Theorem 4.2.** Assume the partition $T_h$ is uniform and $(\sigma, u)$ be the solution to the problem (1.1) with $u \in H^1_0(\Omega; \mathbb{R}^2) \cap H^3(\Omega; \mathbb{R}^2)$. Then,
$$
\sup_{0 \neq \tau_h \in \Sigma_h} \frac{(A\sigma, \tau_h) + \langle \text{div}_h \tau_h, u \rangle}{\|\tau_h\|_{H(\text{div}_h)}} \leq Ch^{3/2}|u|_3. \tag{4.11}
$$

**Proof.** It follows from the first equation of (1.1) that $A\sigma = \frac{1}{2}(\nabla u + \nabla u^T)$ for the exact solution $u \in H^1_0(\Omega; \mathbb{R}^2)$. An elementwise integration by parts gives
$$
(A\sigma, \tau_h) + \langle \text{div}_h \tau_h, u \rangle = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau_h \mathbf{v} \cdot \mathbf{u} \, ds \tag{4.12}
$$
let $\tau_h|_K = \begin{pmatrix} \tau_{11,h} & \tau_{12,h} \\ \tau_{21,h} & \tau_{22,h} \end{pmatrix}$, cf. Fig. 2. Since $\tau_{11,h}$ is continuous in the $x$-direction and $\tau_{22,h}$ is continuous in the $y$-direction, there is a cancellation for these two components on the interior-element boundary. Since $u \in H^1_0(\Omega; \mathbb{R}^2) \cap H^3(\Omega; \mathbb{R}^2),
$$
\sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau_h \mathbf{v} \cdot \mathbf{u} \, ds = \sum_{K \in \mathcal{T}_h} \left( \left( \int_{e_{x,K}} - \int_{e_{x,K}} \right) \tau_{12,h} u_2 \, ds + \left( \int_{e_{y,K}} - \int_{e_{y,K}} \right) \tau_{12,h} u_1 \, ds \right) \tag{4.13}
$$
For any $v \in H^1(\Omega)$, define the $L^2$-projection operator $\Pi_h^0$ on an edge by
$$
\Pi_h^0 v := \frac{1}{|e|} \int_e v \, ds.
$$
Because $\tau_{12,h}$ is continuous at the mid-point of all edges, on the horizontal edges $\partial K_{0,h}$, it follows that,
$$
\sum_{K \in \mathcal{T}_h} \left( \int_{e_{+1,K}} - \int_{e_{1,K}} \right) \tau_{12,h} u_1 \, ds = \sum_{e \in \partial K_{0,h}} \int_e (\tau_{12,h}|_{e_+} - \tau_{12,h}|_{e_-}) u_1 \, ds
$$
$$
= \sum_{e \in \partial K_{0,h}} \int_e (\tau_{12,h}|_{e_+} - \tau_{12,h}|_{e_-})(u_1 - \Pi_h^0 u_1) \, ds
$$
$$
= \sum_{e \in \partial K_{0,h}} \left( \int_{e_{x,K}} \tau_{12,h}(u_1 - \Pi_h^0 u_1) \, ds - \int_{e_{y,K}} \tau_{12,h}(u_1 - \Pi_h^0 u_1) \, ds \right)
$$
$$
= \sum_{e \in \partial K_{0,h}} \left( \int_{e_{x,K}} \tau_{12,h} - \Pi_h^0 \tau_{12,h} \right)(u_1 - \Pi_h^0 u_1) \, ds
$$
$$
- \int_{e_{y,K}} \tau_{12,h} - \Pi_h^0 \tau_{12,h} \, ds
$$
\tag{4.14}
$$
where $e_+$ and $e_-$ are two sides of $e$. There is some superconvergence property for the two terms in (4.14) if they are considered together. In fact, on the reference element $K = [-1, 1] \times [-1, 1],
$$
\hat{\tau}_{12,h}(\hat{x}, \pm 1) = \hat{\tau}_{12,h}(0, 0) + \hat{x}\partial_x \hat{\tau}_{12,h}(0, 0) \pm \hat{y}\partial_y \hat{\tau}_{12,h}(0, 0)
$$
and
$$
\hat{\tau}_{12,h}(\hat{x}, -1) - \Pi_h^0 \hat{\tau}_{12,h}(\hat{x}, -1) = \hat{\tau}_{12,h}(\hat{x}, 1) - \Pi_h^0 \hat{\tau}_{12,h}(\hat{x}, 1) = \hat{x}\partial_x \hat{\tau}_{12,h}(0, 0).
$$
The property of $\Pi_0^0$ gives

$$
\sum_{k \in \mathcal{K}} \left( \int_{e_{1,k}} - \int_{e_{1,k}} \right) \tau_{12,h} u_1 \, ds = \sum_{k \in \mathcal{K}} \frac{h_{k,K}}{2} \int_{-1}^{1} \frac{\partial \tilde{\tau}_{12,h}}{\partial \hat{x}} \left[ \int_{-1}^{1} \frac{\partial}{\partial \hat{y}} \tilde{u}_1(\hat{x}, \hat{y}) \, d\hat{y} - \frac{1}{2} \int_{-1}^{1} \left( \tilde{u}_1(\hat{t}, 1) - \tilde{u}_1(\hat{t}, -1) \right) \, d\hat{t} \right] \, d\hat{x}
$$

For any $f \in L^2(K)$, define the $L^2$-projection operator $\Pi_0^0$ on element $K$ by

$$
\Pi_0^0 f := \frac{1}{|K|} \int_{K} f \, dx \, dy.
$$

Then,

$$
\sum_{k \in \mathcal{K}} \left( \int_{e_{1,k}} - \int_{e_{1,k}} \right) \tau_{12,h} u_1 \, ds = \sum_{k \in \mathcal{K}} \frac{h_{k,K}}{4} \int_{-1}^{1} \frac{\partial \tilde{\tau}_{12,h}}{\partial \hat{x}} \left[ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial^2}{\partial \hat{s} \partial \hat{y}} \tilde{u}_1(\hat{s}, \hat{y}) \, d\hat{s} \, d\hat{y} \, d\hat{t} \right] \, d\hat{x}
$$

A combination of the property of $\Pi_0^0$ and the inverse inequality plus a scaling argument show

$$
|B_1| \leq Ch \sum_{k \in \mathcal{K}} |\tilde{\tau}_{12,h}|_{1,k} |\tilde{u}_1|_{3,k}
$$

$$
\leq Ch^2 |\tau_{12,h}|_0 |u_1|_3.
$$
The term $B_2$ has the following decomposition
\[
B_2 = \sum_{k \in \eta} \frac{h_{k,K}}{6} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x}^2} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \, d\hat{x}d\hat{y} \\
= \sum_{k \in \eta} \frac{h_{k,K}}{6} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x}^2} (\Pi_{2,k}^0 - I) \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \, d\hat{x}d\hat{y} + \sum_{k \in \eta} \frac{h_{k,K}}{6} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x}^2} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \, d\hat{x}d\hat{y}
= : B_3 + B_4.
\]

The term $B_3$ can be bounded by using the scaling argument, the property of $\Pi_{2,k}^0$ and the inverse inequality as follows
\[
|B_3| \leq C_2 |\tau_{12,h}|_0 |u_1|_3.
\]

An integration by parts produces a decomposition for the term $B_4$ as follows
\[
B_4 = \sum_{k \in \eta} \frac{h_{k,K}}{6} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x}} \left( \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \right) \hat{\tau}_{12,h} \, d\hat{x}d\hat{y}
= \sum_{k \in \eta} \left( - \frac{h_{k,K}}{6} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial}{\partial \hat{x}} \left( \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \right) \hat{\tau}_{12,h} \, d\hat{x}d\hat{y} \right)
+ \sum_{k \in \eta} \frac{h_{k,K}}{6} \left( \int_{e_{2,k}} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \hat{\tau}_{12,h} \, d\hat{y} - \int_{e_{4,k}} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \hat{\tau}_{12,h} \, d\hat{y} \right)
= : B_5 + B_6.
\]

A scaling argument shows
\[
|B_5| = \sum_{k \in \eta} \frac{h_{k,K}}{6} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial}{\partial \hat{x}} \left( \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \right) \hat{\tau}_{12,h} \, d\hat{x}d\hat{y}
\leq C \sum_{k \in \eta} |\hat{u}_1|_{1,3,k} |\hat{\tau}_{12,h}|_{0,k}
\leq C_2 |u_1|_1 |\tau_{12,h}|_0,
\]

while the term $B_6$ admits the following estimate
\[
|B_6| = \left| \sum_{k \in \eta} \frac{h_{k,K}}{6} \left( \int_{e_{2,k}} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \hat{\tau}_{12,h} \, d\hat{y} - \int_{e_{4,k}} \frac{\partial^2 \hat{u}_1(\hat{x}, \hat{y})}{\partial \hat{x} \partial \hat{y}} \hat{\tau}_{12,h} \, d\hat{y} \right) \right|
\leq C_2 \sum_{e \in e_h \cap \partial \Omega} \int_{e} \frac{\partial^2 u_1(x, y)}{\partial x \partial y} (\tau_{12,h}|_{e_+} - \tau_{12,h}|_{e_-}) \, dy + C_2 \sum_{e \in e_h \cap \partial \Omega} \int_{e} \frac{\partial^2 u_1(x, y)}{\partial x \partial y} \tau_{12,h} \, dy
= : B_7 + B_8.
\]

Since $\tau_{12,h}$ is continuous at the mid point of all the edges,
\[
B_7 \leq C_2 \sum_{k \in \eta} \left[ \int_{e_{2,k}} \left( \tau_{12,h} - \Pi_{2,k}^0 \tau_{12,h} \right) \left( \frac{\partial^2 u_1}{\partial x \partial y} - \Pi_{2,k}^0 \frac{\partial^2 u_1}{\partial x \partial y} \right) \, dy \right]
+ \int_{e_{4,k}} \left( \tau_{12,h} - \Pi_{e_{4,k}}^0 \tau_{12,h} \right) \left( \frac{\partial^2 u_1}{\partial x \partial y} - \Pi_{e_{4,k}}^0 \frac{\partial^2 u_1}{\partial x \partial y} \right) \, dy
\leq C_2 \sum_{k \in \eta} |h||\tau_{12,h}|_{1,k} |u_1|_{1,3,k}
\leq C_2 |\tau_{12,h}|_0 |u_1|_{1,3}.
\]

For $e \in e_h \cap \partial \Omega$, let $K_e$ be the unique element such that $e$ is one of its edges. The trace theorem and inverse inequality give
\[
\int_{e} \frac{\partial^2 u_1(x, y)}{\partial x \partial y} \tau_{12,h} \, dy = \int_{e} \tau_{12,h} \left( \frac{\partial^2 u_1}{\partial x \partial y} - \Pi_{K_e}^0 \frac{\partial^2 u_1}{\partial x \partial y} \right) \, dy + \int_{e} \Pi_{K_e}^0 \frac{\partial^2 u_1}{\partial x \partial y} \int_{e} \tau_{12,h} \, dy
\leq C \|\tau_{12,h}\|_{0,K_e} |u_1|_{1,3,K_e} + Ch^{-1} |u_1|_{2,K_e} \|\tau_{12,h}\|_{0,K_e},
\]
this and Lemma 4.1 lead to
\[
B_8 \leq Ch^2 \|T_{12,h}\|_{0,\Omega_h}(|u_{13}| + h^{-1/2}|u_{15}|/2) \\
\leq Ch^2 \|T_{12,h}\|_{0,\Omega}(|u_{13}| + h^{-1/2}|u_{13}|) \\
\leq Ch^{3/2} \|T_{12,I}|_0 |u_{13}|.
\]

A substitution of the estimations of \(B_1, B_3, B_5, B_7, \) and \(B_8 \) into (4.14) proves
\[
\sum_{k \in P_h} \left( \int_{e_{1,k}} - \int_{e_{2,k}} \right) T_{12,h} u_4 \, ds \leq Ch^{3/2} |u|_3 |\eta|_0.
\]
A similar argument shows
\[
\sum_{k \in P_h} \left( \int_{e_{2,k}} - \int_{e_{4,k}} \right) T_{12,h} u_2 \, ds \leq Ch^{3/2} |u|_3 |\eta|_0.
\]

Then,
\[
|(A\sigma, \tau_h) + (\text{div}_h \tau_h, u)| \leq Ch^{3/2} |u|_3 |\eta|_0,
\]
which completes the proof. ■

**Remark 4.1.** For general rectangular meshes, a similar argument of the above theorem proves
\[
\sup_{0 \neq \tau_h \in \Sigma_h} \frac{(A\sigma, \tau_h) + (\text{div}_h \tau_h, u)}{\|\tau_h\|_{H(\text{div}, \Omega)}} \leq Ch |u|_2,
\] provided that \( u \in H^1_0(\Omega; \mathbb{R}^2) \cap H^2(\Omega; \mathbb{R}^2). \)

**Theorem 4.3.** Let \((\sigma, u) \in \Sigma \times V\) be the exact solution of problem (1.1) and \((\sigma_h, u_h) \in \Sigma_h \times V_h\) the finite element solution of (2.6). Then
\[
\|\sigma - \sigma_h\|_0 \leq Ch(|u|_2 + |\sigma|_2), \\
\|\text{div}_h(\sigma - \sigma_h)\|_0 \leq Ch(|u|_2 + |\sigma|_2), \\
\|u - u_h\|_0 \leq Ch(|u|_2 + |\sigma|_2).
\]

**Proof.** In fact, a standard argument for both mixed and nonconforming finite element methods, see [51,41], leads to
\[
\|\sigma - \sigma_h\|_{H(\text{div}, \Omega)} \leq C \left\{ \inf_{\eta_h \in \Sigma_h} \|\sigma - \eta_h\|_{H(\text{div}, \Omega)} + \sup_{0 \neq \xi_h \in \Sigma_h} \frac{(A\sigma, \xi_h) + (\text{div}_h \xi_h, u)}{\|\xi_h\|_{H(\text{div}, \Omega)}} \right\}.
\] (4.16)

The first term on the right-hand side of (4.16) is the approximation error. The choice \( \eta_h = H \sigma\) with Theorem 4.1 gives its upper bound. The second term on the right-hand side of (4.16) is the usual consistency error for the nonconforming finite element method, which has already been bounded in Remark 4.1. A combination of these two results implies
\[
\|\sigma - \sigma_h\|_0 \leq Ch(|u|_2 + |\sigma|_2) ,
\]
\[
\|\text{div}_h(\sigma - \sigma_h)\|_0 \leq Ch(|u|_2 + |\sigma|_2).
\]

The rest of the proof is concerned with the estimation of \( u - u_h \). In view of the discrete B–B Condition (3.2), it holds, for any \( v \in V_h \),
\[
\|u_h - v\|_0 \leq C \sup_{0 \neq \tau_h \in \Sigma_h} \frac{(A\sigma, \tau_h) + (\text{div}_h \tau_h, u)}{\|\tau_h\|_{H(\text{div}, \Omega)}} + C(\|\sigma - \sigma_h\|_0 + \|u - v\|_0).
\]

By the error estimation of \( \|\sigma - \sigma_h\|_0 \), the triangle inequality plus \( v = P_h u \) (\( P_h \) is the \( L^2 \) projection into piecewise constant spaces) yield
\[
\|u - u_h\|_0 \leq \|u - P_h u\|_0 + \|P_h u - u_h\|_0 \\
\leq Ch|u|_2 + C(\|\sigma - \sigma_h\|_0 + \|u - P_h u\|_0) \\
\leq Ch(|u|_2 + |\sigma|_2),
\]
which completes the proof of this theorem. ■
5. Superconvergence analysis

This section presents superconvergence analysis for the element in 2D. Since the canonical interpolation operators for the discrete stress space have no commuting properties and the normal stresses are coupled, we adopt a new expansion.

5.1. The superclose property of \((\text{div}_h(\sigma - \Pi_h \sigma), v)\)

We follow the idea of [47] to give a new expansion of the operator \(\Pi_h\). In fact, let \(\Pi_K = \Pi_h|_K\), we have the following crucial result.

**Lemma 5.1.** For any \(\sigma \in P_2(K; \mathbb{S})\) and \(v \in V_K\), it holds that
\[
(\text{div}(\sigma - \Pi_K \sigma), v)_K = 0.
\] (5.1)

**Proof.** We only need to prove the result on the reference element \(K = [-1, 1] \times [-1, 1]\). For any \(\sigma \in P_2(K; \mathbb{S})\), its component \(\sigma_{12}\) can be expressed as
\[
\sigma_{12} = \sigma_{21} = c_0 + c_1x + c_2y + c_{12}xy + c_3x^2 + c_4y^2.
\]

The definition of \(\Pi_{12,K}\) leads to
\[
\sigma_{12} - \Pi_{12,K} \sigma_{12} = c_3(x^2 - 1) + c_4(y^2 - 1) + c_{12}xy.
\] (5.2)

Its components \(\sigma_{11}\) and \(\sigma_{22}\) can be expressed as
\[
\begin{align*}
\sigma_{11} &= a_0 + a_1x + a_2x^2 + a_{12}xy + a_3y + a_4(y^2 - 1/3), \\
\sigma_{22} &= b_0 + b_1y + b_2y^2 + b_{12}xy + b_3x + b_4(x^2 - 1/3).
\end{align*}
\]

The definition of \(\Pi_{1h,K}\) leads to
\[
\begin{align*}
\sigma_{11} - \Pi_{11,K} \sigma_{11} &= a_2(x^2 - 1) + a_4(y^2 - 1/3) + a_{12}xy, \\
\sigma_{22} - \Pi_{22,K} \sigma_{22} &= b_2(x^2 - 1/3) + b_4(y^2 - 1) + b_{12}xy.
\end{align*}
\]

Thus we have
\[
\begin{align*}
\frac{\partial (\sigma_{11} - \Pi_{11,K} \sigma_{11})}{\partial x} &= 2a_3x + a_{12}y, \\
\frac{\partial (\sigma_{22} - \Pi_{22,K} \sigma_{22})}{\partial y} &= 2b_2y + b_{12}x, \\
\frac{\partial (\sigma_{12} - \Pi_{12,K} \sigma_{12})}{\partial x} &= 2c_3x + c_{12}y, \\
\frac{\partial (\sigma_{12} - \Pi_{12,K} \sigma_{12})}{\partial y} &= 2c_4y + c_{12}x.
\end{align*}
\]

They are all polynomials of degree 1 in variable \(x\) or \(y\). Since \(v\) is a constant vector on \(K\),
\[
\int_K \text{div}(\sigma - \Pi_K \sigma) v \, dx dy = 0,
\]
which completes the proof.  

As a consequence of (5.1), we have the following superclose property for the term \((\text{div}_h(\sigma - \Pi_h \sigma), v)\).

**Theorem 5.1.** Suppose that \(\sigma \in H^3(\Omega; \mathbb{S})\). Then it holds that
\[
|(\text{div}_h(\sigma - \Pi_h \sigma), v)| \leq C h^2 |\sigma|_3 \|v\|_0 \text{ for any } v \in V_h.
\] (5.3)

**Proof.** Given element \(K\), let \(I_{2,K} : H(\text{div}, K; \mathbb{S}) \to P_2(K; \mathbb{S})\) be the \(L^2\) projection operator defined as: Given \(\tau \in H(\text{div}, K; \mathbb{S})\), find \(I_{2,K} \tau \in P_2(K; \mathbb{S})\) such that
\[
\int_K (I_{2,K} \tau) q \, dx dy = \int_K \tau q \, dx dy \text{ for any } q \in P_2(K; \mathbb{S}).
\]
This allows for the following decomposition:
\[
(\text{div}_h(\sigma - \Pi_h\sigma), v) = \sum_{K \in \mathcal{T}_h} (\text{div}((I - \Pi_K)I_{2,K}\sigma + (I - \Pi_K)(I - I_{2,K})\sigma), v)_K
\]
\[
= \sum_{K \in \mathcal{T}_h} (\text{div}(I - \Pi_K)(I - I_{2,K})\sigma), v)_K,
\]
where we use (5.1). The desired result follows from the approximation property of $I_{2,K}$. 

5.2. The superclose property of $(A(\sigma - \Pi_h\sigma), \sigma_h - \Pi_h\sigma)$

To deal with the second difficulty, we propose to explore the discrete inf–sup condition presented in Theorem 3.1.

**Lemma 5.2.** For any $\sigma \in P_1(K; \mathcal{S})$ and $\tau \in \Sigma_{n,h}$, it holds that
\[
|A(\sigma - \Pi_K\sigma), \tau)|_K \leq C h^2 \|\text{div} \|_{0,K} |\sigma|_{1,K},
\]
(5.4)
where
\[
\Sigma_{n,h} = \{\tau = \text{diag}(\tau_{11}, \tau_{22}), \tau \in \Sigma_h\}.
\]

**Remark 5.1.** For ease of presentation, we restrict ourselves to the special but practical case where
\[
A\sigma = \frac{1}{2\mu} \left(\sigma - \frac{\lambda}{2\mu + 2\lambda} \text{tr}(\sigma)\delta\right),
\]
(5.5)
where $\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\mu$ and $\lambda$ are the Lamé constants such that $0 < \mu_1 \leq \mu \leq \mu_2$ and $0 < \lambda < \infty$. However, the analysis herein can be extended to the following more general cases
\[
A\sigma = \begin{pmatrix} a_{11}\sigma_{11} + a_{12}\sigma_{12} + a_{22}\sigma_{22} & b_{11}\sigma_{11} + b_{12}\sigma_{12} + b_{22}\sigma_{22} \\ b_{11}\sigma_{11} + b_{12}\sigma_{12} + b_{22}\sigma_{22} & c_{11}\sigma_{11} + c_{12}\sigma_{12} + c_{22}\sigma_{22} \end{pmatrix}
\]
where $a_{ij}, b_{ij},$ and $c_{ij}$, are piecewise constant with respect to the mesh.

**Proof.** We only need to prove the result on the reference element $K = [-1, 1]^2$. For any $\sigma \in P_1(K; \mathcal{S})$, its components can be written as
\[
\sigma_{11} = a_0 + a_1 x + a_2 y,
\]
\[
\sigma_{22} = b_0 + b_1 x + b_2 y,
\]
\[
\sigma_{12} = \sigma_{21} = c_0 + c_1 x + c_2 y.
\]
By the definition of the operators $A$ and $\Pi_K$,
\[
A(\sigma - \Pi_K\sigma) = \frac{1}{2\mu(2\mu + 2\lambda)} \begin{pmatrix} (2\mu + \lambda)a_2 y - \lambda b_1 x & 0 \\ 0 & (2\mu + \lambda)b_1 x - \lambda a_2 y \end{pmatrix}.
\]
Note that the $ii$-th component of $\tau$ can be written as
\[
\tau_{11} = d_0^{11} + d_1^{11} x, \quad \tau_{22} = d_0^{22} + d_1^{22} y.
\]
Therefore,
\[
(A(\sigma - \Pi_K\sigma), \tau|_{11})_K = \frac{1}{2\mu(2\mu + 2\lambda)} ((2\mu + \lambda)a_2 y - \lambda b_1 x, d_0^{11} + d_1^{11} x)_K
\]
\[
= \frac{\lambda}{2\mu(2\mu + 2\lambda)} b_1 d_1^{11}(x, x)_K
\]
\[
= \frac{2\lambda}{3\mu(2\mu + 2\lambda)} \frac{\partial \tau_{11}}{\partial x} \frac{\partial \sigma_{12}}{\partial x}.
\]
(5.6)
Similarly,
\[
(A(\sigma - \Pi_K\sigma), \tau|_{22})_K = -\frac{2\lambda}{3\mu(2\mu + 2\lambda)} \frac{\partial \tau_{22}}{\partial y} \frac{\partial \sigma_{11}}{\partial y}.
\]
(5.7)
A summation over these two components leads to
\[ |(A(\sigma - \Pi h \sigma), \tau)_{\Sigma_h}| \leq C \|\text{div}\tau\|_{0,K} \|\sigma\|_{1,K}. \]

The final result follows from a scaling argument. ■

A combination of the above lemma and (5.3) yields the following result.

**Lemma 5.3.** Suppose that \( \sigma \in H^2(\Omega; \mathbb{S}) \). Then it holds that
\[
\|\Pi h \sigma - \sigma_h\|_0^2 + \|P_h u - u_h\|_0^2 \leq C(\Pi h \sigma - \sigma, \Pi h \sigma - \sigma_h) + Ch^4 \|\sigma\|_2^3 + Ch^4 |u|_2^2.
\]

**Proof.** It follows from Theorem 3.1 that there exists \( \tau = \text{diag}(\tau_{11}, \tau_{22}) \in \Sigma_h \) such that
\[
\text{div}_h \tau = u_h - P_h u \quad \text{and} \quad \|\tau\|_{H(\text{div}_h)} \leq C\|u_h - P_h u\|_0.
\]

This allows for the following decomposition:
\[
(u_h - P_h u, u_h - P_h u) = (u_h - P_h u, \text{div}_h \tau) = (A(\sigma - \sigma_h), \tau) - (A\sigma, \tau) - (u, \text{div}_h \tau) = (A(\sigma - \Pi h \sigma), \tau) + (A(\Pi h \sigma - \sigma_h), \tau) - ((A\sigma, \tau) + (u, \text{div}_h \tau)).
\]

Since \( \tau \) is a diagonal matrix, it follows from (5.4) and (5.9) that
\[
(A(\sigma - \Pi h \sigma), \tau) \leq Ch^2 |\sigma|_1 \|\text{div}_h \tau\|_0 \leq Ch^2 |\sigma|_1 \|u_h - P_h u\|_0.
\]

A substitution of this inequality into the previous equation, by the Cauchy–Schwarz inequality, (4.11), and (5.9), leads to
\[
\|u_h - P_h u\|_0 \leq C(h^2 |\sigma|_1 + \|\Pi h \sigma - \sigma_h\|_0 + h^{3/2} |u|_3).
\]

On the other hand, let \( \tilde{\tau} = \Pi h \sigma - \sigma_h \), we have
\[
(A(\Pi h \sigma - \sigma_h), \tilde{\tau}) \leq (A(\Pi h \sigma - \sigma), \tilde{\tau}) + Ch^2 |\sigma|_3 \|P_h u - u_h\|_0 + Ch^{3/2} |u|_3 \|\Pi h \sigma - \sigma_h\|_0 \leq (A(\Pi h \sigma - \sigma), \tilde{\tau}) + Ch^{3/2} |u|_3 \|\Pi h \sigma - \sigma_h\|_0 + Ch^4 |\sigma|_1 \|\Pi h \sigma - \sigma_h\|_0.
\]

It follows from (5.3) and (5.10),
\[
(A(\Pi h \sigma - \sigma_h), \tilde{\tau}) \leq (A(\Pi h \sigma - \sigma), \tilde{\tau}) + Ch^4 |\sigma|_2 \|\Pi h \sigma - \sigma_h\|_0 + Ch^4 |\sigma|_2 \|\Pi h \sigma - \sigma_h\|_0.
\]

Since there exists a positive constant \( \beta \) such that
\[
\beta \|\Pi h \sigma - \sigma_h\|_0^2 \leq (A(\Pi h \sigma - \sigma_h), \tilde{\tau}),
\]

an application of the Young inequality leads to
\[
\|\Pi h \sigma - \sigma_h\|_0^2 + \|P_h u - u_h\|_0^2 \leq C(A(\Pi h \sigma - \sigma), \tilde{\tau}) + Ch^4 |\sigma|_2^3 + Ch^4 |u|_2^3,
\]

which completes the proof. ■

Since \( \sigma_{12} - \Pi_{12,K} \sigma_{12} = 0 \) for any \( \sigma_{12} \in P_1(K) \), a similar argument of (5.3) can prove the following superstokes result.

**Lemma 5.4.** For any \( \tau_{12} \in \Sigma_{12,h} := \{e_1' \tau e_2, \tau \in \Sigma_h\} \), it holds that
\[
(\sigma_{12} - \Pi_{12,K} \sigma_{12}, \tau_{12}) \leq Ch^2 |\sigma_{12}|_2 \|\tau_{12}\|_0,
\]

provided that \( \tau_{12} \in H^2(\Omega) \). Here \( e_1 \) and \( e_2 \) are the 1-st and 2-nd canonical basis of the space \( \mathbb{R}^2 \), respectively.

**Theorem 5.2.** Let \( (\sigma, u) \) and \( (\sigma_h, u_h) \) be solutions of problems (1.1) and (2.6), respectively. Suppose that \( \sigma \in H^2(\Omega; \mathbb{S}) \) and \( u \in H^4(\Omega; \mathbb{R}^2) \). Then it holds that
\[
(A(\sigma - \Pi h \sigma), \sigma_h - \Pi h \sigma) \leq Ch^{3/2} |\sigma|_2 \|\sigma_h - \Pi h \sigma\|_0.
\]
Proof. Let \( \tau = \sigma_h - \Pi_h \sigma \). Given element \( K \), let \( I_{1,K} : L^2(K; \mathbb{S}) \to P_1(K; \mathbb{S}) \) be the \( L^2 \) projection operator defined as: Given \( v \in L^2(K; \mathbb{S}) \), find \( I_{1,K}v \in P_1(K; \mathbb{S}) \) such that
\[
\int_K (I_{1,K}v) q \, dx \, dy = \int_K v q \, dx \, dy \quad \text{for any } q \in P_1(K; \mathbb{S}).
\]
This leads to the following decomposition:
\[
(A(\sigma - \Pi_h \sigma)_{11}, \tau_{11}) = \sum_{K \in \mathcal{T}_h} (A(\sigma - \Pi_K \sigma)_{11}, \tau_{11})_K
\]
\[
= \sum_{K \in \mathcal{T}_h} (A((I - \Pi_K)I_{1,K} \sigma)_{11}, \tau_{11})_K + \sum_{K \in \mathcal{T}_h} (A((I - \Pi_K)(I - I_{1,K}) \sigma)_{11}, \tau_{11})_K.
\]
By (5.6),
\[
(A((I - \Pi_K)I_{1,K} \sigma)_{11}, \tau_{11})_K = \frac{h^2}{4} \int_K A((I - \hat{h}_K)I_{1,K} \hat{\sigma})_{11} : \hat{\tau}_{11} \, dx \, dy
\]
\[
= -\frac{h^2}{4} \frac{2\lambda}{3(2\mu + 2\lambda)} \frac{\partial \hat{\tau}_{11}}{\partial \hat{x}} \frac{\partial (I_{1,K} \hat{\sigma})_{22}}{\partial \hat{x}}
\]
\[
= -\frac{h^2}{16} \frac{2\lambda}{3(2\mu + 2\lambda)} \left( \frac{\partial (I_{1,K} \sigma)_{22}}{\partial x}, \frac{\partial \tau_{11}}{\partial x} \right)_K.
\]
Summing over all the elements yields
\[
(A(\sigma - \Pi_h \sigma)_{11}, \tau_{11}) \leq Ch^2 \sum_{K \in \mathcal{T}_h} \left( \frac{\partial (I_{1,K} \sigma)_{22}}{\partial x}, \frac{\partial \tau_{11}}{\partial x} \right)_K + Ch^2 \| \sigma \|_2 \| \tau_{11} \|_0.
\]
Since \( \frac{\partial \tau_{11}}{\partial x} \) is a constant by definition, we have \( \left( \frac{\partial (I_{1,K} \sigma)_{22}}{\partial x}, \frac{\partial \tau_{11}}{\partial x} \right)_K = 0 \) for all \( q \in P_2(K) \),
\[
\left( \frac{\partial (I_{1,K} \sigma - \sigma)_{22}}{\partial x}, \frac{\partial \tau_{11}}{\partial x} \right)_K \leq Ch | \sigma |_{2,K} | \tau_{11} |_{1,K}.
\]
A combination of the above two inequalities and an application of inverse inequality yield
\[
(A(\sigma - \Pi_h \sigma)_{11}, \tau_{11}) \leq -Ch^2 \sum_{K \in \mathcal{T}_h} \left( \frac{\partial (I_{1,K} \sigma)_{22}}{\partial x}, \frac{\partial \tau_{11}}{\partial x} \right)_K + Ch^2 \| \sigma \|_2 \| \tau_{11} \|_0.
\]
(5.14)
Since the integral of the jump \( [\tau_{11}]_e \) across the edges \( e \) vanishes for all interior edges \( e \in \mathcal{E}_{h,v} \), we have
\[
\left( \frac{\partial \sigma_{22}}{\partial x}, \frac{\partial \tau_{11}}{\partial x} \right)_e = -\left( \frac{\partial^2 \sigma_{22}}{\partial x^2}, \tau_{11} \right)_e + \sum_{e \in \mathcal{E}_{h,v}} \int_e [\tau_{11}]_e \frac{\partial \sigma_{22}}{\partial x} \, ds
\]
\[
= -\left( \frac{\partial^2 \sigma_{22}}{\partial x^2}, \tau_{11} \right)_e + \sum_{e \in \mathcal{E}_{h,v} \cap \partial \Omega} \int_e \frac{\partial \sigma_{22}}{\partial x} \, ds.
\]
(5.15)
In order to use Lemma 4.1, for any \( e \in \mathcal{E}_{h} \cap \partial \Omega \), let \( K_e \) be the unique element such that \( e \) is one of its edges. This, the trace theorem, inverse estimate and triangle inequality, lead to
\[
\int_e \tau_{11} \frac{\partial \sigma_{22}}{\partial x} \, ds = \int_e \tau_{11}(1 - \Pi_0_{K_e}) \frac{\partial \sigma_{22}}{\partial x} \, ds + \Pi_0_{K_e} \frac{\partial \sigma_{22}}{\partial x} \int_e \tau_{11} \, ds
\]
\[
\leq C \| \tau_{11} \|_{0,K_e} | \sigma_{22} |_{2,K_e} + Ch^{-1} \left\| \Pi_0_{K_e} \frac{\partial \sigma_{22}}{\partial x} \right\|_{0,K_e} \| \tau_{11} \|_{0,K_e}
\]
\[
\leq C \| \tau_{11} \|_{0,K_e} | \sigma_{22} |_{2,K_e} + Ch^{-1} \left\| \frac{\partial \sigma_{22}}{\partial x} \right\|_{0,K_e} \| \tau_{11} \|_{0,K_e}.
\]
(5.16)
Summing over \( e \in \mathcal{E}_{h,v} \cap \partial \Omega \) and taking in account the error estimates presented in Theorem 4.1 for the interpolation operator \( \Pi_0 \) and convergence from Theorem 4.3 for the finite element solution \( \sigma_h \), we arrive at
\[
\sum_{e \in \mathcal{E}_{h,v} \cap \partial \Omega} \int_e \tau_{11} \frac{\partial \sigma_{22}}{\partial x} \, ds \leq C \| \tau_{11} \|_{0,\Omega} (| \sigma |_{2,3,h} + h^{-1} | \sigma |_{1,3,h}).
\]
Hence it follows from Lemma 4.1 that
\[
\sum_{e \in E_h} \int_{\Gamma} \tau_{11} \frac{\partial \sigma_{22}}{\partial x} \, ds \leq C \|\tau_{11}\|_{0,\partial\Gamma_h}(\|\sigma\|_{2,\Gamma} + h^{-1/2}\|\sigma\|_{3,\Gamma})).
\] (5.17)

A summary of (5.14) through (5.17) shows that
\[
(A(\sigma - \Pi_h\sigma), \tau) \leq Ch^{3/2} \|\sigma\|_2 \left(\|\tau_{11}\|_0^2 + \|\tau_{22}\|_0^2\right)^{1/2}.
\] (5.18)

Finally, let \(\sigma_n = \text{diag}(\sigma_{11}, \sigma_{22})\), \(\tau_n = \text{diag}(\tau_{11}, \tau_{22})\), \(\sigma = \sigma - \sigma_n\) and \(\tau = \tau - \tau_n\). The previous equation, the estimate (5.12) for the shear stress and estimates in Theorem 4.1 and Theorem 4.3, yield
\[
(A(\sigma - \Pi_h\sigma), \tau) = (A(\sigma - \Pi_h\sigma_n), \tau_n) + (A(\sigma_n - \Pi_h\sigma_n), \tau_n) \leq C h^{3/2} \|\sigma\|_2 \|\tau\|_0.
\]

This completes the proof. ■

**Theorem 5.3.** Let \((\sigma, u)\) and \((\sigma_n, u_n)\) be solutions of problems (1.1) and (2.6), respectively. Suppose that \(\sigma \in H^3(\Omega; \mathbb{S})\) and \(u \in H^1_h(\Omega; \mathbb{R}^2) \cap H^3(\Omega; \mathbb{R}^2)\). Then there holds that
\[
\|\sigma_n - \Pi_h\sigma\|_{H^2(div_h)}^2 + \|u_n - P_hu\|_1^2 \leq C h^3(\|\sigma\|_3^2 + \|u\|_3^2).
\] (5.19)

**Proof.** It is a consequence of (5.8) and (5.13). ■

### 6. The minimal element in any spatial dimension

Assume the domain \(\Omega\) is a unit hypercube \([0, 1]^n\) in the \(n\)-dimensional space, which is subdivided by a uniform rectangular grid of \(N^n\) cubes:
\[T_h := \{K_1, K_2, \ldots, K_N\} = \{(i_1 - 1)h, i_1h] \times \cdots \times (i_n - 1)h, i_nh] \times [0, h] / i_1, \ldots, i_n \leq N; h = 1/N\}.
\]
The set of all \((n - 1)\)-dimensional face hyperplanes of the triangulation \(T_h\) that are perpendicular to the axis \(x_i\) is denoted by \(E_{n-1,i}\). That is
\[
e_{n-1,i} = \left\{((i_1 - 1)h, i_1h) \times \cdots \times ((i_n - 1)h, i_nh) \times [i_1h] \times \cdots \times [i_nh], \quad \begin{array}{l}
1 \leq i_1, \ldots, i_n \leq N, \\
0 \leq i_i \leq N.
\end{array}
\right\}
\]
The internal hyperplanes are denoted by
\[
e_{n-1,i}(\Omega) = \bigcap_{1 \leq i \leq n} E_{n-1,i} \cap \Omega.
\]
The set of all \((n - 2)\)-dimensional mid-surface hyperplanes (orthogonal to both \(x_i\) and \(x_j\) axes) is denoted by
\[
e_{n-2,ij} := \left\{((i_1 - 1)h, i_1h) \times \cdots \times (i_ih) \times \cdots \times \left[\left(i_j - \frac{1}{2}\right)h\right] \times \cdots \times \begin{array}{l}
(i_1 - 1)h, i_1h), \quad 1 \leq i_1, \ldots, i_n \leq N, \\
0 \leq i_i \leq N
\end{array}\right\}.
\]
In addition, define \(E_{n-2,ij}(K) := E_{n-2,ij} \cap \partial K\) for any \(K \in T_h\). In 2D, these sets are
\[
e_{1,1} = \text{all edges in } T_h \text{ perpendicular to } x_1, \\
e_{1,2} = \text{all edges in } T_h \text{ perpendicular to } x_2, \\
e_{0,1,2} = \text{all mid-points of edges in } T_h.
\]
In 3D, they are
\[
e_{2,1} = \text{all squares in } T_h \text{ perpendicular to } x_1, \quad 1 \leq i \leq 3, \\
e_{1,ij} = \text{all mid-square edges of squares in } E_{2,ij} \text{ and } E_{2,ji} \text{ parallel to } x_i, \\
i \neq j \neq k \in \{1, 2, 3\}.
\]
In \(n\) space-dimension, the symmetric tensor space is defined in (1.2). The discrete stress space is defined by
\[
\Sigma_h := \left\{ \tau_{ij} \in L^2(\Omega; \mathbb{R}^{n \times n}) \mid \tau_{ij} = \tau_{ji}, \\
|\tau_{ij}| \in \text{span}\{1, x_i, x_j\}, \tau_{ij} \text{ is continuous on } E_i \in E_{n-1,i}; \\
|\tau_{ij}| \in \text{span}\{1, x_i, x_j\}, \tau_{ij} \text{ is continuous on } E_j \in E_{n-2,ij}(\Omega) \right\}.
\] (6.1)

Some comments are in order for this family of minimal finite element spaces.
Remark 6.1. The normal stress \( \tau_{ii} \) is a constant on each \((n - 1)\)-dimensional hyper-plane \( E_i \in \mathcal{E}_{n-1,i} \). In addition, for the case \( n = 1 \), \( \Sigma_h \) is

\[
\{ \tau_{ii} \in L^2(\Omega; \mathbb{R}) \mid |\tau|_K \in \text{span}\{1, x\} \text{ is continuous at the nodes} \} \subset H^1(\Omega),
\]

the 1D Raviart–Thomas element space, which is the only conforming space in this family.

Remark 6.2. The dimension of the space

\[
\Sigma_{h,y} := \{ \tau_{ij} \in L^2(\Omega; \mathbb{R}) \mid \tau_{ij}|_K \in \text{span}\{1, x_i, x_j\}, \tau_{ij} \text{ is continuous on } E_{ij} \in \mathcal{E}_{n-2,y}(\Omega) \}
\]

is

\[
N^{n-2}((N + 1)^2 - 1) = N^n + 2N^{n-1}.
\]

see [49,48] for more details for 2D.

Let us give the local basis for \( \tau_{ij} \) and but a local frame (not basis) for \( \tau_{ij} \) on an element \( K := K_{i_1,i_2,...,i_n} \in \mathcal{T}_h \). Define, for \((x_1, \ldots, x_n) \in K\),

\[
\psi^{(k)}_{i,j}(x_1, \ldots, x_n) = \hat{\psi}^{(k)}(\frac{x_i - (i - 1/2)h}{h/2}, x_j - (j - 1/2)h, \frac{h}{2}), \quad k = 0, 1,
\]

where

\[
\hat{\psi}^{(0)}(\hat{x}) = \frac{1 - \hat{x}}{2}, \quad \hat{\psi}^{(1)}(\hat{x}) = \frac{1 + \hat{x}}{2}, \quad \hat{x} \in [-1, 1].
\]

Define, for \( k = 0, 1, 2, 3 \), for \((x_1, \ldots, x_n) \in K\),

\[
\phi^{(k)}_{i,j}(x_1, \ldots, x_n) = \hat{\phi}^{(k)}(\frac{x_i - (i - 1/2)h}{h/2}, x_j - (j - 1/2)h, \frac{h}{2}),
\]

where \( \hat{\phi}^{(k)}(\hat{x}, \hat{y}) \), \( k = 0, \ldots, 3 \), are defined in (2.9) (cf. Fig. 1).

As in 2D, the discrete displacement space is

\[
V_h = \{ v \in L^2(\Omega; \mathbb{R}^n) \mid v|_K \text{ is a constant vector} \}. \tag{6.2}
\]

In the \( n \)-dimension, since \( \text{div}_h \Sigma_h \subset V_h \), the K-ellipticity (3.1) is proved exactly the same way as in 2D. The explicit construction proof of the discrete B–B condition (3.2) can be divided into \( n \) essentially 1-dimensional construction proofs similar to that for the 1D Raviart–Thomas element of the 1D Poisson equation, see Section 3 for more details for 2D. For the consistency error, the proof remains the same except there is a multiple summation instead of 2-index summation. All the analysis in 2D remains the same for \( n \)-D.

7. Numerical tests

Two examples in 2D and one in 3D are presented to demonstrate the methods. These are pure displacement problem with a homogeneous boundary condition that \( u \equiv 0 \) on \( \partial \Omega \). Assume the material is isotropic in the sense that

\[
A\sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + n\lambda} \text{tr}(\sigma) \delta \right), \quad n = 2, 3, \tag{7.1}
\]

where \( \delta \) is the identity matrix, and \( \mu \) and \( \lambda \) are the Lamé constants such that \( 0 < \mu_1 \leq \mu \leq \mu_2 \) and \( 0 < \lambda < \infty \).

In 2D, let the exact solution on the unit square \([0, 1]^2\) be

\[
u = \begin{pmatrix} 4x(1-x)y(1-y) \\ -4x(1-x)y(1-y) \end{pmatrix}, \tag{7.2}
\]

and

\[
u = \begin{pmatrix} e^{x-y}(1-x)y(1-y) \\ \sin(\pi x) \sin(\pi y) \end{pmatrix}. \tag{7.3}
\]

Notice that the second example is from [24].

In 2D, the parameters in (7.1) are chosen as

\[
\lambda = 1 \quad \text{and} \quad \mu = \frac{1}{2}.
\]

Then, the true stress function \( \sigma \) and the load function \( f \) are defined by the equations in (1.1), for the given solution \( u \).
In the computation, the level one grid is the given domain, a unit square or a unit cube. Each grid is refined into a half-size grid uniformly, to get a higher level grid, see the first column in Table 1. In Table 1, the errors and the convergence order in various norms are listed for the true solution (7.2). Here and in rest of the tables in the section, \( I_h \) is the usual nodal interpolation operator. For example, \( I_h u_1(x_i + h/2, y_j + h/2) = u_1(x_i + h/2, y_j + h/2) \), and \( I_h \sigma_{12} = I_{h0} \sigma_{12} \), defined in (2.10). An order 2 convergence is demonstrated, see Table 1. In the second half of Table 1, the true errors and convergence orders are listed. The numerical results match the theory of the first order convergence.

The next example, (7.3), of Yi [24] is implemented for a comparison. The finite element errors and the order of convergence are listed in Table 2. An order 2 convergence is again observed. Notice that the minimal element of this paper has a much less dof than that of Yi, but has one order higher of convergence. In the second half of Table 2, the errors between the true solution and numerical solutions, and the order of convergence, are listed. The numerical results match the theory of the first order convergence.

As a third example, we compute a 3D solution for the following exact solution:

\[
\begin{align*}
    u &= \begin{pmatrix}
        16(1 - x)y(1 - y)z(1 - z) \\
        32x(1 - x)y(1 - y)z(1 - z) \\
        64x(1 - x)y(1 - y)z(1 - z)
    \end{pmatrix},
\end{align*}
\]

(7.4)
on the unit cube \([0, 1]^3\). This time, the parameters in (7.1) are taken as

\[
    \lambda = 1, \quad \mu = \frac{1}{2} \quad \text{and} \quad n = 3.
\]

Again the order of convergence is still one higher than what is proved in this paper, see the top part of Table 3. The proved theorem is confirmed by the second half of Table 3.
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### References


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