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Quadratic divergence-free finite elements on Powell–Sabin tetrahedral grids

Shangyou Zhang

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Abstract Given a tetrahedral grid in 3D, a Powell–Sabin grid can be constructed by refining each original tetrahedron into 12 subtetrahedra. A new divergence-free finite element on 3D Powell–Sabin grids is constructed for Stokes equations, where the velocity is approximated by continuous piecewise quadratic polynomials while the pressure is approximated by discontinuous piecewise linear polynomials on the same grid. To be precise, the finite element space for the pressure is exactly the divergence of the corresponding space for the velocity. Therefore, the resulting finite element solution for the velocity is pointwise divergence-free, including the inter-element boundary. By establishing the inf-sup condition, the finite element is stable and of the optimal order. Numerical tests are provided.

Keywords Mixed finite element · Stokes equations · Divergence-free element · Tetrahedral grid · Powell–Sabin grid

Mathematics Subject Classification (2000) 65M60 · 65N30 · 76M10 · 76D07

1 Introduction

A natural finite element method for the Stokes equations would be the P_k–P_{k-1} element which approximates the velocity in an H^1-subspace of continuous P_k piecewise-polynomials (referred by C_0–P_k), and approximates the pressure in an L^2-subspace of discontinuous P_{k-1} polynomials (referred by C_{-1}–P_{k-1}). This is a truly conforming element as the incompressibility condition is satisfied pointwise and the discrete solution for the velocity is a projection within the space of divergence-free functions. A fundamental study on the method was done by Scott and Vogelius [10,

S. Zhang (✉)
Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
e-mail: szhang@udel.edu

11] that the method is stable and consequently of the optimal order on 2D triangular grids for any $k \geq 4$, provided the grids have no nearly-singular vertex. A 2D vertex of a triangulation is singular if all edges meeting at the vertex form two cross lines. When singular vertices are present, the finite element space for the pressure is not the full C_{-1}–P_{k-1} space, but a proper subspace, that is, the divergence of velocity space of C_0–P_k polynomials. For $k \leq 3$, Scott and Vogelius showed that the P_k–P_{k-1} element would not be stable, and may not produce approximating solutions on general 2D triangular grids in [10, 11]. What is this magic number k in 3D? Scott and Vogelius posted this question explicitly after discovering that $k = 4$ in 2D. The problem is still open.

The geometry of the 3D tetrahedral grids is more complicated. In 2D, by perturbing a vertex or adding a few edges, a singular vertex can be eliminated without altering the triangular grid much. But this cannot be done for tetrahedral grids, to remove a singular vertex or a singular edge, without creating a new singular vertex or singular edge, unless we change grids structurally as in [16]. When a triangular grid is singular-vertex free, it is shown by Scott and Vogelius [10, 11] that the divergence of C_0–P_k vector space is exactly the space of C_{-1}–P_{k-1}. Following the approach, we found previously that this is true, the divergence of a C_0–P_k space is the C_{-1}–P_{k-1} space, on Hsieh–Clough–Tocher tetrahedral grids (cf. Fig. 1, and [16]). Via the macro-element technique of Stenberg [14], it is shown in [16] that the P_k–P_{k-1} element is stable for all $k \geq 3$ on Hsieh–Clough–Tocher tetrahedral grids. Though the P_k–P_{k-1} element is not stable in 2D for $k \leq 3$ in general, as pointed out by Scott and Vogelius [10, 11], some such low order elements can still be stable on certain grids. In [8] and [1], it is shown that P_3–P_2 and P_2–P_1 elements are stable on 2D Hsieh–Clough–Tocher grids. The lowest order element of the family, the P_1–P_0 element, is shown to be stable and to be of the optimal order on 2D Powell–Sabin grids, in [17]. In this work, we study the P_2–P_1 element on 3D Powell–Sabin grids. We will show that the element is stable and provides the optimal order solutions for the Stokes equations. Numerical evidence shows that this is the lowest order working element on the type of grids, as the P_1–P_0 element fails to converge. We note that the analysis here is not a simple extension of [16] or [17]. To solve the 3D Scott–Vogelius problem, the main difficulty is in discovering singular vertices and edges in 3D. Our analysis here is first one in studying a special type of singular vertex, cf. Lemma 3.3.

Given a tetrahedral grid, if we connect the center of maximal inscribed ball of each tetrahedron with its four vertices, and with the four centers of four neighboring tetrahedra, we obtain a corresponding Powell–Sabin grid, where each base tetrahedron is subdivided into 12 tetrahedra, cf. Figs. 1 and 4. Such a Powell–Sabin grid has singular vertices, on the faces of base tetrahedra. Consequently, the space for the pressure is a proper subspace of C_{-1}–P_1 polynomials. It is the divergence of the C_0–P_2 space on the grid. We apply the iterated penalty method [3, 4, 6, 13] for the resulting linear system of equations, where the pressure space is not coded and the finite element solution for the pressure is generated as a byproduct. Therefore we only need to solve a Laplacian-like equation for the velocity. The mixed P_2–P_1 element is reduced to a single P_2 element. This avoids also the difficulty of solving non positive definite problems iteratively.

The rest of the paper is organized as follows. In Sect. 2, we define the P_2–P_1 element on Powell–Sabin grids. We introduce the iterated penalty method for solving the resulting linear systems of equations. In Sect. 3, we will find all the so-called singular vertex, for the discrete pressure space. In Sect. 4, we prove the stability of the P_2–P_1 element, and show the optimal convergence. In Sect. 5, we provide some numerical results.

2 The 3D Powell–Sabin P_2 element

In this section, we shall introduce 3D grids and the Powell–Sabin refinements. We then define the P_2–P_1 Powell–Sabin finite element for the stationary Stokes equations. We will introduce the classic iterated penalty method [3, 4, 6, 13] by which the mixed element is reduced to a single divergence-free element.

We solve a model stationary Stokes problem: Find functions $\mathbf{u}$ (the fluid velocity) and p (the pressure) on a 3D polyhedral domain Ω such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\mathbf{f}$ is the body force. The standard variational form for (2.1) reads: Find $\mathbf{u} \in H_0^1(\Omega)^3$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} &a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad && \forall \mathbf{v} \in H_0^1(\Omega)^3, \\ &b(\mathbf{u}, q) = 0 && \forall q \in L_0^2(\Omega). \end{aligned} \tag{2.2}$$

Here $H_0^1(\Omega)^3$ is the Sobolev space (cf. [5]) with zero boundary trace, $L_0^2(\Omega)$ is the L^2 space with zero mean value, i.e., $L^2(\Omega)/R = \{p \in L^2 \mid \int_\Omega p = 0\}$, and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x}, \\ b(\mathbf{v}, p) &= -\int_\Omega \operatorname{div} \mathbf{u} \, p \, d\mathbf{x}, \\ (\mathbf{f}, \mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}. \end{aligned}$$

Let $\bar{\Omega}_h$ be a family of quasiuniform (with $C > 0$ in (2.4)) tetrahedral grids on Ω (cf. [5]):

$$\bar{\Omega}_h = \left\{ \bar{K} \mid \bar{K} \text{ is a tetrahedron with the longest edge } h_{\bar{K}} \le h \right\}, \tag{2.3}$$

$$C = \min_{\bar{K}, \bar{K}' \in \bar{\Omega}_h} \left\{ \rho_{\bar{K}} / h_{\bar{K}'} \mid \rho_{\bar{K}} \text{ is the radius of maximal ball inscribed in } \bar{K} \right\}. \tag{2.4}$$

Here and throughout the manuscript, C stands for a generic positive constant independent of h. The Hsieh–Clough–Tocher grid is a refinement based on $\bar{\Omega}_h$, where each tetrahedron $\bar{K}$ is refined into 4 tetrahedra by connecting the center O of the maximal inscribed ball in $\bar{K}$ to the four vertices $ABCD$ of $\bar{K}$ (cf. Figs. 1, 5 and [5]):

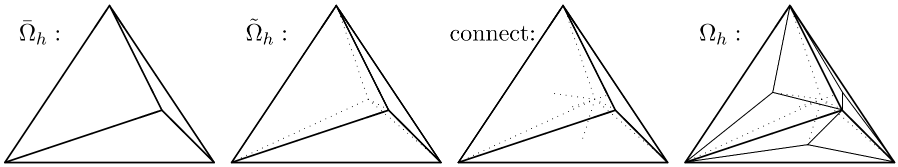

Fig. 1 A base grid $\bar{\Omega}_h$, Hsieh–Clough–Tocher grid $\tilde{\Omega}_h$, and Powell–Sabin grid Ω_h

$$\tilde{\Omega}_h = \left\{OABC, OACD, OADB, OBDC \mid \bar{K} = ABCD \in \bar{\Omega}_h\right\}. \tag{2.5}$$

The Powell–Sabin grid Ω_h is a further refinement based on $\bar{\Omega}_h$. In addition to connecting O to four vertices, we also connect O to each of the four centers of the maximal inscribed balls of four neighboring tetrahedra of $\bar{K}$, which share one face triangle with $\bar{K}$. At a boundary triangle of $\bar{K}$, O is connected to the bary-center, G, of the boundary triangle, cf. Figs. 1, 3 and [7, 17].

$$\Omega_h = \{OABG, OBCG, OCAG \mid \tilde{K} = OABC \in \tilde{\Omega}\}. \tag{2.6}$$

So G is either the intersection of OO' with the common triangle ABC of $\bar{K}$ and its neighbor $\bar{K}'$ (with center O'), or the bary-center of a boundary triangle ABC of $\bar{K}$.

In short, we refine each tetrahedron of $\bar{\Omega}_h$ into 4 to obtain a Hsieh–Clough–Tocher grid, and we refine further each tetrahedron of the Hsieh–Clough–Tocher grid into 3 to obtain a Powell–Sabin grid.

On the quasi-uniform Powell–Sabin grid Ω_h, we define the P_2–P_1 mixed element spaces by

$$\begin{aligned} \mathbf{V}_h &= \left\{\mathbf{u}_h \in C(\Omega) \mid \mathbf{u}_h|_K \in P_2(K)^3 \ \forall K \in \Omega_h, \quad \text{and} \quad \mathbf{u}_h|_{\partial\Omega} = \mathbf{0}\right\} \\ &\subset H_0^1(\Omega)^3, \end{aligned} \tag{2.7}$$

$$P_h = \{\operatorname{div} \mathbf{u}_h \mid \mathbf{u}_h \in \mathbf{V}_h\} \subset L_0^2(\Omega). \tag{2.8}$$

If (2.8) holds, then the discrete solution for $\mathbf{u}$ is pointwise divergence-free. It is widely taken that the pointwise divergence-free mixed method is too complicated to be practical, cf. [4]. A reason is the difficulty in describing the pressure space P_h in (2.8) and in finding a basis for it. We shall use the iterated penalty method to avoid this difficulty so that the discrete pressure function is obtained as a byproduct, without encoding the pressure finite element. This is in turn the advantage of divergence-free element where the mixed element is reduced to a single element for the velocity only.

The resulting system of finite element equations for (2.2) is: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q) &= 0 \qquad\quad \forall q \in P_h. \end{aligned} \tag{2.9}$$

The linear system of (2.9) always has a unique solution, regardless of the uniform inf-sup condition (4.40), in the divergence-free element method, shown in [17].

Proposition 2.1 [17] *The discrete linear system* (2.9) *has a unique solution in* $\mathbf{V}_h \times P_h$, *defined in* (2.7)–(2.8).

We note that P_h is a proper subspace of discontinuous P_1 finite element space (cf. [10]). However, letting $q = -\operatorname{div} \mathbf{u}_h \in P_h$ in (2.9), we still have the (pointwise) divergence-free property for the finite element solution:

$$\int_\Omega (\operatorname{div} \mathbf{u}_h)^2 d\mathbf{x} = b(\mathbf{u}_h, q) = 0. \tag{2.10}$$

In fact, the solution $\mathbf{u}_h$ is the $a(\cdot,\cdot)$ orthogonal projection from the divergence-free space $\mathbf{Z}$ to a subspace $\mathbf{Z}_h$, defined by

$$\mathbf{Z} := \left\{\mathbf{v} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{v} = 0\right\}, \tag{2.11}$$

$$\mathbf{Z}_h := \{\mathbf{v} \in \mathbf{V}_h \mid \operatorname{div} \mathbf{v} = 0\}. \tag{2.12}$$

For computation, we do not need to code P_h. It is the special interest of the divergence-free element method that the discrete solutions approximating the pressure in the Stokes equations can be obtained as byproducts, via the method in Definition 2.1. It does not only simplify the coding work, but also it avoids the difficulty of solving *non positive definite* systems of linear equations encountered in typical mixed element methods. We refer to [3, 4, 6, 13] for more information and the constant speed of convergence (independent of grid size, cf. Theorem 13.1.19 in [3] based on the inf-sup condition (4.40)) on the following iterative method.

Definition 2.1 (The iterated penalty method) Let the initial iterate $\mathbf{u}_h^0 = \mathbf{0}$ for the finite element Stokes equation (2.9). The rest iterates $\mathbf{u}_h^n$ are defined sequentially as the unique solution of

$$a\left(\mathbf{u}_h^n, \mathbf{v}_h\right) + \alpha\left(\operatorname{div} \mathbf{u}_h^n, \operatorname{div} \mathbf{v}_h\right) = (\mathbf{f}, \mathbf{v}_h) + \left(\operatorname{div} \sum\nolimits_{j=0}^{n-1} \mathbf{u}_h^j, \operatorname{div} \mathbf{v}_h\right) \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

$n = 1, 2, \ldots$. Here α is positive constant. At the end of iteration, we let

$$p_h^n = \operatorname{div} \sum\nolimits_{j=0}^{n} \mathbf{u}_h^j. \tag{2.13}$$

3 Singular vertex

In this section, we will study the finite-element pressure space P_h. Defined as $P_h = \operatorname{div} \mathbf{V}_h$, the pressure space is not the full discontinuous P_1 space on the grid Ω_h. Because of continuity constraint on a $\mathbf{V}_h$ function $\mathbf{v}_h$, its divergence function may also have certain constraints at certain grid points. Such a point is called a singular point, or singular vertex, by [10].

Lemma 3.1 *For any* $q \in P_h$, *it holds that*

$$q|_{K_{i,j}}(A) = q|_{K_{i,j+1}}(A), \tag{3.1}$$

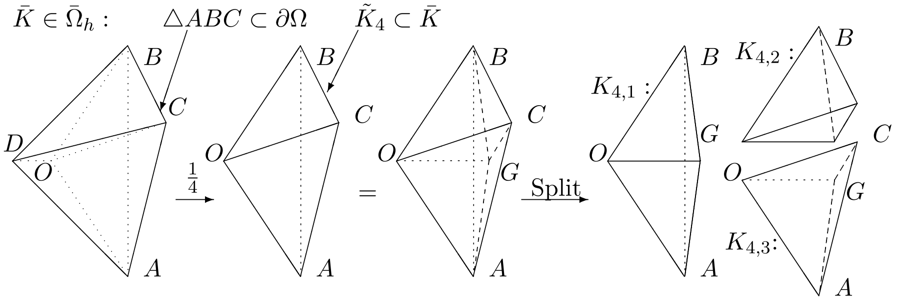

Fig. 2 Splitting one HCT-subtetrahedron into 3 PS sub-subtetrahedra at boundary

where $A \in \partial\Omega$ *is any face vertex of* $K_{i,j}$, $\{K_{i,j}, j = 1, 2, 3\}$ *are three sub-tetrahedra of* $\tilde{K}_i$, $K_{i,4} = K_{i,1}$, *and* $\{\tilde{K}_i, i = 1, 2, 3, 4\}$ *are four subtetrahedra of* $\bar{K}$, *see Fig. 2.*

Proof Let $q \in P_h$. Let $\tilde{K}_4 \subset \bar{K}$ shown in Fig. 2, its outside face triangle ABC is on $\partial\Omega$. By definition, there is a $\mathbf{w}_h \in \mathbf{V}_h$ such that $\operatorname{div} \mathbf{w}_h = q$. As $ABC \subset \partial\Omega$, $\mathbf{w}_h|_{ABC} \equiv 0$. We will compute $\operatorname{div} \mathbf{w}_h$ at A.

Let F be the reference mapping from the reference tetrahedron

$$\hat{K} := \{\hat{x} \geq 0, \hat{y} \geq 0, \hat{z} \geq 0, \hat{x} + \hat{y} + \hat{z} \leq 1\}$$

to one subtetrahedron $K_{4,1} = ABOG$ of $\tilde{\Omega}_h$, shown in Fig. 2. It follows that

$$F(\hat{x}, \hat{y}, \hat{z}) = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} + \begin{pmatrix} \vec{AO} & \vec{AG} & \vec{AB} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}. \tag{3.2}$$

Here (x_A, y_A, z_A) are coordinates of A, and in particular, $F(0, 0, 0) = A$ and $F(1, 0, 0) = O$. For any function $f(x, y, z)$ defined on the tetrahedron $ABOG$, we have

$$\operatorname{grad} f(x, y, z) = \begin{pmatrix} \vec{AO} & \vec{AG} & \vec{AB} \end{pmatrix}^{-T} \widehat{\operatorname{grad}} \hat{f}(\hat{x}, \hat{y}, \hat{z}). \tag{3.3}$$

For example, if $f_{AO}(x, y, z)$ is the P_2 directional-derivative nodal basis function, in the direction $\vec{AO}$ at A, supported on the tetrahedra with vertex A. That is, the directional-derivative $D_{\vec{AO}} f_{AO}(A) = 1$ and $f_{AO} = 0$ at the other 9 Lagrange nodes for P_2 polynomials on $ABOG$ excepting the mid-point Lagrange node of edge AO. By (3.3),

$$\operatorname{grad} f_{AO}|_{ABOG}(A) = \frac{|\vec{AO}|(\vec{AG} \times \vec{AB})}{\vec{AO} \cdot (\vec{AG} \times \vec{AB})}. \tag{3.4}$$

A $\mathbf{V}_h$ function $\mathbf{w}_h$ is a linear combination of such P_2 basis functions:

$$\mathbf{w}_h(\mathbf{x}) = \sum_f \begin{pmatrix} w_{f,1} \\ w_{f,2} \\ w_{f,3} \end{pmatrix} f(\mathbf{x}). \tag{3.5}$$

Therefore, we can compute its divergence at any vertex accordingly. As $\mathbf{w}_h = \mathbf{0}$ on triangles ABG and ACG, see Fig. 2, its coefficients $(w_{f,1}, w_{f,2}, w_{f,3}) = \mathbf{0}$ for the three directional-derivatives basis functions: f_{AB}, f_{AG} and f_{AC}. Thus

$$\begin{aligned}
\operatorname{div} \mathbf{w}_h(A) &= \operatorname{div} \left(\sum_f \begin{pmatrix} w_{f,1} \\ w_{f,2} \\ w_{f,3} \end{pmatrix} \right) f(A) \\
&= \operatorname{div} \begin{pmatrix} w_{f_{AO},1} \\ w_{f_{AO},2} \\ w_{f_{AO},3} \end{pmatrix} f_{AO}(A) \\
&= \begin{cases} \frac{|AO|(D_{\vec{AO}}\mathbf{w}_h)\cdot(\vec{AG}\times\vec{AB})}{\vec{AO}\cdot(\vec{AG}\times\vec{AB})} & \text{in } AGBO, \\ \frac{|AO|(D_{\vec{AO}}\mathbf{w}_h)\cdot(\vec{AC}\times\vec{AG})}{\vec{AO}\cdot(\vec{AC}\times\vec{AG})} & \text{in } ACGO. \end{cases}
\end{aligned} \tag{3.6}$$

Here the computation on $ACGO$ is similar to that on $AGBO$, cf. Fig. 2. The two unit vectors in (3.6) are equal,

$$\frac{\vec{AG} \times \vec{AB}}{|\vec{AG} \times \vec{AB}|} = \frac{\vec{AC} \times \vec{AG}}{|\vec{AC} \times \vec{AG}|}.$$

Thus

$$q|_{AGBO}(A) = \operatorname{div} \mathbf{w}_h|_{AGBO}(A) = \operatorname{div} \mathbf{w}_h|_{ACGO}(A) = q|_{ACGO}(A).$$

The two values do match, i.e., (3.1) holds. □

Lemma 3.2 *For any $q \in P_h$, it holds that*

$$q|_{K_{i,j}}(G) = q|_{K_{i,1}}(G), \quad j = 2, 3, \tag{3.7}$$

where $G \in \partial\Omega$ is any mid-face point of $\tilde{K}_i$, $\{K_{i,j}, j = 1, 2, 3\}$ are three subtetrahedra of $\tilde{K}_i$, and $\{\tilde{K}_i, i = 1, 2, 3, 4\}$ are four subtetrahedra of a $\bar{K} \in \bar{\Omega}_h$, see Fig. 2.

Proof Let $q \in P_h$ and $\mathbf{w}_h \in \mathbf{V}_h$ such that $\operatorname{div} \mathbf{w}_h = q$. Let $\tilde{K}_4 \subset \bar{K}$ shown in Fig. 2. Its outside mid-face point G is on $\partial\Omega$. A $\mathbf{V}_h$ function $\mathbf{w}_h$ is a linear combination of P_2 basis functions, shown in (3.5). As $ABC \subset \partial\Omega$, $\mathbf{w}_h|_{ABC} \equiv 0$. Thus, at point G, $\mathbf{w}_h$ is a vector multiple of basis function f_{GO}, see (3.6). Hence, similarly to (3.2)–(3.6), we can find, cf. Fig. 2,

$$\begin{aligned}
\operatorname{div} \mathbf{w}_h(G) &= \operatorname{div} \left(\sum_f \begin{pmatrix} w_{f,1} \\ w_{f,2} \\ w_{f,3} \end{pmatrix} f \right) (G) \\
&= \operatorname{div} \begin{pmatrix} w_{f_{GO},1} \\ w_{f_{GO},2} \\ w_{f_{GO},3} \end{pmatrix} f_{GO}(G)
\end{aligned}$$

$$= \begin{cases} \frac{|GO|(D_{\vec{GO}}\mathbf{w}_h)\cdot(\vec{GB}\times\vec{GA})}{\vec{GO}\cdot(\vec{GB}\times\vec{GA})} & \text{in } AGBO, \\ \frac{|GO|(D_{\vec{GO}}\mathbf{w}_h)\cdot(\vec{GA}\times\vec{GC})}{\vec{GO}\cdot(\vec{GA}\times\vec{GC})} & \text{in } ACGO, \\ \frac{|GO|(D_{\vec{GO}}\mathbf{w}_h)\cdot(\vec{GC}\times\vec{GB})}{\vec{GO}\cdot(\vec{GC}\times\vec{GB})} & \text{in } BCGO. \end{cases}$$

These three unit vectors are equal, all orthogonal to ABC,

$$\frac{\vec{GB}\times\vec{GA}}{|\vec{GB}\times\vec{GA}|} = \frac{\vec{GA}\times\vec{GC}}{|\vec{GA}\times\vec{GC}|} = \frac{\vec{GC}\times\vec{GB}}{|\vec{GC}\times\vec{GB}|}.$$

Thus

$$q|_{AGBO}(G) = \operatorname{div}\mathbf{w}_h|_{AGBO}(G) = q|_{ACGO}(G) = q|_{BCGO}(G).$$

The three values do match, i.e., (3.7) holds. □

We study the linear vector space formed by the 6 nodal-values of P_h functions at a singular vertex G in the next lemma, see Fig. 3. Here the line OO', connecting two centers of two inscribed balls of two neighboring base tetrahedra ($\bar{K}$ and $\bar{K}'$ in $\bar{\Omega}_h$), cuts the two subtetrahedra, $\tilde{K}_4 \subset \bar{K}$ and $\tilde{K}_4' \subset \bar{K}'$, into 6 sub-subtetrahedra. Let the 6 nodal values of a function f at G be denoted by a vector (see Fig. 3)

$$\mathbf{U}_{f(G)} = \begin{pmatrix} f|_{ABOG}(G) \\ f|_{CAOG}(G) \\ f|_{BCOG}(G) \\ f|_{BAO'G}(G) \\ f|_{ACO'G}(G) \\ f|_{CBO'G}(G) \end{pmatrix}.$$

Let the vector space

$$\mathbf{V}_G = \left\{ \mathbf{U}_{q(G)} \mid q \in P_h \right\}. \tag{3.8}$$

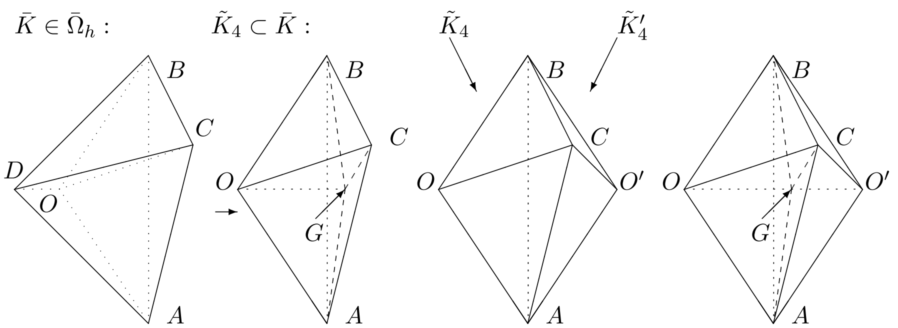

Fig. 3 Splitting of $\tilde{K}_4$, with a boundary or an internal face-triangle ABC

Lemma 3.3 *At any internal singular (mid-face) point G of Ω_h, cf. Fig. 3,* $\dim \mathbf{V}_G \neq 6$,

$$\dim \mathbf{V}_G = 4, \quad \textit{and} \quad \mathbf{V}_G = \operatorname{span}\left\{ \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\\1\end{pmatrix} \right\}.$$

Proof By the definition of P_h in (2.8), we have also

$$\mathbf{V}_G = \left\{ \mathbf{U}_{\operatorname{div} \mathbf{w}_h(G)} \mid \mathbf{w}_h \in \mathbf{V}_h \right\}.$$

$\operatorname{div} \mathbf{w}_h(G)$ is determined by the directional derivatives (in 5 directions) of the three components of $\mathbf{w}_h$. There are 15 degrees of freedom:

$$\mathbf{U}_{\operatorname{div} \mathbf{w}_h(G)} = \mathbf{U}_{\operatorname{div} \sum_{j=1}^{5} \mathbf{w}_j f_j(G)}, \quad \text{where } \mathbf{w}_j = \begin{pmatrix} w_{j,1} \\ w_{j,2} \\ w_{j,3} \end{pmatrix} \text{ are coefficients,} \tag{3.9}$$

and f_j are C_0–P_2 functions such that $D_{\mathbf{m}_i} f_j(G) = \delta_{ij}$, $f_j(G) = 0$ and $f_j \in H_0^1(\tilde{K}_4 \cup \tilde{K}_4')$, cf. (4.16). Here $\mathbf{m}_i$ denote the five vectors, $\vec{GO}$, $\vec{GA}$, $\vec{GB}$, $\vec{GC}$ and $\vec{GO'}$. We used a short notation here, that is, $f_1 = f_{GO}$ by the notation used in last two lemmas. By (3.3) and (3.4), we can find grad $f_j(G)$ on all 6 subtetrahedra. For example, we get these two typical grad $f_j(G)$:

$$\operatorname{grad} f_1(G) = \begin{cases} \frac{|\vec{OG}|(\vec{GB}\times\vec{GA})}{\vec{GO}\cdot(\vec{GB}\times\vec{GA})}, & \text{in } OGBA, \\ \frac{|\vec{OG}|(\vec{GA}\times\vec{GC})}{\vec{GO}\cdot(\vec{GA}\times\vec{GC})}, & \text{in } OGAC, \\ \frac{|\vec{OG}|(\vec{GC}\times\vec{GB})}{\vec{GO}\cdot(\vec{GC}\times\vec{GB})}, & \text{in } OGCB, \\ \mathbf{0}, & \text{in } O'GBA,\ O'GAC\ \&\ O'GCB, \end{cases}$$

$$\operatorname{grad} f_2(G) = \begin{cases} \frac{|\vec{GA}|(\vec{GO}\times\vec{GB})}{\vec{GA}\cdot(\vec{GO}\times\vec{GB})}, & \text{in } OGBA, \\ \frac{|\vec{GA}|(\vec{GC}\times\vec{GO})}{\vec{GA}\cdot(\vec{GC}\times\vec{GO})}, & \text{in } OGAC, \\ \frac{|\vec{GA}|(\vec{GB}\times\vec{GO'})}{\vec{GA}\cdot(\vec{GB}\times\vec{GO'})}, & \text{in } O'GAB, \\ \frac{|\vec{GA}|(\vec{GO'}\times\vec{GC})}{\vec{GA}\cdot(\vec{GO'}\times\vec{GC})}, & \text{in } O'GCA, \\ \mathbf{0}, & \text{in } OGCB\ \&\ O'GCB. \end{cases} \tag{3.10}$$

Hence, by (3.9),

$$\mathbf{U}_{\operatorname{div}\mathbf{w}_h(G)} = \begin{pmatrix} \mathbf{w}_1\cdot\mathbf{m}_{132}+\mathbf{w}_2\cdot\mathbf{m}_{213}+\mathbf{w}_3\cdot\mathbf{m}_{321} \\ \mathbf{w}_1\cdot\mathbf{m}_{124}+\mathbf{w}_2\cdot\mathbf{m}_{241}+\mathbf{w}_4\cdot\mathbf{m}_{412} \\ \mathbf{w}_1\cdot\mathbf{m}_{143}+\mathbf{w}_3\cdot\mathbf{m}_{314}+\mathbf{w}_4\cdot\mathbf{m}_{431} \\ \mathbf{w}_2\cdot\mathbf{m}_{235}+\mathbf{w}_3\cdot\mathbf{m}_{352}+\mathbf{w}_5\cdot\mathbf{m}_{523} \\ \mathbf{w}_2\cdot\mathbf{m}_{254}+\mathbf{w}_4\cdot\mathbf{m}_{425}+\mathbf{w}_5\cdot\mathbf{m}_{542} \\ \mathbf{w}_3\cdot\mathbf{m}_{345}+\mathbf{w}_4\cdot\mathbf{m}_{453}+\mathbf{w}_5\cdot\mathbf{m}_{534} \end{pmatrix}$$
$$= (a_1+a_2+a_4)\begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix} + (a_1+a_3+a_6)\begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix}$$
$$+ (a_1+a_5+a_7)\begin{pmatrix}0\\0\\1\\0\\0\\1\end{pmatrix} + (-a_1+a_8)\begin{pmatrix}0\\0\\0\\1\\1\\1\end{pmatrix}, \tag{3.11}$$

where

$$\mathbf{m}_{ijk} = \frac{|\mathbf{m}_i|(\mathbf{m}_j\times\mathbf{m}_k)}{\mathbf{m}_i\cdot(\mathbf{m}_j\times\mathbf{m}_k)} \quad \text{and} \quad \begin{cases} a_1 = \mathbf{w}_1\cdot\mathbf{m}_{132} = \mathbf{w}_1\cdot\mathbf{m}_{124} = \mathbf{w}_1\cdot\mathbf{m}_{143}, \\ a_2 = \mathbf{w}_2\cdot\mathbf{m}_{213} = \mathbf{w}_2\cdot\mathbf{m}_{235}, \\ a_3 = \mathbf{w}_2\cdot\mathbf{m}_{241} = \mathbf{w}_2\cdot\mathbf{m}_{254}, \\ a_4 = \mathbf{w}_3\cdot\mathbf{m}_{321} = \mathbf{w}_3\cdot\mathbf{m}_{352}, \\ a_5 = \mathbf{w}_3\cdot\mathbf{m}_{314} = \mathbf{w}_3\cdot\mathbf{m}_{345}, \\ a_6 = \mathbf{w}_4\cdot\mathbf{m}_{412} = \mathbf{w}_4\cdot\mathbf{m}_{425}, \\ a_7 = \mathbf{w}_4\cdot\mathbf{m}_{431} = \mathbf{w}_4\cdot\mathbf{m}_{453}, \\ a_8 = \mathbf{w}_5\cdot\mathbf{m}_{523} = \mathbf{w}_5\cdot\mathbf{m}_{534} = \mathbf{w}_5\cdot\mathbf{m}_{542}. \end{cases} \tag{3.12}$$

Note that OG and GO' are on the same line and that points $ABCG$ form one plane. This is why we have, in (3.12),

$$\mathbf{m}_{213} = \frac{|\mathbf{m}_2|(\mathbf{m}_1\times\mathbf{m}_3)}{\mathbf{m}_2\cdot(\mathbf{m}_1\times\mathbf{m}_3)} = \frac{\mathbf{n}_{OO'B}}{\cos\theta} = \frac{|\mathbf{m}_2|(\mathbf{m}_3\times\mathbf{m}_5)}{\mathbf{m}_2\cdot(\mathbf{m}_3\times\mathbf{m}_5)} = \mathbf{m}_{235},$$

where θ is the angle between $\mathbf{m}_2$ and the following unit normal vector to the plane $OO'B$ (see Fig. 3)

$$\mathbf{n}_{OO'B} = (\mathbf{m}_1\times\mathbf{m}_3)/|\mathbf{m}_1\times\mathbf{m}_3| = (\mathbf{m}_3\times\mathbf{m}_5)/|\mathbf{m}_3\times\mathbf{m}_5|.$$

By (3.11),

$$\mathbf{V}_G \subset \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\\1\end{pmatrix}\right\} \tag{3.13}$$

and that $\dim \mathbf{V}_G \le 4$. Though there are 15 coefficients in defining $\mathbf{w}_h$ and $\mathbf{U}_{\operatorname{div}\mathbf{w}_h}$, we can choose 4 sets of coefficients for the whole space $\mathbf{U}_{\operatorname{div}\mathbf{w}_h(G)}$. Let $\mathbf{w}_j$ in (3.9) be defined as in the following 4 cases.

$$\mathbf{U}_{\operatorname{div}\mathbf{w}_h(G)} = \begin{cases} (1\;0\;0\;1\;0\;0)^T & \text{if } \mathbf{w}_2 = \mathbf{n}_{\mathbf{m}_{213}}^{\perp\mathbf{m}_{241}}, \text{ other } \mathbf{w}_i = \mathbf{0}, \\ (0\;1\;0\;0\;1\;0)^T & \text{if } \mathbf{w}_2 = \mathbf{n}_{\mathbf{m}_{241}}^{\perp\mathbf{m}_{213}}, \text{ other } \mathbf{w}_i = \mathbf{0}, \\ (0\;0\;1\;0\;0\;1)^T & \text{if } \mathbf{w}_3 = \mathbf{n}_{\mathbf{m}_{314}}^{\perp\mathbf{m}_{321}}, \text{ other } \mathbf{w}_i = \mathbf{0}, \\ (0\;0\;0\;1\;1\;1)^T & \text{if } \mathbf{w}_5 = \mathbf{m}_{523}/|\mathbf{m}_{523}|^2, \text{ other } \mathbf{w}_i = \mathbf{0}, \end{cases} \tag{3.14}$$

where $\mathbf{n}_{\mathbf{m}_{213}}^{\perp\mathbf{m}_{241}}$, for example, is a scaled component of $\mathbf{m}_{213}$, in the plane formed by $\mathbf{m}_{213}$ and $\mathbf{m}_{241}$, orthogonal to $\mathbf{m}_{241}$:

$$\mathbf{n}_{\mathbf{m}_{213}}^{\perp\mathbf{m}_{241}} = \frac{\mathbf{m}_{213}|\mathbf{m}_{241}|^2 - (\mathbf{m}_{213}\cdot\mathbf{m}_{241})\mathbf{m}_{241}}{|\mathbf{m}_{213}|^2|\mathbf{m}_{241}|^2 - (\mathbf{m}_{213}\cdot\mathbf{m}_{241})^2}. \tag{3.15}$$

The denominator in (3.14) is non-zero:

$$|\mathbf{m}_{523}| = \frac{2|\vec{GO'}||\triangle GAB|}{6V_{GABO'}} \ge C > 0$$

for some C depending on the grid regularity (2.4). Also the denominator in (3.15) is bounded below by a positive constant C, as the intersection G of OO' and ABC is bounded away from the three edges of triangle ABC and the angle between $\mathbf{m}_{213}$ and $\mathbf{m}_{241}$ is bounded away from 0. In fact,

$$|\mathbf{m}_{213}|^2|\mathbf{m}_{241}|^2 - (\mathbf{m}_{213}\cdot\mathbf{m}_{241})^2 = V_{OABG}V_{OACG}\sin^2\theta \ge C > 0,$$

where θ is the angle between planes OGB and OGC. $\sin\theta = 0$ would imply either $\triangle GBC$ degenerates to an edge GB ($\theta = \pi$) or $\triangle GBC$ degenerates to an edge BC ($\theta = 0$). The lemma is proved as we found 4 basis vectors of $\mathbf{V}_G$. □

We study next the linear vector space formed by the 4 nodal-values of P_h functions at a corner singular vertex A, see the right figure in Fig. 3. Let $\mathbf{V}_A$ be the space of the 4-nodal value vectors:

$$\mathbf{V}_A = \left\{ \mathbf{U}_q(A) = \begin{pmatrix} q|_{AGBO}(A) \\ q|_{AGOC}(A) \\ q|_{AGCO'}(A) \\ q|_{AGO'B}(A) \end{pmatrix} \;\middle|\; q \in P_h \right\}. \tag{3.16}$$

Lemma 3.4 *At any A of Ω_h, cf. Fig. 3,* $\dim \mathbf{V}_A \neq 4$,

$$\dim \mathbf{V}_A = 3, \quad \textit{and} \quad \mathbf{V}_A = \operatorname{span}\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}. \tag{3.17}$$

Proof By the definition of P_h in (2.8), we have also

$$bV_A = \left\{ \mathbf{U}_{\operatorname{div}\mathbf{w}_h}(A) \mid \mathbf{w}_h \in \mathbf{V}_h \right\}.$$

$\operatorname{div}\mathbf{w}_h(A)$ is determined by the directional derivatives (in 5 directions) of the three components of $\mathbf{w}_h$. There are 15 degrees of freedom:

$$\mathbf{U}_{\operatorname{div}\mathbf{w}_h}(A) = \mathbf{U}_{\operatorname{div}\sum_{j=1}^{5}\mathbf{w}_j f_j}(A), \quad \text{where } \mathbf{w}_j = \begin{pmatrix} w_{j,1}\\ w_{j,2}\\ w_{j,3}\end{pmatrix} \text{ are coefficients,} \tag{3.18}$$

and f_j are C_0–P_2 functions such that $D_{\mathbf{m}_i} f_j(A) = \delta_{ij}$, $f_j(A) = 0$ and $f_j \in H_0^1(\tilde{K}_4 \cup \tilde{K}_4')$, cf. (4.16). Here $\mathbf{m}_i$ denote the five vectors, $\vec{AG}$, $\vec{AO}$, $\vec{AC}$, $\vec{AO'}$ and $\vec{AB}$. As in Lemma 3.3, we can find, by (3.18),

$$\begin{aligned}\mathbf{U}_{\operatorname{div}\mathbf{w}_h(A)} = &\begin{pmatrix}\mathbf{w}_1\cdot\mathbf{m}_{152}\\ \mathbf{w}_1\cdot\mathbf{m}_{123}\\ \mathbf{w}_1\cdot\mathbf{m}_{134}\\ \mathbf{w}_1\cdot\mathbf{m}_{145}\end{pmatrix} + \begin{pmatrix}\mathbf{w}_2\cdot\mathbf{m}_{215}\\ \mathbf{w}_2\cdot\mathbf{m}_{231}\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \mathbf{w}_3\cdot\mathbf{m}_{312}\\ \mathbf{w}_3\cdot\mathbf{m}_{341}\\ 0\end{pmatrix}\\ &+ \begin{pmatrix}0\\ 0\\ \mathbf{w}_4\cdot\mathbf{m}_{413}\\ \mathbf{w}_4\cdot\mathbf{m}_{451}\end{pmatrix} + \begin{pmatrix}\mathbf{w}_5\cdot\mathbf{m}_{521}\\ 0\\ 0\\ \mathbf{w}_5\cdot\mathbf{m}_{514}\end{pmatrix},\end{aligned} \tag{3.19}$$

where $\mathbf{m}_{ijk}$ are defined as in (3.12). By the same reason for (3.12), we have

$$\mathbf{m}_{215} = \mathbf{m}_{231}, \qquad \mathbf{m}_{312} = \mathbf{m}_{341}, \qquad \mathbf{m}_{413} = \mathbf{m}_{451}, \qquad \mathbf{m}_{521} = \mathbf{m}_{514}.$$

As $AOGO'$ are on one plane and $ABGC$ on another plane, we have the following relations

$$\begin{aligned}\mathbf{m}_1 &= \frac{v_{145}}{v_{245}}\mathbf{m}_2 + \frac{v_{125}}{v_{245}}\mathbf{m}_4,\\ \mathbf{m}_1 &= \frac{v_{134}}{v_{234}}\mathbf{m}_2 + \frac{v_{123}}{v_{234}}\mathbf{m}_4,\\ \mathbf{m}_1 &= \frac{v_{125}}{v_{234}}\mathbf{m}_3 + \frac{v_{123}}{v_{234}}\mathbf{m}_5,\end{aligned}$$

where we used volume notations that v_{125} is the volume of tetrahedron formed by the three vectors $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_5$ at A, i.e., $v_{125} = V_{AGOB}$. Using these relations, we get

$$
\begin{aligned}
\mathbf{m}_{152} &= \frac{|\mathbf{m}_1|(\mathbf{m}_5 \times \mathbf{m}_2)}{6v_{125}} \\
&= \frac{|\mathbf{m}_1|}{6v_{125}} \mathbf{m}_5 \times \left(\frac{v_{245}}{v_{145}} \mathbf{m}_1 - \frac{v_{125}}{v_{145}} \mathbf{m}_4 \right) \\
&= \frac{|\mathbf{m}_1| v_{245}}{6v_{125} v_{145}} \mathbf{m}_5 \times \mathbf{m}_1 + \mathbf{m}_{145}, \\
\mathbf{m}_{123} &= \frac{|\mathbf{m}_1|(\mathbf{m}_2 \times \mathbf{m}_3)}{6v_{123}} \\
&= \frac{|\mathbf{m}_1|}{6v_{123}} \left(\frac{v_{234}}{v_{134}} \mathbf{m}_1 - \frac{v_{123}}{v_{134}} \mathbf{m}_4 \right) \times \mathbf{m}_3 \\
&= \frac{|\mathbf{m}_1| v_{234}}{6v_{123} v_{134}} \mathbf{m}_1 \times \mathbf{m}_3 + \mathbf{m}_{134} \\
&= \frac{|\mathbf{m}_1| v_{234}}{6v_{123} v_{134}} \mathbf{m}_1 \times \left(\frac{v_{234}}{v_{125}} \mathbf{m}_1 - \frac{v_{123}}{v_{125}} \mathbf{m}_5 \right) + \mathbf{m}_{134} \\
&= \frac{|\mathbf{m}_1| v_{234}}{6v_{125} v_{134}} \mathbf{m}_5 \times \mathbf{m}_1 + \mathbf{m}_{134}.
\end{aligned}
$$

Noting that $v_{245}/v_{145} = v_{234}/v_{134}$, we find

$$
\mathbf{m}_{152} = \mathbf{m}_{123} - \mathbf{m}_{134} + \mathbf{m}_{145}.
$$

Hence we can rewrite (3.19) as

$$
\begin{aligned}
\mathbf{U}_{\operatorname{div} \mathbf{w}_h(A)} = (a_1 - a_2 + a_3 + a_4 + a_7) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
+ (a_2 - a_3 + a_5 - a_7) \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + (a_3 + a_6 + a_7) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{cases}
a_1 = \mathbf{w}_1 \cdot \mathbf{m}_{123}, \\
a_2 = \mathbf{w}_1 \cdot \mathbf{m}_{134}, \\
a_3 = \mathbf{w}_1 \cdot \mathbf{m}_{145}, \\
a_4 = \mathbf{w}_2 \cdot \mathbf{m}_{215}, \\
a_5 = \mathbf{w}_3 \cdot \mathbf{m}_{312}, \\
a_6 = \mathbf{w}_4 \cdot \mathbf{m}_{413}, \\
a_7 = \mathbf{w}_5 \cdot \mathbf{m}_{514}.
\end{cases}
$$

We can choose three sets of vectors:

$$\begin{cases} (1) & \mathbf{w}_2 = \mathbf{m}_{215}/|\mathbf{m}_{215}|^2, & \text{the rest } \mathbf{w}_i = \mathbf{0}, \\ (2) & \mathbf{w}_3 = \mathbf{m}_{312}/|\mathbf{m}_{312}|^2, & \text{the rest } \mathbf{w}_i = \mathbf{0}, \\ (3) & \mathbf{w}_4 = \mathbf{m}_{413}/|\mathbf{m}_{413}|^2, & \text{the rest } \mathbf{w}_i = \mathbf{0}, \end{cases} \tag{3.21}$$

to obtain 3 linearly independent vectors $\mathbf{U}_{\operatorname{div}\mathbf{w}_h(A)}$ in (3.20), $(1, 1, 0, 0)$, $(0, 1, 1, 0)$ and $(0, 0, 1, 1)$. The lemma is proven as we found 3 basis vectors of $\mathbf{V}_A$. □

4 Stability and convergence

In this section, we will prove the inf-sup condition (4.40), i.e., the stability of the divergence-free P_2–P_1 mixed element on Powell–Sabin grids. The analysis is done by construction. The convergence follows the stability routinely, at the end of the section.

Lemma 4.1 *For any $q \in P_h$ (defined in (2.8)), there is a function $\mathbf{v}_1 \in \mathbf{V}_h$ (defined in (2.7)) such that*

$$\int_{\bar K} \operatorname{div} \mathbf{v}_1 d\mathbf{x} = \int_{\bar K} q\, d\mathbf{x} \quad \forall \bar K \in \bar\Omega_h, \quad \text{and} \quad \|\mathbf{v}_1\|_{H^1(\Omega)^3} \le C\|q\|_{L^2(\Omega)}. \tag{4.1}$$

Proof The proof here is standard, by constructing a Fortin operator for $\mathbf{V}_h$ functions, against the piecewise constant pressure space. For any $q \in P_h$, let a C_{-1}–P_0 function $\bar q$ be

$$\bar q|_{\bar K} = \frac{1}{|\bar K|} \int_{\bar K} q\, d\mathbf{x} \quad \forall \bar K \in \bar\Omega_h.$$

Then $\bar q \in L^2_0(\Omega)$ as $\int_\Omega \bar q\, d\mathbf{x} = \sum_{\bar K \in \bar\Omega_h} \int_{\bar K} \bar q\, d\mathbf{x} = \int_\Omega q\, d\mathbf{x}$. By the inf-sup condition for the continuous functions, cf. [9], there is a $\mathbf{u}_{\bar q} \in H^1_0(\Omega)^3$ such that

$$\operatorname{div} \mathbf{u}_{\bar q}(\mathbf{x}) = \bar q(\mathbf{x}) \quad \text{a.e. for } \mathbf{x} \in \Omega, \tag{4.2}$$

and

$$\|\mathbf{u}_{\bar q}\|_{H^1} \le C\|\bar q\|_{L^2} \le C\|q\|_{L^2}.$$

We modify the P_2 Lagrange interpolation operator slightly to define a Fortin operator (cf. [4]):

$$\mathbf{I}_h\colon \quad C(\Omega) \cap H^1_0(\Omega)^3 \to \mathbf{V}_h, \qquad \mathbf{I}_h\colon \quad \mathbf{u}_{\bar q} \to \mathbf{I}_h \mathbf{u}_{\bar q},$$

$\mathbf{I}_h \mathbf{u}_{\bar q}(\mathbf{x}_i) = \mathbf{u}_{\bar q}(\mathbf{x}_i)$ at all P_2 nodes except the four internal face nodes,

$$\int_{(\partial \bar K)_i} \mathbf{I}_h \mathbf{u}_{\bar q} d\mathbf{x} = \int_{(\partial \bar K)_i} \mathbf{u}_{\bar q} d\mathbf{x}, \quad i = 1, 2, 3, 4,$$

where the mid-face values $\mathbf{I}_h \mathbf{u}_{\bar q}(\mathbf{c}_i)$ (see Figs. 4 and 1) are chosen so that the integrals of the three components of $\mathbf{I}_h \mathbf{u}_{\bar q}$ on each of the four face triangles $(\partial \bar K)_i$ match those

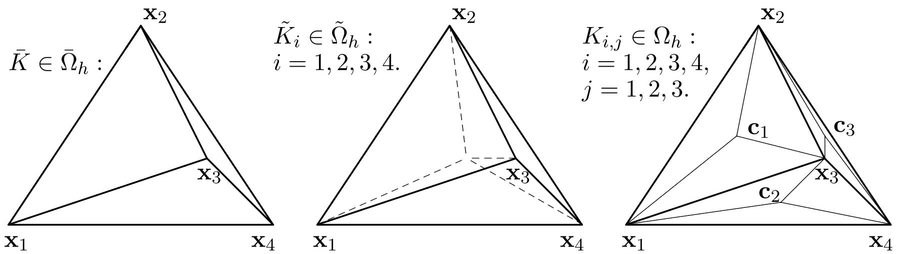

Fig. 4 12 subtetrahedra of a Powell–Sabin grid Ω_h, inside one $\bar K \in \bar\Omega_h$

of $\mathbf{u}_{\bar q}$. We note that an averaging interpolation can be adopted if the function $\mathbf{u}_{\bar q}$ is not continuous, as usual, see [12]. Also following [12], it is standard to show the stability of such an interpolation operator by scaling:

$$\|\mathbf{I}_h \mathbf{u}_{\bar q}\|_{H^1} \le C \|\mathbf{u}_{\bar q}\|_{H^1}.$$

Thus, letting $\mathbf{v}_1 = \mathbf{I}_h \mathbf{u}_{\bar q}$, the lemma is proved as the interpolant also preserves the divergence element-wise:

$$\int_{\bar K} \operatorname{div} \mathbf{v}_1 d\mathbf{x} = \int_{\partial \bar K} \mathbf{v}_1 \cdot \mathbf{n} d\mathbf{x} = \int_{\partial \bar K} \mathbf{u}_{\bar q} \cdot \mathbf{n} d\mathbf{x} = \int_{\bar K} \operatorname{div} \mathbf{u}_{\bar q} d\mathbf{x} = \int_{\bar K} q d\mathbf{x}. \qquad \square$$

We note that the above analysis for the lemma in defining $\mathbf{I}_h \mathbf{u}_{\bar q} \in \mathbf{V}_h$ is well known in showing the stability of P_3–P_0 element in 3D, cf. [14, 16]. Further, we note that it is not necessary to introduce $\bar q$ in the proof. The analysis remains the same when $\bar q$ is replaced by q, that is, using q in (4.2).

After matching the integral values of q elementwise by $\operatorname{div} \mathbf{v}_1$, we next match the vertex-values of $q - \operatorname{div} \mathbf{v}_1$, which is piecewise P_1 and discontinuous. It is done in two steps, for C_{-1}–P_1 functions on $\tilde\Omega_h$ and on Ω_h. Corresponding to (2.7), we define a C_0–P_2 space on a Hsieh–Clough–Tocher grid $\tilde\Omega_h$ instead of the Powell–Sabin grid Ω_h:

$$\tilde{\mathbf{V}}_h = \left\{ \mathbf{u}_h \in C(\Omega) \mid \mathbf{u}_h|_{\tilde K} \in P_2(\tilde K)^3 \ \forall \tilde K \in \tilde\Omega_h, \quad \text{and} \quad \mathbf{u}_h|_{\partial\Omega} = \mathbf{0} \right\} \subset H_0^1(\Omega)^3. \tag{4.3}$$

We introduce a C_{-1}–P_1 space on the Hsieh–Clough–Tocher grid:

$$\tilde P_h = \left\{ q_h \mid q_h|_{\tilde K} \in P_1(\tilde K) \ \forall \tilde K \in \tilde\Omega_h, \quad \text{and} \quad \int_\Omega q_h = 0 \right\} \subset L_0^2(\Omega). \tag{4.4}$$

It is apparent that $\tilde P_h \supset \operatorname{div} \tilde{\mathbf{V}}_h$. But we do not know if

$$\tilde P_h \ne \operatorname{div} \tilde{\mathbf{V}}_h$$

or not. We do know $\tilde P_h = \operatorname{div} \tilde{\mathbf{V}}_h$ when the polynomial degree k is greater than 2, cf. [16]. Even if $\tilde P_h = \operatorname{div} \tilde{\mathbf{V}}_h$ holds for polynomial degree $k = 2$, the inf-sup condition (4.40) may not hold independently of grid size, cf. [16]. This is shown also by a numerical test below in Table 3, where there is no convergence for the pressure solution. Therefore we do not know at the moment if, note that $\operatorname{div} \tilde{\mathbf{V}}_h \subset \operatorname{div} \mathbf{V}_h$,

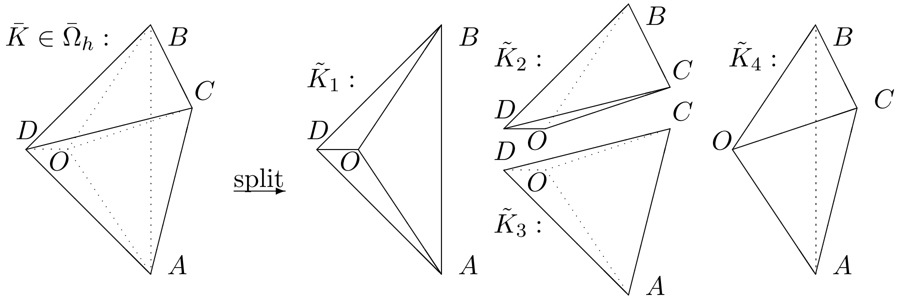

Fig. 5 Splitting one tetrahedron $\bar K$ in $\bar\Omega_h$ into 4 subtetrahedra of $\tilde\Omega_h$

$$\tilde P_h \subset \operatorname{div} \mathbf{V}_h. \tag{4.5}$$

But we will prove (4.5) at the end of the section, in Corollary 4.1. The next lemma is essentially proved in [16]. Here we give a different, constructive proof.

Lemma 4.2 *On the Hsieh–Clough–Tocher grids, for any $\tilde q \in \tilde P_h$ (defined in (4.4)) such that $\int_{\bar K} \tilde q = 0 \quad \forall \bar K \in \bar\Omega_h$, there is a function $\mathbf{v}_2 \in \tilde{\mathbf{V}}_h$ (defined in (4.3)) such that*

$$\operatorname{div} \mathbf{v}_2 = \tilde q \quad \forall \tilde K \in \tilde\Omega_h, \tag{4.6}$$

$$\int_{\bar K} \operatorname{div} \mathbf{v}_2 = 0 \quad \forall \bar K \in \bar\Omega_h, \tag{4.7}$$

$$\|\mathbf{v}_2\|_{H^1(\Omega)^3} \le C \|\tilde q\|_{L^2(\Omega)}. \tag{4.8}$$

Proof Here we construct $\mathbf{v}_2$ inside each big tetrahedron $\bar K$ of $\bar\Omega_h$ to match $\tilde q$ on the four subtetrahedra $\tilde K_i$ in $\tilde\Omega_h$, cf. Fig. 5. That is,

$$\mathbf{v}_2|_{\bar K} \in \tilde V_h \cap H_0^1(\bar K) \; \forall \bar K \in \bar\Omega_h.$$

Let $f(x, y, z)$ be the P_2 nodal basis function on $DABO$ such that f has a value 1 at the middle point of edge DO and vanishes at the other 9 Lagrange nodes for P_2 polynomials on $DABO$. We find the directional derivative $D_{\vec{DO}} f(D) = \frac{4}{|DO|}$. By the reference mapping (3.2)–(3.3), we have

$$\operatorname{grad} f|_{DABO}(D) = \frac{4(\vec{DA} \times \vec{DB})}{\vec{DO} \cdot (\vec{DA} \times \vec{DB})}. \tag{4.9}$$

The above calculation remains the same on the other two tetrahedra, $DBCO = \tilde K_2$ and $DCAO = \tilde K_3$, cf. Fig. 5. Now, let $f(x, y, z)$ also denote the combination, i.e. the continuous, piecewise P_2, nodal basis function at the middle point of DO, supported on the three subtetrahedra, $\tilde K_1$, $\tilde K_2$ and $\tilde K_3$. As the big tetrahedron $\bar K = DABC$ is non-degenerate, its three face normal vectors are linearly independent, that is,

$$\begin{aligned}
&\det\left(\vec{DA} \times \vec{DB} \;\; \vec{DB} \times \vec{DC} \;\; \vec{DC} \times \vec{DA}\right) \\
&\quad = (\vec{DA} \times \vec{DB}) \cdot \left[(\vec{DB} \times \vec{DC}) \times (\vec{DC} \times \vec{DA})\right]
\end{aligned}$$

$$
\begin{aligned}
&= (\vec{DA} \times \vec{DB}) \cdot \big[\big((\vec{DB} \times \vec{DC}) \cdot \vec{DA}\big)\vec{DC}\big] \\
&= 6V_{DABC} \cdot \big[(\vec{DA} \times \vec{DB}) \cdot \vec{DC}\big] \\
&= 36V_{DABC}^2 \geq Ch^6,
\end{aligned}
$$

where C is a positive constant depending on the quasiuniform constant in (2.4) only (see [5]). We denote the inverse matrix by

$$
P = (p_{ij}) = \left(\frac{4(\vec{DA} \times \vec{DB})}{6V_{DABO}} \; \frac{4(\vec{DB} \times \vec{DC})}{6V_{DBCO}} \; \frac{4(\vec{DC} \times \vec{DA})}{6V_{DCAO}} \right)^{-1}, \tag{4.10}
$$

where the three column vectors are from three Jacobian matrices of reference mappings for $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$. Here we used the formula that the volume of tetrahedron $V_{DABO} = \vec{DO} \cdot (\vec{DA} \times \vec{DB})/6$. We now define a nodal basis $\mathbf{v}_{2,1,1}$ of $\tilde{\mathbf{V}}_h$, $\mathbf{v}_{2,1,1} \in \tilde{\mathbf{V}}_h \cap H_0^1(\bar{K})^3$, by

$$
\mathbf{v}_{2,1,1} = \begin{cases} \begin{pmatrix} p_{11} \\ p_{12} \\ p_{13} \end{pmatrix} f & \text{in } \tilde{K}_1 \cup \tilde{K}_2 \cup \tilde{K}_3, \\ \mathbf{0} & \text{elsewhere.} \end{cases} \tag{4.11}
$$

By (4.9) and (4.10), it follows that

$$
\begin{aligned}
&\operatorname{div} \mathbf{v}_{2,1,1}|_{DABO}(D) = 1, && \operatorname{div} \mathbf{v}_{2,1,1}|_{DABO}(A) = 0, && \operatorname{div} \mathbf{v}_{2,1,1}|_{DABO}(B) = 0, \\
&\operatorname{div} \mathbf{v}_{2,1,1}|_{DBCO}(D) = 0, && \operatorname{div} \mathbf{v}_{2,1,1}|_{DBCO}(B) = 0, && \operatorname{div} \mathbf{v}_{2,1,1}|_{DBCO}(C) = 0, \\
&\operatorname{div} \mathbf{v}_{2,1,1}|_{DCAO}(D) = 0, && \operatorname{div} \mathbf{v}_{2,1,1}|_{DCAO}(C) = 0, && \operatorname{div} \mathbf{v}_{2,1,1}|_{DCAO}(A) = 0.
\end{aligned}
$$

Furthermore, by mapping the nodal basis f to the reference element $\hat{K}$ for each subtetrahedron, it is standard to verify that

$$
\|\mathbf{curl}\, \mathbf{v}_{2,1,1}\|_{L^2(\Omega)^3} \leq C \|\operatorname{div} \mathbf{v}_{2,1,1}\|_{L^2(\Omega)},
$$

for some constant C depending on the quasiuniform constant of the grid. By Lemma 2.1 in [17], we get that

$$
|\mathbf{v}_{2,1,1}|_{H^1(\Omega)^3} \leq C \|\operatorname{div} \mathbf{v}_{2,1,1}\|_{L^2(\Omega)}. \tag{4.12}
$$

Similarly to $\mathbf{v}_{2,1,1}$ in (4.11), we can define $\mathbf{v}_{2,1,2}$ and $\mathbf{v}_{2,1,3}$ so that they have divergence value 1 at vertices D on one subtetrahedron, $\tilde{K}_2$ or $\tilde{K}_3$, respectively, while $\operatorname{div} \mathbf{v}_{2,1,i} = 0$ at all other vertices of $\bar{K}$ restricted on each $\tilde{K}_j$. Repeating the construction above at A, B, and C, instead of D, we can find $\mathbf{v}_{2,i,j}$, $1 \leq i \leq 4$, $1 \leq j \leq 3$, such that

$$
\operatorname{div} \mathbf{v}_{2,i,j}|_{\tilde{K}_l}\big(a_m^{\tilde{K}_l}\big) = \delta_{il}\delta_{jm}, \quad 1 \leq l \leq 4,\; 1 \leq m \leq 3,
$$

where $a_m^{\tilde{K}_l}$ is the m-th vertex ($m < 4$, also a vertex of $\bar{K}$, but not inner vertex O, cf. Fig. 5) of subtetrahedron $\tilde{K}_l$. This way, we match the $\bar{K}$ vertex values of $\tilde{q}$ by letting

$$\mathbf{v}_{2,v}(\mathbf{x}) = \sum_{i=1}^{4}\sum_{j=1}^{3} \tilde{q}|_{\tilde{K}_i}\left(a_j^{\tilde{K}_i}\right)\mathbf{v}_{2,i,j}(\mathbf{x}). \tag{4.13}$$

By (4.12), it follows that

$$|\mathbf{v}_{2,v}|_{H^1(\Omega)^3} \le C\|\operatorname{div}\mathbf{v}_{2,v}\|_{L^2(\Omega)}. \tag{4.14}$$

Let the C_{-1}–P_1 function $\tilde{q}_2$ be

$$\tilde{q}_2 = \tilde{q} - \operatorname{div}\mathbf{v}_{2,v}. \tag{4.15}$$

By (4.13) $\tilde{q}_2$ vanishes at the vertex of $\bar{K}$, i.e., at all vertices of four subtetrahedra $\tilde{K}_i$ except the center point O, see Fig. 5. To match further the vertex values of C_{-1}–P_1 function $\tilde{q}_2$ at O, we use a Hermite type nodal basis at point O of C_0–P_2 functions. Let f_O be a P_2 polynomial on $\tilde{K}_1 = DABO$ (see Fig. 5) such that

$$\begin{aligned} &f_O(O) = 1, && f_O|_{DAB} = 0, \\ &D_{\vec{DO}} f_O(D) = 0, && D_{\vec{AO}} f_O(A) = 0, \qquad D_{\vec{BO}} f_O(B) = 0. \end{aligned} \tag{4.16}$$

That is, f_O vanishes on triangle DAB, has its three directional derivatives 0 at three vertices of DAB, and has its nodal value 1 at O. Thus, f_O is uniquely defined by

$$f_O(x,y,z) = \begin{cases} L^2_{DAC}(x,y,z)/L^2_{DAC}(x_O,y_O,z_O) & \text{in } ODAC, \\ L^2_{DCB}(x,y,z)/L^2_{DCB}(x_O,y_O,z_O) & \text{in } ODCB, \\ L^2_{DBA}(x,y,z)/L^2_{DBA}(x_O,y_O,z_O) & \text{in } ODBA, \\ L^2_{ABC}(x,y,z)/L^2_{ABC}(x_O,y_O,z_O) & \text{in } OABC, \end{cases} \tag{4.17}$$

where (x_O, y_O, z_O) are coordinates of point O and, for example, $L_{DAC} \in P_1$ such that $L_{DAC} = 0$ is an equation for the plane DAC. To find the gradient of f_O, we map each tetrahedron $\tilde{K}_i$ back to the reference element $\hat{K}$ by (3.2). By (3.3) and $\widehat{\operatorname{grad}}\,\hat{f}_O(0,0,1) = (0\ 0\ 2)^T$, (4.9) would become

$$\operatorname{grad} f_O|_{DABO}(O) = \frac{2(\vec{DA}\times\vec{DB})}{\vec{DO}\cdot(\vec{DA}\times\vec{DB})}.$$

Repeating the computation above on $\tilde{K}_2$ and $\tilde{K}_3$, we can find the gradient vector of f_O at O. We define

$$\tilde{P} = (\tilde{p}_{ij}) = \frac{1}{2}P,$$

where P is defined in (4.10). Following the idea in (4.11), let

$$\mathbf{v}_{2,2,i} = \begin{cases} \begin{pmatrix} \tilde{p}_{i1} \\ \tilde{p}_{i2} \\ \tilde{p}_{i3} \end{pmatrix} f_O & \text{on } \tilde{K}_1 \cup \tilde{K}_2 \cup \tilde{K}_3 \cup \tilde{K}_4, \\ \mathbf{0} & \text{elsewhere,} \end{cases}$$

for $i = 1, 2, 3$. On $\tilde{K}_4$, $\mathbf{v}_{2,2,i}$ is a P_2 polynomial identically zero on triangle ABC while matching the inter-element values on the other three faces of $\tilde{K}_4$. Then $\operatorname{div} \mathbf{v}_{2,2,i} = 0$ at all vertices of four $\tilde{K}_i$, except $\operatorname{div} \mathbf{v}_{2,2,i}|_{\tilde{K}_i}(O)$ (which is 1) and $\operatorname{div} \mathbf{v}_{2,2,i}|_{\tilde{K}_4}(O)$ (which may not be zero). Now we define

$$\mathbf{v}_{2,O} = \sum_{i=1}^{3} \tilde{q}_2|_{\tilde{K}_i}(O)\mathbf{v}_{2,2,i},$$

where $\tilde{q}_2$ is defined in (4.15). By the same argument used in (4.14), we also have that

$$|\mathbf{v}_{2,O}|_{H^1(\Omega)^3} \le C \| \operatorname{div} \mathbf{v}_{2,O} \|_{L^2(\Omega)}. \tag{4.18}$$

Let $\mathbf{v}_2 = \mathbf{v}_{2,v} + \mathbf{v}_{2,O}$. By the construction above, $\operatorname{div} \mathbf{v}_2$ matches $\tilde{q}$ at all vertices of $\tilde{K}_i$ except at O on $\tilde{K}_4$. As the difference is a C_{-1}–P_1 function, we have

$$\tilde{q} - \operatorname{div} \mathbf{v}_2 = \begin{cases} \alpha \frac{L_{ABC}(x,y,z)}{L_{ABC}(x_O,y_O,z_O)} & \text{in } \tilde{K}_4, \\ 0 & \text{in } \bar{K} \setminus \tilde{K}_4, \end{cases} \tag{4.19}$$

where $L_{ABC}(x, y, z) = 0$ is an equation for the plane ABC and $\alpha = \tilde{q}|_{\tilde{K}_4}(O) - \operatorname{div} \mathbf{v}_2|_{\tilde{K}_4}(O)$. Because $\mathbf{v}_2 \in H_0^1(\bar{K})^3$, we have

$$\int_{\bar{K}} \operatorname{div} \mathbf{v}_2 d\mathbf{x} = \int_{\partial \bar{K}} \mathbf{v}_2 \cdot \mathbf{n} dS = 0.$$

That is, (4.7) holds. By the condition $\int_{\bar{K}} \tilde{q} = 0$ and (4.19),

$$0 = \int_{\bar{K}} \tilde{q} - \operatorname{div} \mathbf{v}_2 = \int_{\tilde{K}_4} \alpha \frac{L_{ABC}(x, y, z)}{L_{ABC}(x_O, y_O, z_O)} = \frac{\alpha}{4} V_{OABC}.$$

This concludes $\alpha = 0$ and $\tilde{q} = \operatorname{div} \mathbf{v}_2$ on $\bar{K}$. As we can construct $\mathbf{v}_2 \in H_0^1(\bar{K})^3$ simultaneously on all $\bar{K} \in \bar{\Omega}_h$, by (4.14) and (4.18), the lemma is proven. □

On one macro-tetrahedron $\bar{K} = \bigcup K_{i,j}$, a piecewise P_1 pressure function $q \in P_h$ has $4 \times 3 \times 4 = 48$ nodal values on the (4×3) sub-tetrahedra. We will match all 6 values at vertices $\{\mathbf{x}_l, l = 1, 2, 3, 4\}$, 12 values at the internal point $\{\mathbf{b}_1\}$, and 3 values at 4 mid-face vertices $\{\mathbf{c}_l, l = 1, 2, 3, 4\}$ of a P_1 pressure function within each $\bar{K} \in \bar{\Omega}_h$, cf. Fig. 6. That is, there are

$$4 \times 6(\text{at 4 corners}) + 4 \times 3(\text{at the center } \mathbf{b}_1) + 4 \times 3(\text{at the face-center } \mathbf{c}_i) = 48$$

values of $q \in P_h$ to be matched by $\operatorname{div} \mathbf{w}_h$ for some $\mathbf{w}_h \in \mathbf{V}_h$. The matching is done in the following three correction steps.

1. Correct q at each vertex $\mathbf{x}_j$ of $\tilde{K}_i$ so that its two values there are same, in Lemma 4.3.
2. Correct q at the mid-face vertex $\mathbf{c}_l$ of $\tilde{K}_i$ so that its three values there are same, in Lemma 4.4.

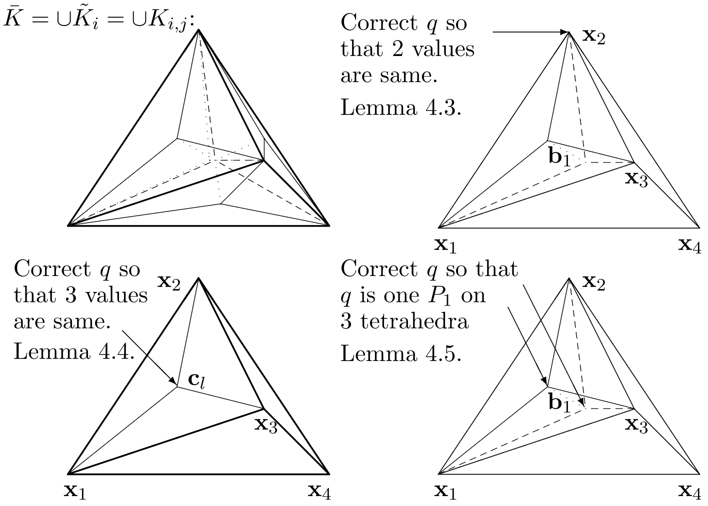

Fig. 6 Three steps (lemmas) to find $\mathbf{v}_6$: $(q - \operatorname{div}\mathbf{v}_6) \in \tilde{P}_h$

3. Correct q simultaneously at the inner vertex $\{\mathbf{b}_1\}$ and the mid-face vertex $\mathbf{c}_l$ of $\tilde{K}_i$, so that it becomes a whole P_1 function on $\tilde{K}_i$, instead of three P_1 functions on $\bigcup\{K_{i,j}\} = \tilde{K}_i$.

After these three corrections, such a q function is piecewise P_1 on the Hsieh–Clough–Tocher grid, i.e., a $\tilde{P}_h$ function, defined in (4.4). Then the inf-sup condition is proved by Lemma 4.2.

Lemma 4.3 *For any $q \in P_h$ such that $\int_{\bar{K}} q = 0$ for all $\bar{K} \in \bar{\Omega}_h$, there is a function $\mathbf{v}_3 \in \mathbf{V}_h$ such that,*

$$[q - \operatorname{div}\mathbf{v}_3]_{K_{i,j}}(\mathbf{x}_l) = [q - \operatorname{div}\mathbf{v}_3]_{K_{i,j+1}}(\mathbf{x}_l), \quad i,l = 1,2,3,4, \ \ j = 1,2, \tag{4.20}$$

$$\int_{\bar{K}} \operatorname{div}\mathbf{v}_3 = 0 \quad \forall \bar{K} \in \bar{\Omega}_h, \tag{4.21}$$

$$\|\mathbf{v}_3\|_{H^1(\Omega)^3} \le C\|q\|_{L^2(\Omega)}. \tag{4.22}$$

Here $\mathbf{x}_l$ is a vertex on the outside face-triangle of $\tilde{K}_i$ shared by two subtetrahedra $K_{i,j}$ and $K_{i,j+1}$ of $\tilde{K}_i$, cf. Fig. 6, that is, $\mathbf{x}_l$ is a vertex of $\bar{K}$, $\{K_{i,j},\ j = 1,2,3\}$ the three subtetrahedra of $\tilde{K}_i$, and $\bigcup_{i=1}^4 \tilde{K}_i = \bar{K}$.

Proof Let $q \in P_h$, $\int_{\bar{K}} q = 0$ for all $\bar{K} \in \bar{\Omega}_h$. We consider two cases in constructing $\mathbf{v}_3$. $\mathbf{v}_3$ is constructed on each subtetrahedron $\tilde{K}_i$, which may have its outside face triangle (the other three face-triangles are inside $\bar{K}$) on the boundary, or shared by a neighboring subtetrahedron $\tilde{K}_i'$. That is, for a subtetrahedron $\tilde{K}_4 \subset \bar{K}$, its outside face triangle ABC is either on $\partial\Omega$ (Fig. 7), or case two, an internal triangle shared by a neighboring tetrahedron $\tilde{K}_4'$ (Fig. 3). In the first case, we let $\mathbf{v}_3|_{\tilde{K}_4} = \mathbf{0}$, as the two values of q are same already at A, by (3.1).

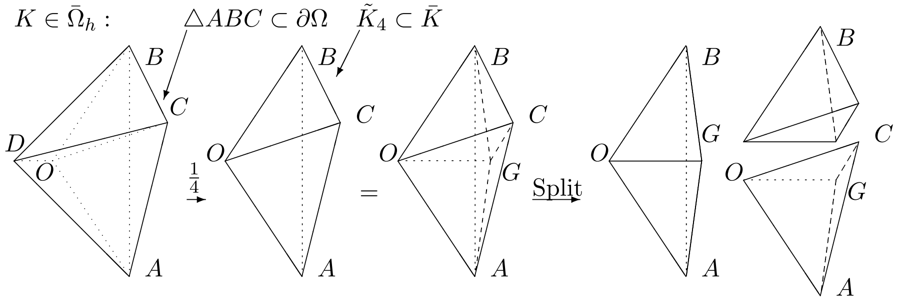

Fig. 7 Splitting each of 4 HCT subtetrahedra into 3 PS sub-subtetrahedra

For the second case, shown in Fig. 3, the outside face triangle ABC of $\tilde{K}_4$ is shared by a neighboring tetrahedron $\tilde{K}'_4 \in \tilde{\Omega}_h$. At vertex A, the values of q are studied in Lemma 3.4. By (3.20), using the three basis functions in (3.21), we can easily define a $\mathbf{v}_3$ so that the four values $\operatorname{div} \mathbf{v}_3(A)$ match those of $q(A)$ on the four subtetrahedra of $\tilde{K}_4$ and $\tilde{K}'_4$ meeting at A. But (3.21) uses basis functions on the boundary of $\tilde{K}_4 \cup \tilde{K}'_4$. Using such basis functions would destroy the matching made in neighboring tetrahedra. We choose another set of basis functions for $\mathbf{U}_{\operatorname{div} \mathbf{w}_h(A)}$ in (3.20). Define an inverse matrix similarly to (4.10)

$$P = (p_{ij}) = (\mathbf{m}_{123} \quad \mathbf{m}_{134} \quad \mathbf{m}_{145})^{-1}.$$

Here the matrix is invertible, cf. Fig. 3, because its determinant is non-zero:

$$\begin{aligned}
&\det\big(\mathbf{m}_{123} \quad \mathbf{m}_{134} \quad \mathbf{m}_{145}\big)\\
&\quad = \mathbf{m}_{13} \cdot (\mathbf{m}_{134} \times \mathbf{m}_{145})\\
&\quad = \frac{|AG|(\vec{AO} \times \vec{AC})}{\vec{AG} \cdot (\vec{AO} \times \vec{AC})} \cdot \left(\frac{|AG|(\vec{AC} \times \vec{AO'})}{\vec{AG} \cdot (\vec{AC} \times \vec{AO'})} \times \frac{|AG|(\vec{AO'} \times \vec{AB})}{\vec{AG} \cdot (\vec{AO'} \times \vec{AB})} \right)\\
&\quad = \frac{|AG|^3}{6^3 V_{AGOC} V_{AGCO'} V_{AGO'B}} (\vec{AO} \times \vec{AC}) \cdot \big[(\vec{AC} \times \vec{AO'}) \times (\vec{AO'} \times \vec{AB}) \big]\\
&\quad = \frac{|AG|^3}{6^3 V_{AGOC} V_{AGCO'} V_{AGO'B}} (\vec{AO} \times \vec{AC}) \cdot \big[6 V_{ACO'B} \vec{AO'} \big]\\
&\quad = \frac{|AG|^3 V_{ACO'B} V_{AOCO'}}{6 V_{AGOC} V_{AGCO'} V_{AGO'B}} \ge C > 0.
\end{aligned}$$

Again, the constant C depends only on the mesh quality (2.4). Now we define

$$\mathbf{v}_{3,\tilde{K}_4,A}(\mathbf{x}) = -\big(q|_{AGBO}(A) - q|_{AGOC}(A)\big) \left(\begin{pmatrix} p_{11} \\ p_{12} \\ p_{13} \end{pmatrix} + \begin{pmatrix} p_{21} \\ p_{22} \\ p_{23} \end{pmatrix} \right) f_1(\mathbf{x}),$$
$$\mathbf{x} \in (\tilde{K}_4 \cup \tilde{K}'_4).$$

By (3.20),

$$(q - \operatorname{div} \mathbf{v}_{3,\tilde{K}_4,A})(A) = \begin{cases} q|_{AGBO}(A) & \text{in } AGBO \,\&\, AGOC, \\ q|_{AGO'B}(A) & \text{in } AGO'B \,\&\, AGCO'. \end{cases}$$

Add such 3 corrections to define

$$\mathbf{v}_{3,\tilde{K}_4} = \mathbf{v}_{3,\tilde{K}_4,A} + \mathbf{v}_{3,\tilde{K}_4,B} + \mathbf{v}_{3,\tilde{K}_4,C}.$$

We would match the values of q on three vertices A, B and C, inside each of $\tilde{K}_4$ and $\tilde{K}_4'$. But $\mathbf{v}_{3,\tilde{K}_4}$ may violate the element divergence-free condition (4.21). To satisfy (4.21), we need to correct $\mathbf{v}_{3,\tilde{K}_4}$ further on its supported region $\tilde{K}_4 \cup \tilde{K}_4'$, see Fig. 3. Similarly to constructing f_O and $\mathbf{v}_{2,O}$ in Lemma 4.2, we define a P_2 nodal basis function on $\tilde{K}_4 \cup \tilde{K}_4'$,

$$f_G(x,y,z) = \begin{cases} L_{OAB}^2(x,y,z)/L_{OAB}^2(x_G,y_G,z_G) & \text{in } OBAG, \\ L_{OBC}^2(x,y,z)/L_{OBC}^2(x_G,y_G,z_G) & \text{in } OCBG, \\ L_{OCA}^2(x,y,z)/L_{OCA}^2(x_G,y_G,z_G) & \text{in } OACG, \\ L_{O'BA}^2(x,y,z)/L_{O'BA}^2(x_G,y_G,z_G) & \text{in } O'ABG, \\ L_{O'AC}^2(x,y,z)/L_{O'AC}^2(x_G,y_G,z_G) & \text{in } O'CAG, \\ L_{O'CB}^2(x,y,z)/L_{O'CB}^2(x_G,y_G,z_G) & \text{in } O'BCG, \end{cases} \tag{4.23}$$

where (x_G, y_G, z_G) are the coordinates of G, cf. Fig. 3, and $L_{OAB} = 0$ is an equation for the plane OAB, for example. $\operatorname{grad} f_G$ is a piecewise linear function with zero trace on the boundary of $\tilde{K}_4 \cup \tilde{K}_4'$. Let

$$\mathbf{v}_{3,G}' = \frac{\vec{OG}}{12|\vec{OG}|} f_G. \tag{4.24}$$

Then, because $V_{OABC} \neq 0$,

$$\begin{aligned} \operatorname{div} \mathbf{v}_{3,G}'(\mathbf{x}) &> 0 \quad \text{if } \mathbf{x} \in \tilde{K}_4, \\ \operatorname{div} \mathbf{v}_{3,G}'(\mathbf{x}) &= 0 \quad \text{if } \mathbf{x} \in \partial\tilde{K}_4, \\ \operatorname{div} \mathbf{v}_{3,G}'(\mathbf{x}) &= 0 \quad \text{if } \mathbf{x} \in \partial\tilde{K}_4', \\ \operatorname{div} \mathbf{v}_{3,G}'(\mathbf{x}) &< 0 \quad \text{if } \mathbf{x} \in \tilde{K}_4'. \end{aligned}$$

Again, we note that $\operatorname{div} \mathbf{v}_{3,G}'(A) = 0$, which does not destroy the matching we made by $\mathbf{v}_{3,\tilde{K}_4}$. We correct $\mathbf{v}_{3,\tilde{K}_4}$ (on $\tilde{K}_4 \cup \tilde{K}_4'$ for the three vertices of both tetrahedra) by

$$\mathbf{v}_{3,G} = \mathbf{v}_{3,\tilde{K}_4} - \mathbf{v}_{3,G}' \frac{\int_{\tilde{K}_4} \operatorname{div} \mathbf{v}_{3,\tilde{K}_4}}{\int_{\tilde{K}_4} \operatorname{div} \mathbf{v}_{3,G}'}.$$

Then $\int_{\tilde{K}_4} \operatorname{div} \mathbf{v}_{3,G} = \int_{\tilde{K}_4'} \operatorname{div} \mathbf{v}_{3,G} = 0$. Add all such $\mathbf{v}_{3,G}$ for all internal mid-face points of $\bar{K}$ of $\bar{\Omega}_h$ to define $\mathbf{v}_3$. Then $\mathbf{v}_3$ satisfies (4.20) and (4.21). As in Lemma 4.2, because $\mathbf{v}_3$ is constructed locally, (4.22) holds. $\square$

Lemma 4.4 *For any $q \in P_h$ such that $\int_{\bar K} q = 0$ for all $\bar K \in \bar\Omega_h$, there is a function $\mathbf{v}_4 \in \mathbf{V}_h$ such that,*

$$\operatorname{div} \mathbf{v}_4(\mathbf{x}_l) = 0, \quad \forall \mathbf{x}_l \in \bar K \in \bar\Omega_h, \tag{4.25}$$

$$[q - \operatorname{div} \mathbf{v}_4]_{K_{i,1}}(\mathbf{c}_l) = [q - \operatorname{div} \mathbf{v}_4]_{K_{i,j}}(\mathbf{c}_l), \quad j = 2, 3, \ i = 1, 2, 3, 4, \tag{4.26}$$

$$\int_{\bar K} \operatorname{div} \mathbf{v}_4 = 0 \ \forall \bar K \in \bar\Omega_h, \tag{4.27}$$

$$\|\mathbf{v}_4\|_{H^1(\Omega)^3} \le C \|q\|_{L^2(\Omega)}. \tag{4.28}$$

Here $\mathbf{x}_l$ is any one vertex of $\bar K$, shown in Fig. 6, $\mathbf{c}_l$ is the mid-face point G on the outside face-triangle of $\tilde K_i$, $i = 1, 2, 3, 4$, $\bigcup_{i=1}^4 \tilde K_i = \bar K$, cf. Fig. 3, and $\{K_{i,j}, \ j = 1, 2, 3\}$ the three subtetrahedra of $\tilde K_i$.

Proof Let $q \in P_h$, $\int_{\bar K} q = 0$ for all $\bar K \in \bar\Omega_h$. There are two cases in constructing $\mathbf{v}_4$. $\mathbf{v}_4$ is constructed on each subtetrahedron $\tilde K_i$, which may have its outside face triangle (the other three face-triangles are inside $\bar K$) on the boundary (Fig. 7), or shared by a neighboring subtetrahedron $\tilde K_i'$ (Fig. 3). In the first case, we let $\mathbf{v}_4|_{\tilde K_4} = \mathbf{0}$, as the three values of q are same already at G, by (3.7), in Fig. 7.

For the second case, shown in Fig. 3, the face triangle ABC is shared by $\tilde K_4$ and $\tilde K_4' \in \tilde\Omega_h$. Using the notations in Lemma 3.3, if

$$\begin{aligned}\mathbf{v}_{4,G}'' &= q|_{GOAB}(G)\mathbf{n}_{\mathbf{m}_{213}}^{\perp \mathbf{m}_{241}} f_2 + q|_{GOAC}(G)\mathbf{n}_{\mathbf{m}_{241}}^{\perp \mathbf{m}_{213}} f_2 \\ &\quad + q|_{GOCB}(G)\mathbf{n}_{\mathbf{m}_{314}}^{\perp \mathbf{m}_{321}} f_3 + \big(q|_{O'GAB}(G) - q|_{GOAB}(G)\big)\frac{\mathbf{m}_{523}}{|\mathbf{m}_{523}|^2} f_5,\end{aligned}$$

by Lemma 3.3,

$$\operatorname{div} \mathbf{v}_{4,G}''(G) = q(G) \text{ at all 6 subtetrahedra meeting at } G.$$

That is, we have a total match at G. However, $\mathbf{v}_{4,G}''$ would violate condition (4.25), that is, it would destroy the matching of Lemma 4.3 made at vertices such as A in Fig. 3. We will construct a $\mathbf{v}_{4,G}'$ using only one nodal function f_G at point G, defined in (4.23). At point G, cf. (3.12) for notations,

$$-\frac{1}{2}\operatorname{grad} f_G(G) = \begin{cases} \frac{\mathbf{m}_{132}}{|\mathbf{m}_1|} + \frac{\mathbf{m}_{321}}{|\mathbf{m}_3|} + \frac{\mathbf{m}_{213}}{|\mathbf{m}_2|} & \text{in } OBAG, \\ \frac{\mathbf{m}_{143}}{|\mathbf{m}_1|} + \frac{\mathbf{m}_{431}}{|\mathbf{m}_4|} + \frac{\mathbf{m}_{314}}{|\mathbf{m}_3|} & \text{in } OCBG, \\ \frac{\mathbf{m}_{124}}{|\mathbf{m}_1|} + \frac{\mathbf{m}_{241}}{|\mathbf{m}_2|} + \frac{\mathbf{m}_{412}}{|\mathbf{m}_4|} & \text{in } OACG, \\ \frac{\mathbf{m}_{523}}{|\mathbf{m}_5|} + \frac{\mathbf{m}_{235}}{|\mathbf{m}_2|} + \frac{\mathbf{m}_{352}}{|\mathbf{m}_3|} & \text{in } O'ABG, \\ \frac{\mathbf{m}_{534}}{|\mathbf{m}_5|} + \frac{\mathbf{m}_{345}}{|\mathbf{m}_3|} + \frac{\mathbf{m}_{453}}{|\mathbf{m}_4|} & \text{in } O'CAG, \\ \frac{\mathbf{m}_{542}}{|\mathbf{m}_5|} + \frac{\mathbf{m}_{425}}{|\mathbf{m}_4|} + \frac{\mathbf{m}_{254}}{|\mathbf{m}_2|} & \text{in } O'BCG. \end{cases}$$

Define an inverse matrix by

$$P = (p_{ij}) = \left(\frac{\mathbf{m}_{132}}{|\mathbf{m}_1|} + \frac{\mathbf{m}_{321}}{|\mathbf{m}_3|} + \frac{\mathbf{m}_{213}}{|\mathbf{m}_2|} \ \frac{\mathbf{m}_{143}}{|\mathbf{m}_1|} + \frac{\mathbf{m}_{431}}{|\mathbf{m}_4|} + \frac{\mathbf{m}_{314}}{|\mathbf{m}_3|} \ \frac{\mathbf{m}_{124}}{|\mathbf{m}_1|} + \frac{\mathbf{m}_{241}}{|\mathbf{m}_2|} + \frac{\mathbf{m}_{412}}{|\mathbf{m}_4|}\right)^{-1}.$$

Here the matrix is invertible because its vectors are linearly independent, to be shown below. Let

$$c_1\left(\frac{\mathbf{m}_{132}}{|\mathbf{m}_1|}+\frac{\mathbf{m}_{321}}{|\mathbf{m}_3|}+\frac{\mathbf{m}_{213}}{|\mathbf{m}_2|}\right)+c_2\left(\frac{\mathbf{m}_{143}}{|\mathbf{m}_1|}+\frac{\mathbf{m}_{431}}{|\mathbf{m}_4|}+\frac{\mathbf{m}_{314}}{|\mathbf{m}_3|}\right)$$
$$+c_3\left(\frac{\mathbf{m}_{124}}{|\mathbf{m}_1|}+\frac{\mathbf{m}_{241}}{|\mathbf{m}_2|}+\frac{\mathbf{m}_{412}}{|\mathbf{m}_4|}\right)=\mathbf{0}.$$

Its dot product with vector $(\mathbf{m}_2-\mathbf{m}_1)$ is

$$c_1\left(-\frac{v_{132}}{v_{132}}+\frac{v_{132}}{v_{132}}\right)+c_2\left(-\frac{v_{213}}{v_{431}}-\frac{v_{241}}{v_{314}}-\frac{v_{143}}{v_{143}}\right)+c_3\left(\frac{v_{241}}{v_{241}}-\frac{v_{124}}{v_{124}}\right)=0,$$
$$-c_2\frac{V_{OABC}}{V_{OGBC}}=0 \quad\Rightarrow\quad c_2=0.$$

Here v_{ijk} denotes the (positive) volume of tetrahedron formed by vectors $\mathbf{m}_i$, $\mathbf{m}_j$ and $\mathbf{m}_k$. By computing the dot products with $(\mathbf{m}_3-\mathbf{m}_1)$ and $(\mathbf{m}_4-\mathbf{m}_1)$, we find $c_1=0$ and $c_3=0$. Let

$$\mathbf{v}'_{4,G}=\left(\frac{1}{2}\big(q|_{GOAB}(G)-q|_{GOCA}(G)\big)\begin{pmatrix}p_{21}\\p_{22}\\p_{23}\end{pmatrix}\right.$$
$$\left.+\frac{1}{2}\big(q|_{GOAB}(G)-q|_{GOBC}(G)\big)\begin{pmatrix}p_{31}\\p_{32}\\p_{33}\end{pmatrix}\right)f_G.$$

Then

$$(q-\operatorname{div}\mathbf{v}'_{4,G})(G)=\begin{cases}q|_{GOAB}(G) & \text{in } GOAB, GOCA \,\&\, GOBC,\\ c_0 & \text{in } GO'AB, GO'CA \,\&\, GO'BC.\end{cases}$$
$$(q-\operatorname{div}\mathbf{v}'_{4,G})(\mathbf{x})=q(\mathbf{x}) \quad \mathbf{x}=A,B,C,O \,\&\, O'.$$

Here we obtain a same constant c_0 in the three subtetrahedral of $\tilde{K}'$ because of (3.11). We correct $\mathbf{v}'_{4,G}$ (on $\tilde{K}_4\cup\tilde{K}'_4$) to keep (4.27) by the function $\mathbf{v}'_{3,G}$ defined in (4.24):

$$\mathbf{v}_{4,G}=\mathbf{v}'_{4,G}-\mathbf{v}'_{3,G}\frac{\int_{\tilde{K}_4}\operatorname{div}\mathbf{v}'_{4,G}d\mathbf{x}}{\int_{\tilde{K}_4}\operatorname{div}\mathbf{v}'_{3,G}d\mathbf{x}}.$$

Note that, by (3.11), the three values of $\operatorname{div}\mathbf{v}'_{3,G}(G)$ are same on the side of $OABC$. This correction does not perturb the matching (4.26) by $\mathbf{v}'_{4,G}$. Now, we let $\mathbf{v}_4$ be the sum of $\mathbf{v}_{4,G}$ for all mid-face, internal points G on $\bar{K}$. Finally, as in Lemma 4.2, (4.28) holds as each $\mathbf{v}_{4,G}$ is constructed locally. □

Lemma 4.5 *For any $q\in P_h$ satisfying that*

$$\int_{\bar{K}} q=0\;\forall\bar{K}\in\bar{\Omega}_h,$$

$$q|_{K_{i,j}}(\mathbf{x}_l) = q|_{K_{i,j+1}}(\mathbf{x}_l) \quad \forall \mathbf{x}_l \in \bar{K} \in \bar{\Omega}_h,$$
$$q|_{K_{i,1}}(\mathbf{c}_l) = q|_{K_{i,j}}(\mathbf{c}_l), \quad j=2,3,\ i=1,2,3,4,$$

there is a function $\mathbf{v}_5 \in \mathbf{V}_h$ *such that,*

$$(q - \operatorname{div}\mathbf{v}_5) \in P_1(\tilde{K}_i), \quad i=1,2,3,4, \tag{4.29}$$

$$\int_{\bar{K}} \operatorname{div}\mathbf{v}_5 = 0 \quad \forall \bar{K} \in \bar{\Omega}_h, \tag{4.30}$$

$$\|\mathbf{v}_5\|_{H^1(\Omega)^3} \le C\|q\|_{L^2(\Omega)}. \tag{4.31}$$

Here $\mathbf{c}_l$ *is the mid-face point* G *on the outside face-triangle of* $\tilde{K}_i$, $i=1,2,3,4$, $\bigcup_{i=1}^4 \tilde{K}_i = \bar{K}$, *cf. Fig. 6, and* $\mathbf{x}_l$ *is a vertex of* $\bar{K}$.

Proof Let $q \in P_h$ be a given P_h function in the lemma. $\mathbf{v}_5$ is constructed on each HCT tetrahedron such as $\tilde{K}_4$ in Figs. 7 and 3. That is, to satisfy (4.29),

$$(q - \operatorname{div}\mathbf{v}_5)(\mathbf{x}) = \begin{cases} \gamma & \text{in } OGBA = K_{4,1} \subset \tilde{K}_4,\ \mathbf{x}=O, \\ \gamma & \text{in } OGAC = K_{4,2} \subset \tilde{K}_4,\ \mathbf{x}=O, \\ \gamma & \text{in } OGCB = K_{4,3} \subset \tilde{K}_4,\ \mathbf{x}=O, \\ \gamma_0 & \text{in } K_{4,i}, i=1,2,3,\ \mathbf{x}=G, \\ q(\mathbf{x}) & \text{in } K_{4,i}, i=1,2,3,\ \mathbf{x}=A,B,C, \end{cases} \tag{4.32}$$

where the constant γ is to be specified in (4.37) and γ_0 in (4.36).

Let $f_{\tilde{K}_4}$ be a scaled internal nodal basis function on $\tilde{K}_4$, a piecewise P_2 function with zero trace.

$$f_{\tilde{K}_4}(x,y,z) = \begin{cases} \dfrac{|\vec{OG}|L_{ABO}(\mathbf{x})L_{ABC}(\mathbf{x})}{L_{ABO}(\mathbf{x}_G)L_{ABC}(\mathbf{x}_O)} & \text{in } OGBA, \\ \dfrac{|\vec{OG}|L_{CAO}(\mathbf{x})L_{ABC}(\mathbf{x})}{L_{CAO}(\mathbf{x}_G)L_{ABC}(\mathbf{x}_O)} & \text{in } OGAC, \\ \dfrac{|\vec{OG}|L_{BCO}(\mathbf{x})L_{ABC}(\mathbf{x})}{L_{BCO}(\mathbf{x}_G)L_{ABC}(\mathbf{x}_O)} & \text{in } OGCB. \end{cases}$$

Here $L_{ABO}=0$ is an equation for the plane ABO. By the reference mappings, cf. (3.2) and (3.3), it follows that

$$\operatorname{grad} f_{\tilde{K}_4}(O) = \begin{cases} \dfrac{|\vec{OG}|(\vec{OA}\times\vec{OB})}{\vec{OG}\cdot(\vec{OA}\times\vec{OB})} & \text{in } OGBA, \\ \dfrac{|\vec{OG}|(\vec{OC}\times\vec{OA})}{\vec{OG}\cdot(\vec{OC}\times\vec{OA})} & \text{in } OGAC, \\ \dfrac{|\vec{OG}|(\vec{OB}\times\vec{OC})}{\vec{OG}\cdot(\vec{OB}\times\vec{OC})} & \text{in } OGCB, \end{cases}$$

$$\operatorname{grad} f_{\tilde{K}_4}(\mathbf{x}) = \mathbf{0} \quad \text{in } K_{4,i},\ i=1,2,3,\ \mathbf{x}=A,B,C,$$

$$\operatorname{grad} f_{\tilde{K}_4}(G) = \frac{|\vec{GO}|(\vec{GB}\times\vec{GA})}{\vec{GO}\cdot(\vec{GB}\times\vec{GA})} \quad \text{in } K_{4,i},\ i=1,2,3. \tag{4.33}$$

Similarly to the construction of $\mathbf{v}_{2,1,1}$ in (4.10) and (4.11), let matrix Q be the inverse matrix

$$Q=(q_{ij})_{3\times 3}=\left(\frac{|\vec{OG}|(\vec{OA}\times\vec{OB})}{\vec{OG}\cdot(\vec{OA}\times\vec{OB})}\;\frac{|\vec{OG}|(\vec{OC}\times\vec{OA})}{\vec{OG}\cdot(\vec{OC}\times\vec{OA})}\;\frac{|\vec{GO}|(\vec{GB}\times\vec{GA})}{\vec{GO}\cdot(\vec{GB}\times\vec{GA})}\right)^{-1}.$$

Here the matrix is invertible as its determinant is non-zero:

$$\begin{aligned}\det(Q^{-1})&=\frac{|\vec{OG}|^3}{6^3V_{GOAB}^2V_{GOAC}}(\vec{OA}\times\vec{OB})\times(\vec{OC}\times\vec{OA})\cdot(\vec{GB}\times\vec{GA})\\&=\frac{|\vec{OG}|^3}{6^3V_{GOAB}^2V_{GOAC}}(-V_{OABC}\vec{OA})\cdot(\vec{GB}\times\vec{GA})\\&=\frac{|\vec{OG}|^3V_{OABC}}{6^3V_{GOAB}V_{GOAC}}\geq C>0.\end{aligned}$$

We then use the row vectors to define three $\mathbf{v}_{5,i}\in H_0^1(\tilde{K}_4)^3$:

$$\mathbf{v}_{5,i}=\begin{cases}\begin{pmatrix}q_{i1}\\q_{i2}\\q_{i3}\end{pmatrix}f_{\tilde{K}_4} & \text{in } \tilde{K}_4=OGAC\cup OGBA\cup OGCB,\\ \mathbf{0} & \text{elsewhere.}\end{cases}\tag{4.34}$$

It follows that, $i=1,2$, $j=1,2$, $l=1,2,3$,

$$\begin{aligned}&\operatorname{div}\mathbf{v}_{5,i}(O)=\begin{cases}\delta_{ij} & \text{in } OGBA=K_{4,1},\quad OGAC=K_{4,2},\\ c_i & \text{in } OGCB=K_{4,3},\end{cases}\\&\operatorname{div}\mathbf{v}_{5,i}(\mathbf{x})=0\quad\text{in } K_{4,l},\ \mathbf{x}=A,B,C,G,\\&\operatorname{div}\mathbf{v}_{5,3}(O)=\begin{cases}0 & \text{in } OGBA=K_{4,1},\quad OGAC=K_{4,2},\\ c_3 & \text{in } OGCB=K_{4,3},\end{cases}\\&\operatorname{div}\mathbf{v}_{5,3}(\mathbf{x})=0\quad\text{in } K_{4,l},\ \mathbf{x}=A,B,C,\\&\operatorname{div}\mathbf{v}_{5,3}(G)=1\quad\text{in } K_{4,l},\end{aligned}$$

where $\{c_i, i=1,2,3\}$ are constants, defined by the matrix-vector product $Q[|\vec{OG}|(\vec{OB}\times\vec{OC})/(\vec{OG}\cdot(\vec{OB}\times\vec{OC}))]$. To find out these constants, we rewrite the vector, cf. Fig. 3, as a linear combination of the three vectors in defining Q:

$$\begin{aligned}\vec{OB}\times\vec{OC}&=(\vec{OA}+\vec{AB})\times\vec{OC}\\&=\vec{OA}\times\vec{OC}+\vec{AB}\times(\vec{OA}+\vec{AC})\\&=-\vec{OC}\times\vec{OA}+\vec{AB}\times(\vec{OA}+\vec{AC})\\&=-\vec{OC}\times\vec{OA}+\vec{AB}\times\vec{OA}-\vec{AC}\times\vec{AB}\end{aligned}$$

$$= -\vec{OC} \times \vec{OA} + (\vec{AO} + \vec{OB}) \times \vec{OA} - \vec{AC} \times \vec{AB}$$
$$= -\vec{OA} \times \vec{OB} - \vec{OC} \times \vec{OA} - \vec{AC} \times \vec{AB}.$$

The two unit vectors, orthogonal to triangle ABC, are the same:

$$\frac{\vec{GB} \times \vec{GA}}{|\vec{GB} \times \vec{GA}|} = \frac{\vec{AC} \times \vec{AB}}{|\vec{AC} \times \vec{AB}|}.$$

The ratio of two vector lengths is also the ratio of the volume of big tetrahedron to that of its subtetrahedron:

$$\frac{|\vec{AC} \times \vec{AB}|}{|\vec{GB} \times \vec{GA}|} = \frac{V_{OABC}}{V_{OABG}}.$$

Then, we find out that

$$\begin{aligned}
\operatorname{grad} f_{\tilde{K}_4}|_{OGCB}(O) &= \frac{|\vec{OG}|(\vec{OB} \times \vec{OC})}{\vec{OG} \cdot (\vec{OB} \times \vec{OC})} \\
&= -\frac{\vec{OG} \cdot (\vec{OA} \times \vec{OB})}{\vec{OG} \cdot (\vec{OB} \times \vec{OC})} \frac{|\vec{OG}|(\vec{OA} \times \vec{OB})}{\vec{OG} \cdot (\vec{OA} \times \vec{OB})} \\
&\quad - \frac{\vec{OG} \cdot (\vec{OC} \times \vec{OA})}{\vec{OG} \cdot (\vec{OB} \times \vec{OC})} \frac{|\vec{OG}|(\vec{OC} \times \vec{OA})}{\vec{OG} \cdot (\vec{OC} \times \vec{OA})} \\
&\quad - \frac{V_{OABC}}{V_{OABG}} \frac{\vec{OG} \cdot (\vec{GB} \times \vec{GA})}{\vec{OG} \cdot (\vec{OB} \times \vec{OC})} \frac{|\vec{OG}|(\vec{GB} \times \vec{GA})}{\vec{OG} \cdot (\vec{GB} \times \vec{GA})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
c_1 &= -\frac{V_{OABG}}{V_{OBCG}}, \\
c_2 &= -\frac{V_{OCAG}}{V_{OBCG}}, \\
c_3 &= -\frac{V_{OABC}}{V_{OBCG}} = -1 + c_1 + c_2 < 0.
\end{aligned} \tag{4.35}$$

Now, we define

$$\mathbf{v}_5 = \sum_{i=1}^{3} v_i \mathbf{v}_{5,i},$$

where

$$\begin{aligned}
v_3 &= q|_{\tilde{K}_4}(G) - \gamma_0, \\
v_2 &= q|_{GOAC}(O) - \gamma, \\
v_1 &= q|_{GOBA}(O) - \gamma.
\end{aligned}$$

Here γ_0 is the same as that in (4.32), defined by

$$\gamma_0 = q|_{GOAB}(C)\frac{V_{GOAB}}{V_{OABC}} + q|_{GOBC}(A)\frac{V_{GOBC}}{V_{OABC}} + q|_{GOBC}(B)\frac{V_{GOCA}}{V_{OABC}}, \tag{4.36}$$

and γ is the same as that in (4.32) too, defined by

$$\gamma = \frac{q|_{OGCB}(O) - c_1 q|_{OGBA}(O) - c_2 q|_{OGAC}(O) - v_3 c_3}{1 - c_1 - c_2}. \tag{4.37}$$

By (4.35), γ is well defined in (4.37). Then

$$\operatorname{div} \mathbf{v}_5(\mathbf{x}) = \begin{cases} q|_{GOBA}(O) - \gamma & \text{if } \mathbf{x} = O, \quad \text{in } OGAB, \\ q|_{GOAC}(O) - \gamma & \text{if } \mathbf{x} = O, \quad \text{in } OGCA, \\ c_1 v_1 + c_2 v_2 + c_3 v_3 & \\ \quad = q|_{GOBC}(O) - \gamma & \text{if } \mathbf{x} = O, \quad \text{in } OGBC, \\ q|_{\tilde{K}_4}(G) - \gamma_0 & \text{if } \mathbf{x} = G, \quad \text{in } OGAB, OGBC \;\&\; OGCA, \\ 0 & \text{if } \mathbf{x} = A, B \;\&\; C. \end{cases}$$

Now we sum all such $\mathbf{v}_5$ over all HCT tetrahedra $\tilde{K}_i$ to define the global $\mathbf{v}_5$. Equation (4.30) holds as $\int_{\tilde{K}_4} \operatorname{div} \mathbf{v}_5 d\mathbf{x} = \int_{\partial \tilde{K}_4} \mathbf{v}_5 \cdot \mathbf{n} dS = 0$. Again, as $\mathbf{v}_5$ is defined locally, (4.31) holds. □

By the construction of $\mathbf{v}_5$ in this lemma, we already proved the inclusion $\tilde{P}_h \subset \operatorname{div} \mathbf{V}_h$ (4.5). But we will prove it as a corollary later. We are ready to prove the main theorem.

Theorem 4.1 (The inf-sup condition) *For any $q \in P_h$, there is a function $\mathbf{v}_h \in \mathbf{V}_h$ such that*

$$\operatorname{div} \mathbf{v}_h|_K(\mathbf{x}) = q|_K(\mathbf{x}) \; \forall K \in \Omega_h, \tag{4.38}$$

$$\|\mathbf{v}_h\|_{H^1(\Omega)^3} \le C\|q\|_{L^2(\Omega)}. \tag{4.39}$$

That is,

$$\inf_{q \in P_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q)}{\|\mathbf{v}_h\|_{H^1(\Omega)^3}\|q\|_{L^2(\Omega)}} \ge C. \tag{4.40}$$

Proof Let $q \in P_h$.

1. By Lemma 4.1, there is a $\mathbf{v}_1$ such that $\int_{\tilde{K}} q - \operatorname{div} \mathbf{v}_1 = 0$.
2. By Lemma 4.3, applied to $q - \operatorname{div} \mathbf{v}_1$, we get a $\mathbf{v}_3$ such that $\int_{\tilde{K}}(q - \operatorname{div} \mathbf{v}_1 - \operatorname{div} \mathbf{v}_3) = 0$ and $(q - \operatorname{div} \mathbf{v}_1 - \operatorname{div} \mathbf{v}_3)$ assumes a same value at all vertex $\mathbf{x}_i$ on the two subtetrahedra of all $\tilde{K}_i \in \tilde{\Omega}_h$.

3. Applying Lemma 4.4 to $(q - \operatorname{div} \mathbf{v}_1 - \operatorname{div} \mathbf{v}_3) \in P_h$, there is a $\mathbf{v}_4$ such that $(q - \operatorname{div} \mathbf{v}_1 - \operatorname{div} \mathbf{v}_3 - \operatorname{div} \mathbf{v}_4)$ assumes a same value at any vertex $\mathbf{x}_i$ of $\bar{K}$ on the two subtetrahedra, and a same value at the mid-face point $\mathbf{c}_l$ on the three subtetrahedra, of all $\tilde{K}_i \in \tilde{\Omega}_h$.
4. Applying Lemma 4.5 to $(q - \operatorname{div} \mathbf{v}_1 - \operatorname{div} \mathbf{v}_3 - \operatorname{div} \mathbf{v}_4) \in P_h$, we find a $\mathbf{v}_5$ such that $(q - \operatorname{div} \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5)) \in P_1(\tilde{K}_i)$ for all $\tilde{K}_i \in \tilde{\Omega}_h$. We call $\mathbf{v}_6 = \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5$.
5. Applying Lemma 4.2 to $(q - \operatorname{div} \mathbf{v}_6) \in P_h$, we get an $\mathbf{v}_2$ such that $(q - \operatorname{div} \mathbf{v}_6 - \operatorname{div} \mathbf{v}_2) = 0$.

Let $\mathbf{v}_h = \mathbf{v}_2 + \mathbf{v}_6$. Then (4.38) holds. (4.39) follows (4.1), (4.8), (4.22), (4.28) and (4.31). □

Corollary 4.1 *The pressure space P_h defined in (2.8) can be characterized as a C_{-1}–P_1 space with additional constraints at each mid-face vertex G and at each corner vertex A:*

$$\begin{cases} q(G_1^-) - q(G_2^-) = q(G_1^+) - q(G_2^+) \\ q(G_2^-) - q(G_3^-) = q(G_2^+) - q(G_3^+) \\ q(A_1^-) - q(A_2^-) = q(A_1^+) - q(A_2^+) \end{cases} \quad \text{and} \quad \begin{array}{l} \forall A \in \tilde{K}_4 \cup \tilde{K}_4', \quad \tilde{K}_4 \in \tilde{\Omega}_h, \\ \forall G \in \tilde{K}_4, \quad \tilde{K}_4 \in \tilde{\Omega}_h, \end{array} \tag{4.41}$$

where G_1^-, G_2^- and G_3^- are three vertices at the mid-face point of the outside face triangle of $\tilde{K}_4$, cf. Figs. 3 and 8, and G_i^+ are corresponding vertices on the neighboring tetrahedron of $\tilde{K}_4$, A_1^- and A_2^- are two vertices at point A of two subtetrahedra of $\tilde{K}_4$. If $G_i^- \in \partial\Omega$ or $A_i^- \in \partial\Omega$, we denote $q(G_i^+) = 0$ or $q(A_i^+) = 0$, respectively. Further, the following inclusions hold,

$$\bar{P}_h \subset \tilde{P}_h \subset P_h, \tag{4.42}$$

where $\tilde{P}_h$ (defined in (4.4)) is the C_{-1}–P_1 space on the Hsieh–Clough–Tocher grid $\tilde{\Omega}_h$, and $\bar{P}_h$ is the C_0–P_1 space on the base grid $\bar{\Omega}_h$,

$$\bar{P}_h = \left\{ q \in L_0^2(\Omega) \cap C(\Omega) \mid q|_{\bar{K}} \in P_1 \ \forall \bar{K} \in \bar{\Omega}_h \right\}. \tag{4.43}$$

Proof Let P_h^* be the space of C_{-1}–P_1 functions q on Ω_h, such that $q \in L_0^2(\Omega)$ and q satisfies (4.41):

$$P_h^* = \left\{ q \in L_0^2(\Omega) \mid q|_K \in P_1 \ \forall K \in \Omega_h; \ q \text{ satisfies (4.41)} \right\}.$$

$\bar{P}_h \subset P_h^*$ holds simply by their definitions. We note that $\tilde{P}_h$ is the full C_{-1}–P_1 L_0^2-space on the grid $\tilde{\Omega}_h$, by Lemma 4.2. Therefore, $\bar{P}_h \subset \tilde{P}_h$.

Next, the inclusion $\tilde{P}_h \subset P_h^*$ follows (4.41) directly as a $\tilde{P}_h$-function is one polynomial at all vertices in (4.41), i.e., $q(G_1^-) = q(G_2^-) = q(G_3^-)$ and $q(A_1^-) = q(A_2^-)$ for all $q \in \tilde{P}_h$. (4.42) would follow after we show $P_h^* = P_h$.

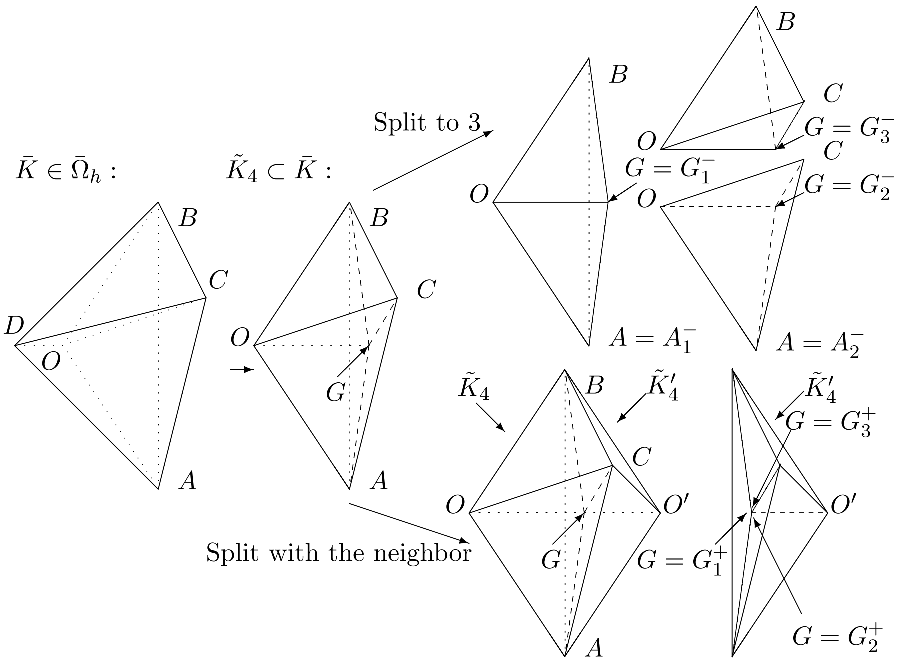

Fig. 8 The vertices and the Lagrange interpolation nodes

Let $q \in P_h$. By Lemmas 3.2 and 3.3, at each singular point $G(= G_i^- = G_j^+)$, see Fig. 8, we have

$$\mathbf{U}_q(G) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus q satisfies the first condition in (4.41), i.e.,

$$\mathbf{U}_q(G) \perp \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{U}_q(G) \perp \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Further, by Lemmas 3.1 and 3.4, we showed that

$$\mathbf{U}_q(A) = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{U}_q(A) \perp \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

i.e., q satisfies the second condition in (4.41). Therefore

$$P_h \subset P_h^*.$$

For the other direction of inclusion, for each $q \in P_h^*$, q satisfies the restrictions (3.1), (3.7), (3.13) and (3.17). By repeating the proof for Theorem 4.1, we construct a $\mathbf{v}_h \in \mathbf{V}_h$ such that $\operatorname{div} \mathbf{v}_h = q$. Thus $q \in \operatorname{div} \mathbf{V}_h = P_h$. So $P_h^* \subset P_h$. □

We remark that it is usually difficult to characterize the divergence space $\operatorname{div} \mathbf{V}_h$, i.e., the pressure space in the divergence-free finite element method. In 2D, Scott and Vogelius [10] found the space $\operatorname{div} \mathbf{V}_h$ for P_k ($k \geq 4$) which is the full C_{-1}–P_{k-1} L_0^2-space except a constraint like (4.41) at each internal and boundary singular vertex. For the 3D Hsieh–Clough–Tocher C_0–P_k H_0^1-finite element space, the divergence is the full C_{-1}–P_{k-1} L_0^2-space, if $k > 2$, i.e., there is no so-called singular vertex, cf. [16]. But it is not known if $\operatorname{div} \tilde{\mathbf{V}}_h = \tilde{P}_h$ when the polynomial degree $k = 2$ (see the comments before (4.5) and in the section on numerical tests). Let us make a brief comment on the 3D singular-vertex by an example where the domain Ω consists of one tetrahedron $\bar{K} \in \bar{\Omega}_h$. The P_2 H_0^1-space has a dimension 0 on the grid $\bar{\Omega}_h = \{\bar{K}\}$, while the P_1 L_0^2-space has a dimension $4 - 1 = 3$. So, in this case, all 4 vertices of $\bar{K}$ are singular, and the divergence of this velocity space (consists of 0) is not the full C_{-1}–P_1 L_0^2-space on the grid. However, the HCT C_0–P_2 H_0^1-space has a dimension $4 \times 3 + 3 = 15$ on the grid $\tilde{\Omega}_h$ (consisting of 4 subtetrahedra $\{\tilde{K}_i\}$), while the C_{-1}–P_1 L_0^2-space, $\tilde{P}_h$ in (4.4), has a dimension $4 \times 3 - 1 = 11$. It is shown in [16] that $\operatorname{div} \tilde{\mathbf{V}}_h = \tilde{P}_h$ even when $k = 2$, but on one macro-element. So we may say there is no singular vertex in HCT grids. But we do have such singular vertices G and A, on Powell–Sabin grids, listed in the constraints (4.41).

We conclude the analysis by the convergence theorem which follows the inf-sup condition as usual.

Theorem 4.2 *The discrete solutions* $(\mathbf{u}_h, p_h)$ *of* (2.9) *approximate that of* (2.2) *in the optimal order*:

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)^3} + \|p - p_h\|_{L^2(\Omega)} \leq C h^{\min\{2,r\}} \big(\|\mathbf{u}\|_{H^{r+1}(\Omega)^3} + \|p\|_{H^r(\Omega)} \big), \quad r \geq 1. \tag{4.44}$$

Proof By the inf-sup condition (4.40) and the standard mixed finite element theory [9], it follows that

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)^3}+\|p-p_h\|_{L^2(\Omega)}\\
&\quad\le C\Big(\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}_h\|_{H^1(\Omega)^3}+\inf_{q_h\in P_h}\|p-q_h\|_{L^2(\Omega)}\Big)\\
&\quad\le C\Big(\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}_h\|_{H^1(\Omega)^3}+\inf_{q_h\in \bar P_h}\|p-q_h\|_{L^2(\Omega)}\Big)
\end{aligned}$$

where $\bar P_h$ is the C_0–P_1 space defined in (4.43). Here in the last step, we applied (4.42). In fact, our theory covers the 3D Taylor-Hood element (C_0–P_2 for the velocity and C_0–P_1 for the pressure [2]) on Powell–Sabin grids. The theorem is proven as both spaces $\mathbf{V}_h$ and $\bar P_h$ provide the optimal order approximation properties. □

5 Numerical tests

In this section, we report some numerical tests on the P_k–P_{k-1} divergence-free elements, including the Powell–Sabin P_2 divergence-free element, for the stationary Stokes equations (2.1) on the unit cube, $\Omega=(0,1)^3$. The base grids $\bar\Omega_h$ are obtained by the standard multigrid refinement, cf. [15]. The first three base grids are depicted in Fig. 9, where the 4th grid is the Powell–Sabin grid based on the level 3 grid.

We choose the right hand side function $\mathbf{f}$ for (2.1) as

$$\begin{aligned}
\mathbf{f}&=-\Delta\,\mathbf{curl}\begin{pmatrix}0\\ g\\ g\end{pmatrix}+\frac{1}{9}\nabla g_{xy}\\
&=\begin{pmatrix}-g_{xxy}-g_{yyy}-g_{yzz}+g_{xxz}+g_{yyz}+g_{zzz}+g_{xxy}/9\\ -g_{xxx}-g_{xyy}-g_{xzz}+g_{xyy}/9\\ g_{xxx}+g_{xyy}+g_{xzz}+g_{xyz}/9\end{pmatrix},
\end{aligned}\tag{5.1}$$

where

$$g=2^{12}\big(x-x^2\big)^2\big(y-y^2\big)^2\big(z-z^2\big)^2.$$

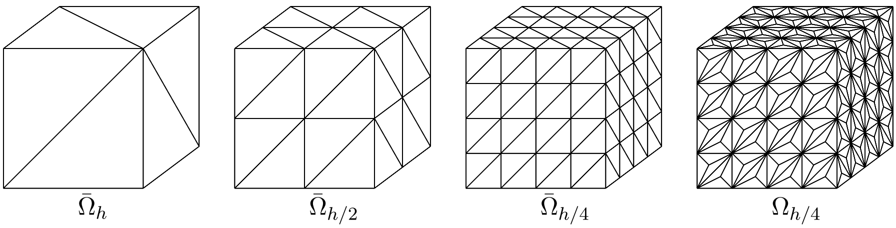

Fig. 9 The first 3 level base grids, and the Powell–Sabin level 4 grid

Table 1 The error and convergence order (P_2 Powell–Sabin element) on Fig. 9 grids

	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1}$	h^n	$\|p-p_h\|_{L^2}$	h^n
2	9.39211		10.50260	
3	9.54791	–	14.29001	–
4	3.06616	1.64	5.42907	1.40
5	0.87072	1.82	1.76084	1.62

Table 2 The error and convergence order (P_2–P_1 element) on uniform Fig. 9 grids

	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1}$	h^n	$\|p-p_h\|_{L^2}$	h^n
2	14.30606		41.81048	
3	8.31682	0.78	54.91746	–
4	4.15555	1.00	55.44895	–
5	2.06199	1.01	55.34440	–

Table 3 The error and convergence order (P_2–P_1 HCT element) on Fig. 9 grids

	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1}$	h^n	$\|p-p_h\|_{L^2}$	h^n
2	11.70759		12.30094	
3	17.52195	–	40.83355	–
4	7.40612	1.24	34.08396	0.26
5	2.95999	1.32	32.15647	0.08

The exact solution for the Stokes equations (2.1) is

$$\mathbf{u} = \mathbf{curl}\begin{pmatrix}0\\ g\\ g\end{pmatrix}, \qquad p = \frac{1}{9}g_{xy}. \tag{5.2}$$

In Table 1 we list the errors and the orders of convergence, for the Powell–Sabin P_2 divergence-free element solutions on several levels. The iterated penalty method defined in Definition 2.1 is used to solve the discrete linear equations. Here the pressure p_h is obtained as a byproduct by (2.13). The order of convergence seems to match the estimate (4.44).

Next, we apply the P_2–P_1 element on the standard grids $\bar{\Omega}_h$, not the Powell–Sabin grids, to solve (2.1) with data (5.1) again. As one expected, the discrete pressure solutions do not converge to the true solution at all. But the discrete velocity solutions seem to converge to the exact solution, but one order lower than the optimal order. On general quasiuniform grids, we may not have a convergence for the velocity either.

Next, in Table 3, we test the P_2–P_1 divergence-free element on Hsieh–Clough–Tocher grids. It is shown in [16] that the P_k–P_{k-1} divergence-free element is stable and optimal on HCT grids, if the polynomial degree $k > 2$. From Table 3, the pressure solutions may not convergent. It is quite convincing that the inf-sup condition (4.40) would not hold uniformly. This also shows the condition $k > 2$ in [16] is truly necessary.

Table 4 The errors for the P_1 Powell–Sabin element on Fig. 9 grids

	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1}$	h^n	$\|p-p_h\|_{L^2}$	h^n
2	13.26842		48.50482	
3	16.04441	–	31.82411	–
4	16.36783	–	113.87327	–
5	15.23631	–	306.59689	–

Finally we test the P_1 Powell–Sabin element. In 2D, it is shown in [17], both analytically and numerically, that the P_1 Powell–Sabin element on triangular grids is stable and provides the optimal-order approximation. However, in 3D, this is no longer true. The minimal polynomial degree for P_k–P_{k-1} mixed element to be stable is $k=2$, on Powell–Sabin grids. That is, our analysis for (4.40) and (4.44) is sharp. In Table 4, when applying P_1 Powell–Sabin element for (2.1) and (5.1), the discrete velocity does not converge, and worse, the discrete pressure diverges to infinity.

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