A superconvergence of the Morley element via postprocessing

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Abstract. Via an equivalence between the Morley element for the biharmonic equation and the Crouzeix-Raviart element for the Stokes equations, we establish a postprocessing algorithm which produces an half order superconvergence for the Morley element. Numerical tests verify the theory.

1. Introduction

The Morley finite element is a nonconforming element for fourth order elliptic partial differential equations ([2, 10, 15]), in particular, for the biharmonic equation:

\[
\begin{align*}
\Delta^2 \varphi &= f \quad \text{in } \Omega, \\
\varphi &= \frac{\partial}{\partial n} \varphi = 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (1.1)

Here \( f \in L^2(\Omega) \), \( \Omega \) is a polygonal domain and \( n \) is the unit outward normal vector at a boundary point. Due to simplicity, the Morley element is widely used and analyzed. It is interesting to mention that the extended Morley elements for higher order equations and in higher space dimensions were constructed and discussed in [11, 12, 17, 18].

In this paper, we study a superconvergence property of the Morley element. Superconvergence is discovered for many continuous finite elements [1, 5, 6, 8, 13, 16, 23, 25, 24, 26]. But for nonconforming elements, such as the Morley element, it is difficult to show the superconvergence. So far, there is only one superconvergence result for the Morley element. In [9], an half order superconvergence is proved for the Morley element solution on a type of uniform grids, cf. Figure 1. If the solution of (1.1) \( \varphi \in H^4(\Omega) \cap H_0^2(\Omega) \),

\[
|\varphi - I_{3h}^3 \varphi|_{2,h} \leq C h^{3/2}(|\varphi|_{4,\Omega} + \|f\|_{0,\Omega}),
\]
where $I^3_{3h}$ is the nodal, bicubic interpolation operator, defined on squares formed by 18 triangles (cf. Figure 1).

\[ \begin{array}{c}
\text{Morley} \rightarrow C^0-Q_3
\end{array} \]

**Figure 1.** A nodal interpolation operator $I^3_{3h}$ used by [9].

Instead of analyzing the Morley solution, we derive a superconvergence from its equivalent solution to the Stokes equations. We note that the solution $\varphi$ of (1.1) is related by $u (= \text{curl} \varphi)$ to the unique solution of the Stokes equation:

\[ \begin{aligned}
-\Delta u + \nabla \lambda &= g \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned} \tag{1.2} \]

where $g$ satisfies curl $g = f$, for example, $g = (0, \int f \, dx)$. Not only the continuous solutions are related, but also discrete solutions. Falk and Morley [7] showed that the curl of the Morley solution to (1.1) is also the Crouzeix-Raviart solution to (1.2). Based on this equivalence and a superconvergence result of Ye [20] on the Crouzeix-Raviart element, we derive an half order superconvergence ($k = 3$ in Theorem 4.2) for the Morley element. Here, the curl of a Morley solution is locally projected (cf. Figure 2) to provide a superconvergence approximation of $\varphi$ in (semi, discrete) $H^1$ and $H^2$ norms. Neither [9] or our result is optimal yet. Numerical results in [9] and here in last section indicate that a full order superconvergence is possible. A further study should be done.

\[ \begin{array}{c}
Q^k_H : C^{-1}-P_1 \rightarrow C^{-1}-P_k
\end{array} \]

**Figure 2.** A local $L^2$-projection operator $Q^k_H$, from $C^{-1}-P_1$ to $C^{-1}-P_2$.

### 2. The continuous equivalent formulation

Consider the biharmonic problem (1.1). Suppose that there exists a vector field $g \in (H^1(\Omega))^2$ satisfying $f = \text{curl } g := \partial_2 g_2 - \partial_1 g_1$. Then, by integration by parts, the variational formulation for (1.1) is available: Finding $\varphi \in H^2_0(\Omega)$ such that

\[ \int_\Omega \nabla^2 \varphi : \nabla^2 \vartheta \, dx = \int_\Omega f \vartheta \, dx = \int_\Omega g \cdot \text{curl} \vartheta \, dx \quad \forall \vartheta \in H^2_0(\Omega), \tag{2.1} \]

where $\text{curl } \vartheta := (\partial_2 \vartheta,-\partial_1 \vartheta)^T$. 

2.1. Let \( \phi \in H^2_0(\Omega) \) be the solution of (2.1), and \( (u, \lambda) \in (H^1_0(\Omega))^2 \times L^2_0(\Omega) \) be the solution of the Stokes equation

\[
\begin{cases}
\int_\Omega \nabla u : \nabla v \, dx + \int_\Omega \lambda \nabla \cdot v \, dx = \int_\Omega g \cdot v \, dx & \forall v \in (H^1_0(\Omega))^2, \\
\int_\Omega \mu \nabla \cdot u \, dx = 0 & \forall \mu \in L^2_0(\Omega).
\end{cases}
\]

Then

\[ u = \text{curl} \phi \]

**Proof.** From the second equation of (2.2), we find that \( \text{div} u = 0 \). Thus, there exists a certain \( \tilde{\phi} \in H^2_0(\Omega) \) such that \( u = \text{curl} \tilde{\phi} \). Substituting the expression into the first equation of (2.2) and taking \( v = \text{curl} \vartheta \) for any \( \vartheta \in H^2_0(\Omega) \), we have

\[
\int_\Omega \nabla^2 \tilde{\phi} : \nabla^2 \vartheta \, dx = \int_\Omega g \cdot \text{curl} \vartheta \, dx \quad \forall \vartheta \in H^2_0(\Omega).
\]

Therefore, \( \tilde{\phi} \) is identically equal to \( \phi \), in terms of the unique solvability of (2.1), leading to the desired result. \( \square \)

### 3. The discrete equivalent formulation

Theorem 2.1 establishes an equivalence between the biharmonic equation and the Stokes equation. We are going to obtain a discrete analogue of this equivalence.

Let \( \{T_h(\Omega)\}_{h > 0} \) be a non-degenerate family of triangulations of \( \Omega \) \([3, 4]\); \( h := \max_{\tau \in T_h} h_\tau \) and \( h_\tau := \text{diam}(\tau) \). Let \( E_h \) be the union of all edges of the triangulation \( T_h(\Omega) \) and \( \partial E_h^\circ \) the union of all edges of the triangulation \( T_h(\Omega) \) on \( \partial \Omega \). Set \( (\tau, P_\tau, \Sigma_\tau) \) be the Morley element with \( \tau \) the geometric shape, \( P_\tau \) the shape function space \( P_k(\tau) \) and \( \Sigma_\tau \) the set of nodal parameters \( \{\vartheta(p_i),\partial_n \vartheta(m_i), 1 \leq i \leq 3\} \) where \( P_k(\tau) \) is the space of all \( k \)-th order polynomials on \( \tau \) for any nonnegative integer \( k \), \( p_i \) is the vertex of \( \tau \) and \( m_i \) the edge midpoint of \( \tau \). \( m_i \) is opposed to \( p_i \).

![Figure 3. The nodal definition of the Morley element, cf. (3.1).](image)

The Morley finite element space is defined, on the grid \( T_h(\Omega) \), as follows, cf. Figure 3.

\[
V^M_h = \{\vartheta; \partial \vartheta|_\tau \in P_2(\tau), \forall \tau \in T_h(\Omega), \vartheta(\text{resp. } \partial_n \vartheta) \text{ is continuous at each vertex } p \text{ of } \tau(\text{resp. each edge midpoint } m \text{ of } \tau)\},
\]

\[
V^M_{h,0} = \{\vartheta \in V^M_h(\Omega); \vartheta(p) = \partial_n \vartheta(m) = 0, \forall p, m \in \partial \Omega\}.
\]
Let the Crouzeix-Raviart element (P₁-nonconforming) spaces be defined as follows.

\begin{align}
V_{CR}^h &= \{ v \mid v|_\tau \in P_1, \text{continuous at each edge midpoint } m \in \mathcal{T}_h(\Omega) \}, \\
V_{CR}^h;0 &= \{ v \in V_{CR}^h(\Omega); v(m) = 0, \forall m \in \partial \Omega \}.
\end{align}

For convenience, we let the recovery space for superconvergence be

\begin{align}
V^k_h &= \{ v \in L^2(\Omega); v|_\tau \in P_k(\tau), \forall \tau \in \mathcal{T}_h(\Omega) \}, \quad \text{for any } k \geq 0.
\end{align}

And let \( V^0_{h,0} = V^0_h \cap L^2(\Omega) \) be the finite element space for the pressure, i.e., piecewise constant space on the grid \( \mathcal{T}_h(\Omega) \). For any function space \( B \), let \( B = B \times B \). For example, \( V_{CR}^h \times V_{CR}^h \). Then, the mixed finite element space for the Stokes equations (2.2) is \( V_{CR}^h \times V^0_{h,0} \), cf. Figure 4.

With these notations, the Morley element method for problem (2.1) is defined as follows. Find \( \phi_h \in V^M_h \) such that

\begin{align}
\int_\Omega \nabla^2 \phi_h : \nabla^2 \vartheta \, dx = \int_\Omega g : \text{curl}_h \vartheta \, dx \quad \forall \vartheta \in V^M_{h,0}.
\end{align}

Here and later, with discrete \( \nabla_h \) operator, an integral is understood as its piecewise sum, i.e.,

\begin{align}
\int_\Omega \nabla_h^2 \phi_h : \nabla^2_h \vartheta \, dx = \sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla_h^2 \phi_h : \nabla^2_h \vartheta \, dx.
\end{align}

The discrete variational formulation for (2.2) is to find \( \mathbf{u}_h \in V_{CR}^h \times V^0_{h,0} \) such that

\begin{align}
\int_\Omega \nabla_h \mathbf{u}_h : \nabla_h \mathbf{v} \, dx - \int_\Omega \lambda_h \nabla_h \cdot \mathbf{v} \, dx = \int_\Omega g : \mathbf{v} \, dx \quad \forall \mathbf{v} \in V_{CR}^h, \\
\int_\Omega \mu \nabla_h \cdot \mathbf{u}_h \, dx = 0 \quad \forall \mu \in V^0_{h,0}.
\end{align}

Before proving the equivalence between (3.5) and (3.6), we recall an important result, given in [7]. Define

\begin{align}
Z_h = \left\{ \mathbf{v}_h \in V_{h,0}^{CR}; \int_\Omega \mu \nabla_h \cdot \mathbf{v}_h \, dx = 0, \forall \mu \in V^0_h \right\}.
\end{align}

Then we have

\begin{align}
Z_h = \text{curl}_h V^M_{h,0}.
\end{align}

**Theorem 3.1**. Let \( \phi_h \in V^M_{h,0} \) be the solution of (3.5) and \( (\mathbf{u}_h, \lambda_h) \in V_{CR}^h \times V^0_{h,0} \) the solution of (3.6). The discrete variational formulation (3.5) is equivalent to (3.6):

\begin{align}
\mathbf{u}_h = \text{curl}_h \phi_h.
\end{align}
where (2.2) has a unique solution $\mathbf{u}_h = \mathbf{curl}_h \tilde{u}_h$. Substituting it into the first equation of (3.6) and taking

$$
\mathbf{v} = \mathbf{curl}_h \vartheta
$$

for any $\vartheta \in V_{h,0}^M$, we have

$$
\int_{\Omega} \nabla^2 \tilde{u}_h : \nabla^2 \vartheta \, dx = \int_{\Omega} \mathbf{g} \cdot \mathbf{curl}_h \vartheta \, dx \quad \forall \vartheta \in V_{h,0}^M.
$$

Using the unique solvability of (3.5), we get $\varphi_h = \varphi$. Therefore $\mathbf{u}_h = \mathbf{curl}_h \varphi_h$. \hfill $\square$

4. A postprocessing algorithm

In what follows, we always assume that $\varphi \in H^2_0(\Omega)$ is the solution of (2.1), and $(\mathbf{u}, \lambda) \in (H^1_0(\Omega))^2 \times L^2_0(\Omega)$ is the solution of the Stokes equation (2.2); $\varphi_h \in V_{h,0}^M$ is the solution of (3.5), and $(\mathbf{u}_h, \lambda_h) \in V_{h,0}^{CR} \times V_h^0$ is the solution of (3.6). Assume the triangulation $T_h(\Omega)$ is quasi-uniform in this section. Let $T_H(\Omega)$ be a coarser quasi-uniform triangulation with mesh size $H$ satisfying $h \ll H$ and

$$
H = h^\alpha,
$$

where $\alpha \in (0, 1)$. Denote $Q_H^k$ to be the $L^2$-orthogonal projector from $L^2(\Omega)$ onto the space $V_H^k$.

Hereafter we assume $k > 1$ and let $\alpha = \frac{2}{k+1}$. From [20], we have the following superconvergence result for problem (3.6).

Lemma 4.1. Assume $\Omega$ is so regular that for any $\mathbf{g} \in H^{s-2}(\Omega)$, the problem (2.2) has a unique solution $\mathbf{u} \in H^s_0(\Omega) \cap H^s(\Omega)$ and $\lambda \in H^{s-1}(\Omega) \cap L^2_0(\Omega)$ satisfying a priori estimate

$$
\|\mathbf{u}\|_s + \|\lambda\|_{s-1} \leq C\|\mathbf{g}\|_{s-2},
$$

where $1 \leq s \leq 2$ and $C$ is a constant independent of $\mathbf{g}$. If $(\mathbf{u}, \lambda) \in H^2(\Omega) \cap H^{k+1}(\Omega) \cap H^k_0(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega)$, then

$$
\|\mathbf{u} - Q_H^k \mathbf{u}_h\|_{1,H} \leq C h^{2k/(k+1)} (\|\mathbf{u}\|_2 + \|\mathbf{u}\|_{k+1} + \|\lambda\|_1).
$$

Moreover, if $\Omega$ is convex, then

$$
\|\mathbf{u} - Q_H^k \mathbf{u}_h\|_{1,H} \leq C h^{2k/(k+1)} (\|\mathbf{g}\|_0 + \|\mathbf{u}\|_{k+1}).
$$

Theorem 4.2. Assume $\Omega$ is convex. If $\varphi \in H^2_0(\Omega) \cap H^{k+2}(\Omega)$, then

$$
\|\mathbf{curl}\varphi - Q_H^k \mathbf{curl}_h \varphi_h\|_{1,H} \leq C h^{2k/(k+1)} (\|\mathbf{g}\|_0 + \|\varphi\|_{k+2}).
$$

Proof. It follows from Theorems 2.1-3.1 and Lemma 4.1 that

$$
\|\mathbf{curl}\varphi - Q_H^k \mathbf{curl}_h \varphi_h\|_{1,H} = \|\mathbf{u} - Q_H^k \mathbf{u}_h\|_{1,H} \leq C h^{2k/(k+1)} (\|\mathbf{g}\|_0 + \|\varphi\|_{k+2}).
$$

5. Numerical tests

We solve the biharmonic equation (1.1) on a unit square, $\Omega = [0, 1]^2$, by the Morley element. The exact solutions of the biharmonic equation is

$$
\varphi = 2^8(x - x^2)^2(y - y^2)^2.
$$

We also solve the Stokes equations (1.2) on the unit square by the Crouzeix-Raviart mixed element, where the exact solution is, for $\varphi$ being the solution above,

$$
\mathbf{u} = \mathbf{curl} \varphi, \quad \lambda = 0.
$$
The domain is subdivided by a multigrid refinement that the level 3 grid and level 1 grid are plotted in Figure 2. First, we compute the Crouzeix-Raviart solution for (5.2). As the mixed-element is locally divergence-free, cf. [14, 21, 22], the resulting linear systems of equations can be solved efficiently by the iterated penalty method (IPM). In Table 1, we list the number of iterated penalty iterations and the number of conjugate gradient iteration within each IPM iteration. By the second column, we can see there is a natural superconvergence for the CR element in \( H^1 \)-norm for solving the Stokes equations. Here, the optimal order is supposed to be 1 for the \( H^1 \)-norm convergence. But we get 2. In fact, the discrete \( H^1 \)-convergence is even better than that of \( L^2 \).

Table 1. The error \( (e_u = I_h u - u_h) \) of CR element for (5.2).

<table>
<thead>
<tr>
<th>grid</th>
<th>( |e_u|_{L^2} )</th>
<th>( h^k )</th>
<th>( |e_u|_{H^1} )</th>
<th>( h^k )</th>
<th>( |e_\lambda|_{L^2} )</th>
<th>( h^k )</th>
<th>#cg</th>
<th>#ipm</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.5374</td>
<td>1.6</td>
<td>0.3199</td>
<td>1.7</td>
<td>3.72</td>
<td>0.7</td>
<td>41</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0.1921</td>
<td>1.5</td>
<td>0.0945</td>
<td>1.8</td>
<td>1.94</td>
<td>0.9</td>
<td>187</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>0.0596</td>
<td>1.7</td>
<td>0.0255</td>
<td>1.9</td>
<td>0.98</td>
<td>1.0</td>
<td>1048</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>0.0167</td>
<td>1.8</td>
<td>0.0066</td>
<td>2.0</td>
<td>0.49</td>
<td>1.0</td>
<td>3128</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>0.0044</td>
<td>1.9</td>
<td>0.0017</td>
<td>2.0</td>
<td>0.25</td>
<td>1.0</td>
<td>8266</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>0.0011</td>
<td>2.0</td>
<td>0.0004</td>
<td>2.0</td>
<td>0.12</td>
<td>1.0</td>
<td>18626</td>
<td>3</td>
</tr>
</tbody>
</table>

Next, we compute the Morley element solution and its equivalence to the Crouzeix-Raviart solution. When solving the biharmonic equation, there are two ways to compute the right hand side vector:

\[
\begin{align*}
\int_{\Omega} \nabla^2 \varphi_h : \nabla^2 \phi_i \, dx &= \int_{\Omega} f \phi_i \, dx, \quad \text{(5.3)} \\
\int_{\Omega} \nabla^2 \varphi_h : \nabla^2 \phi_i \, dx &= \int_{\Omega} g \cdot \text{curl} \phi_i \, dx, \quad \text{(5.4)}
\end{align*}
\]

where \( f = \text{curl} g \), cf. (2.1). If we use the first method (5.3), then the curl of Morley solution is not the same as the Crouzeix-Raviart solution. This is shown in the data on the left of Table 2. That is, the quadrature formula for (5.3) and for the Stokes equations (3.6) cannot produce equivalent integral values. But if we use method (5.4), the difference between the two solutions is \( 10^{-7} \) times that by (5.3). This can be seen from the last two columns in Table 2. The difference is caused by the iterative error (used the Iterated Penalty Method) in the Stokes equations, not by the two finite elements. Table 2 verifies Theorem 3.1.

Table 2. The difference between two solutions, \( e_h = \text{curl}_h \varphi_h - u_h \).

<table>
<thead>
<tr>
<th>grid</th>
<th>( |e_h|_\infty )</th>
<th>( h^k )</th>
<th>( |e_h|_{L^2} )</th>
<th>( h^k )</th>
<th>( |e_h|_\infty )</th>
<th>( |e_h|_{L^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.25780</td>
<td>0.7225</td>
<td>0.1161E-12</td>
<td>0.1887E-13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.41507</td>
<td>1.6</td>
<td>0.2276</td>
<td>1.7</td>
<td>0.1821E-13</td>
<td>0.1069E-13</td>
</tr>
<tr>
<td>5</td>
<td>0.11893</td>
<td>1.8</td>
<td>0.0611</td>
<td>1.9</td>
<td>0.8661E-11</td>
<td>0.5307E-11</td>
</tr>
<tr>
<td>6</td>
<td>0.03412</td>
<td>1.9</td>
<td>0.0156</td>
<td>2.0</td>
<td>0.2633E-11</td>
<td>0.1589E-11</td>
</tr>
<tr>
<td>7</td>
<td>0.00806</td>
<td>2.0</td>
<td>0.0039</td>
<td>2.0</td>
<td>0.1886E-11</td>
<td>0.1029E-11</td>
</tr>
</tbody>
</table>
Next, we compute the Morley element solution to the biharmonic equation. In Table 3, we list the error and the order of convergence for the Morley element. It converges at a linear order in $H^2$-seminorm. This is the optimal order. Finally, we locally $L^2$-project $\text{curl}\varphi_h$ to $P_2$ polynomials (on the grids two levels lower.) Then we obtain one order higher convergence than the optimal order, both in $H^1$ and in $H^2$ semi-norms. The error and the order of superconvergence are listed in Table 4. This is partially confirms our main theorem, Theorem 4.2, where the order can be only $1/2$ higher. We note that people usually express superconvergence as $|I_h\varphi - \varphi_h|_{H^2}$ before post-processing. But we have post-processed $\varphi_h$ in our method. Thus, from Table 4, $|I_h\varphi - Q^2_H\varphi_h|_{H^2} \leq |I_h\varphi - \varphi|_{H^2} + |\varphi - Q^2_H\varphi_h|_{H^2} = O(h^2)$.

Table 3. The error $(e_h = \varphi - \varphi_h)$ of the Morley element for (5.1).

| grid | $\|e_h\|_{L^2}$ | $h^k$ | $|e_h|_{H^1}$ | $h^k$ | $|e_h|_{H^2}$ | $h^k$ | #cg | #dof |
|------|----------------|------|--------------|------|--------------|------|------|------|
| 4    | 0.04158155     | 1.8  | 0.140097     | 1.8  | 2.68762      | 0.9  | 61   | 225  |
| 5    | 0.01083490     | 1.9  | 0.037031     | 1.9  | 1.36707      | 1.0  | 140  | 961  |
| 6    | 0.00274282     | 2.0  | 0.009430     | 2.0  | 0.68720      | 1.0  | 407  | 3969 |
| 7    | 0.00068801     | 2.0  | 0.002369     | 2.0  | 0.34409      | 1.0  | 1402 | 16129|
| 8    | 0.00017215     | 2.0  | 0.000593     | 2.0  | 0.17211      | 1.0  | 5773 | 65025|

Table 4. The superconvergence ($\hat{e}_h = \varphi - Q^2_H\varphi_h$) of Morley element.

| grid | $|\hat{e}_h|_{H^1}$ | $h^k$ | $|\hat{e}_h|_{H^2}$ | $h^k$ |
|------|-------------------|------|-------------------|------|
| 4    | 0.39262           | 2.0  | 3.17743           | 1.2  |
| 5    | 0.06400           | 2.6  | 1.13909           | 1.5  |
| 6    | 0.00899           | 2.8  | 0.30133           | 1.9  |
| 7    | 0.00131           | 2.8  | 0.07651           | 2.0  |
| 8    | 0.00023           | 2.5  | 0.01945           | 2.0  |

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