Superconvergence of the $Q_{k+1,k}$-$Q_{k,k+1}$ divergence-free finite element

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Abstract

By the standard theory, the stable $Q_{k+1,k}$-$Q_{k,k+1}/Q_k^{dc'}$ divergence-free element converges with the optimal order of approximation for the Stokes equations, but only order k for the velocity in H^1-norm and the pressure in L^2-norm. This is due to one polynomial degree less in y direction for the first component of velocity, a $Q_{k+1,k}$ polynomial. In this manuscript, we will show a superconvergence of the divergence free element that the order of convergence is truly $k+1$, for both velocity and pressure. Numerical tests are provided confirming the sharpness of the theory.

Keywords. mixed finite element, Stokes equations, divergence-free element, quadrilateral element, rectangular grids, superconvergence.

AMS subject classifications (2000). 65M60, 65N30, 76D07.

1 Introduction

The divergence-free finite element method is mainly for solving incompressible flow problems, such as Stokes or Navier-Stokes equations, where the finite element space for the pressure is exactly the divergence of the finite element space for the velocity. In such a method, the finite element velocity is divergence-free pointwise, i.e. the incompressibility condition is enforced strongly. Traditional finite elements enforce the incompressibility weakly, cf. [17, 9]. That is, in order to satisfy the inf-sup stability condition, the incompressibility condition is weakened by either enlarging the velocity space or decreasing the pressure space. This often leads to some sub-optimal methods, or a waste of computation, due to the imperfect matching of two spaces. It may lead to inaccurate mass conservation, which is critical in certain computational problems.

A fundamental study on the divergence-free element method was done by Scott and Vogelius ([18, 19]) that the P_{k+1}/P_k^{dc} method is stable and consequently of the optimal order on 2D triangular grids, for $k \geq 3$. Here the velocity space is the continuous piecewise-polynomials of degree $(k+1)$ or less while the the pressure space is the discontinuous piecewise-polynomials of degree k or less, or the divergence of the velocity, to be precise. There are several other such divergence-free finite elements, cf. [2, 14, 15, 16, 23, 24, 25].

Starting from the most popular element, the Q_1/P_0 element ([6, 7]), there is a series of work on Q_k mixed finite elements on rectangular grids in 2D and 3D. Brezzi and Falk showed that the Q_{k+1}/Q_k^{dc} element is unstable in [10], for any $k \geq 0$. Here Q_k^{dc} denotes the space of discontinuous piecewise-polynomials. In [21], Stenberg and Suri showed the stability, but a sub-optimal order of approximation, for the Q_{k+1}/Q_{k-1}^{dc} element for all $k \geq 1$ in 2D. Bernardi and Maday proved the stability and the optimal order of convergence for the Q_{k+1}/P_k^{dc} element, cf.

[4]. Ainsworth and Coggins established [1] the stability and the optimal order of convergence for the Taylor-Hood Q_{k+1}/Q_k element, where the pressure space is continuous too. The Bernardi-Raugel element ([5]) optimizes the Q_{k+1}/Q_{k-1}^{dc} element, when $k = 1$, by reducing the velocity space to $Q_{1,2}$-$Q_{2,1}$ polynomials. Here the first component of velocity in the Bernardi-Raugel element is a polynomial of degree 1 in x direction, but of degree 2 in y direction. To be precise, the Bernardi-Raugel element enrich the Q_1 velocity space by face-bubble functions. Similar to the Bernardi-Raugel element, a divergence-free finite element, $Q_{k+1,k}$-$Q_{k,k+1}/Q_k^{dc'}$ ($k \geq 2$), was proposed in [25], which further optimizes the Bernardi-Raugel element by increasing the polynomial degree of pressure from $(k-1)$ to k. The nodal degrees of freedom of this divergence-free element and the Bernardi-Raugel element are plotted in Figure 1. This divergence-free element was extended to its lowest-order form, $k = 1$, i.e., $Q_{2,1}$-$Q_{1,2}/Q_1^{dc'}$, in [14]. Here the space $Q_k^{dc'}$ for the pressure is the space of discontinuous Q_k polynomials with all spurious modes removed, i.e., eliminating one degree of freedom at each vertex. In the construction, the pressure space is exactly the divergence of the velocity. Thus, the resulting finite element is divergence-free pointwise. In such a case, the discrete pressure space can be omitted in the computation. By an iterated penalty method, we obtain the pressure solution as a byproduct, cf. [24] and Section 4 below. However, by the standard finite element theory developed in [14, 25], this divergence-free element converges at order k only, due to a degree k polynomial in y for the first component of $\mathbf{u}_h$. This cannot be improved by the standard theory, where the optimal order of convergences is derived from the inf-sup stability. In this manuscript, we further study this $Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element and show its superconvergence, that it does converge at order $k+1$. Further the velocity of the $Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element may be ultraconvergent, i.e., two orders higher than the standard convergence, provided the interpolation polynomial is divergence-free. The extension of this divergence-free element to 3D is straightforward, so is its superconvergence property.

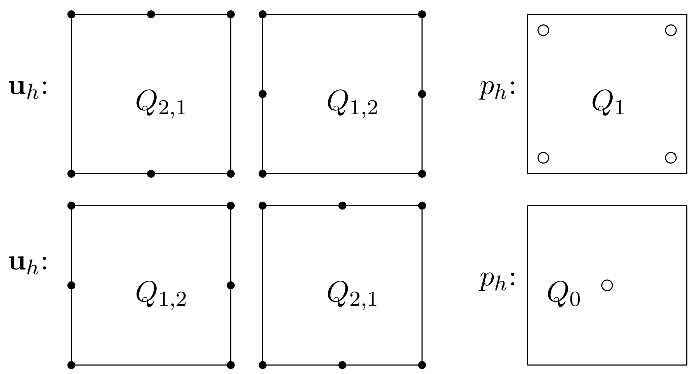

Figure 1: Nodes of $\mathbf{u}_h/p_h$ for divergence-free (top) and Bernardi-Raugel elements.

The rest of the paper is organized as follows. In Section 2, we define the finite element for the Stationary Stokes equations. In Section 3, we establish a superconvergence for the divergence-free element. In Section 4, we provide some test results confirming the analysis. In particular, we show the order of convergence of the divergence-free element is one higher than that of the rotated Bernardi-Raugel element.

2 The $Q_{k+1,k}$-$Q_{k,k+1}$ divergence-free element

In this section, we shall define the divergence-free finite element for the stationary Stokes equations on rectangular grids. The resulting finite element solutions for the velocity are divergence-free point wise.

We consider a model stationary Stokes problem: Find the velocity $\mathbf{u}$ and the pressure p on a 2D polygonal domain Ω, which can be subdivided into rectangles, such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The weak form for (2.1) is: Find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega) := L^2(\Omega)/C = \{p \in L^2 \mid \int_\Omega p = 0\}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ b(\mathbf{u}, q) &= 0 && \forall q \in L_0^2(\Omega). \end{aligned} \tag{2.2}$$

Here $H_0^1(\Omega)^2$ is the subspace of the Sobolev space $H^1(\Omega)^2$ (cf. [11]) with zero boundary trace, and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx, \\ b(\mathbf{v}, p) &= -\int_\Omega \operatorname{div} \mathbf{v} \, p \, dx, \\ (\mathbf{f}, \mathbf{v}) &= \int_\Omega \mathbf{f} \, \mathbf{v} \, dx. \end{aligned}$$

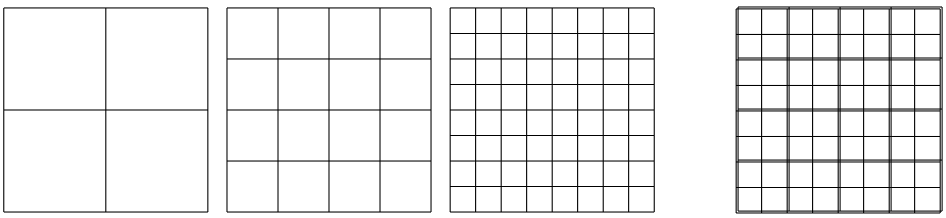

Figure 2: Three levels of grids, and a macro-element grid (for $k = 1$ only).

The finite element grids are defined by, cf. Figure 2,

$$\mathcal{T}_h = \left\{ K \mid \cup K = \overline{\Omega}, \ K = [x_a, x_b] \times [y_c, y_d] \ \text{ with size } \ h_K = \max\{x_b - x_a, y_d - y_c\} \le h \right\}.$$

We further assume, only for the lowest-order element $k = 1$ in (2.3), that the rectangles in grid $\mathcal{T}_h$ can be combined into groups of four to form a macro-element grid:

$$\mathcal{M}_h = \left\{ M \mid M = \cup_{i=1}^4 K_i = [x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}], \ K_i \in \mathcal{T}_h, \ \cup K_i = \Omega \right\}.$$

See the 4th diagram in Figure 2. The polynomial spaces are defined by

$$Q_{k,l} = \{ \sum_{i \le k, j \le l} c_{ij} x^i y^j \}, \qquad Q_k = Q_{k,k}.$$

The $Q_{k+1,k}$-$Q_{k,k+1}(k \geq 1)$ element spaces are

$$\mathbf{V}_h = \left\{\mathbf{v}_h \in C(\Omega)^2 \mid \mathbf{v}_h|_K \in Q_{k+1,k} \times Q_{k,k+1} \; \forall K \in \mathcal{T}_h, \text{ and } \quad \mathbf{u}_h|_{\partial\Omega} = 0\right\}, \tag{2.3}$$
$$P_h = \{\operatorname{div} \mathbf{u}_h \mid \mathbf{u}_h \in \mathbf{V}_h\}. \tag{2.4}$$

Since $\int_\Omega p_h = \int_\Omega \operatorname{div} \mathbf{u}_h = \int_{\partial\Omega} \mathbf{u}_h = 0$ for any $p_h \in P_h$, we conclude that

$$\mathbf{V}_h \subset H_0^1(\Omega)^2, \quad P_h \subset L_0^2(\Omega),$$

i.e., the mixed-finite element pair is conforming. The resulting system of finite element equations for (2.2) is: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q) &= 0 \qquad\quad\; \forall q \in P_h. \end{aligned} \tag{2.5}$$

Traditional mixed-finite elements require the inf-sup condition to guarantee the existence of discrete solutions. As (2.4) provides a compatibility between the discrete velocity and the discrete pressure spaces, the linear system of equations (2.5) always has a unique solution, cf. [24]. Furthermore, such a solution $\mathbf{u}_h$ is divergence-free: by the second equation in (2.5) and the definition of P_h in (2.4),

$$b(\mathbf{u}_h, q) = b(\mathbf{u}_h, -\operatorname{div} \mathbf{u}_h) = \|\operatorname{div} \mathbf{u}_h\|^2_{L^2(\Omega)^2} = 0. \tag{2.6}$$

In this case, i.e., $\mathbf{V}_h \subset \mathbf{Z} := \{\operatorname{div} \mathbf{v} \mid \mathbf{v} \in H_0^1(\Omega)^2\}$, we call the mixed finite element a divergence-free element. It is apparent that the discrete velocity solution is divergence-free if and only if the discrete pressure finite element space is the divergence of the discrete velocity finite element space, i.e., (2.4).

We note that by (2.4), P_h is a subspace of discontinuous, piecewise bilinear polynomials. As singular vertices are present (see [18, 19, 14, 25]), P_h is a proper subset of the discontinuous piecewise Q_1 polynomials. It is possible, but difficult to find a local basis for P_h. But on the other side, it is the special interest of the divergence-free finite element method that the space P_h can be omitted in computation and the discrete solutions approximating the pressure function in the Stokes equations can be obtained as byproducts, if an iterated penalty method is adopted to solve the system (2.5), cf. [13, 9, 8, 20, 24] for more information.

3 Superconvergence

As usual, the superconvergence is obtained by the method of integration by parts, cf. [12, 22]. But we have a long series of lemmas dealing with each term in the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$.

For a convenience in referring components of the vector velocity, we define the two inhomogeneous polynomial spaces:

$$V_{h,1} = \{\phi \in H_0^1(\Omega) \mid \phi|_K \in Q_{k+1,k} \; \forall K \in \mathcal{T}_h\}, \tag{3.1}$$
$$V_{h,2} = \{\phi \in H_0^1(\Omega) \mid \phi|_K \in Q_{k,k+1} \; \forall K \in \mathcal{T}_h\}, \tag{3.2}$$

$k \geq 1$. That is,

$$\mathbf{V}_h = V_{h,1} \times V_{h,2}, \quad k \geq 1.$$

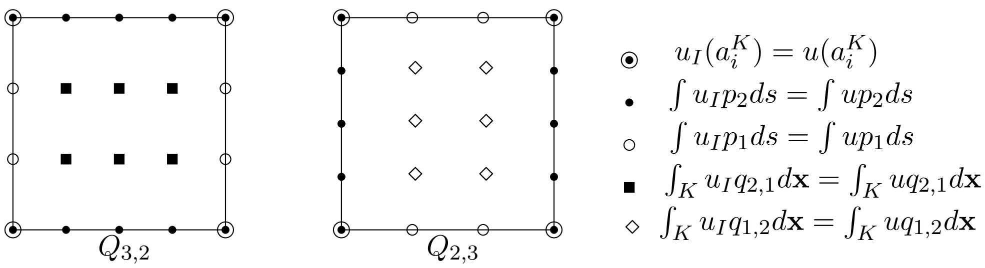

Figure 3: Three types of interpolation nodes, $k = 3$.

The interpolation operator $\mathbf{I}_h$ is defined for the two components of $\mathbf{u}$:

$$\begin{aligned} \mathbf{I}_h : H_0^1(\Omega) \times H_0^1(\Omega) &\to V_{h,1} \times V_{h,2}, \\ \mathbf{I}_h \mathbf{u} = \mathbf{I}_h \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u_I \\ v_I \end{pmatrix}. \end{aligned} \tag{3.3}$$

To define u_I at the Lagrange nodes, we define its vertex nodal values, then internal edge values, and finally internal values, by solving the following equations sequentially (see Figure 3):

$$(u - u_I)(a_i^K) = 0 \qquad \text{at four vertices of } K, \ \forall K \in \mathcal{T}_h, \tag{3.4}$$

$$\int_{y=y_j} (u - u_I) p_{k-1}(x) dx = 0 \qquad \text{on the top and bottom edges of } K, \tag{3.5}$$

$$\int_{x=x_i} (u - u_I) p_{k-2}(y) dy = 0 \qquad \text{on the left and right edges of } K. \tag{3.6}$$

$$\int_K (u - u_I) q_{k-1,k-2} d\mathbf{x} = 0 \qquad \text{on the square } K, \tag{3.7}$$

where $p_k \in P_k$, the space of 1D polynomials of degree k or less, and $q_{k,l} \in Q_{k,l}$. By rotating x and y, v_I is defined similarly/symmetrically to u_I.

Lemma 3.1 *(two-order superconvergence) For any $Q_{k+1,k}$ function $\psi \in V_{h,1}$, defined in* (3.1), *for any $u \in H^{k+3}(\Omega)$, and for all $k > 1$,*

$$\left| \int_\Omega (u - u_I)_x \psi_x d\mathbf{x} \right| \le C h^{k+2} \|u\|_{H^{k+3}} \|\psi\|_{H^1}. \tag{3.8}$$

Proof. We first consider the estimation on the reference element $\hat{K} = [-1, 1]^2$. Since $\psi \in Q_{k+1,k}$, we have an exact Taylor expansion:

$$\psi_x(x, y) = \psi_x(x, 0) + y\psi_{xy}(x, 0) + \cdots + \frac{y^{k-1}}{(k-1)!}\psi_{xy^{k-1}}(x, 0) + \frac{y^k}{k!}\psi_{xy^k}(x, 0), \tag{3.9}$$

where $\psi_x(x, 0)$ and all $\psi_{xy^j}(x, 0)$ are P_k polynomials in x only. We will perform the integration by parts repeatedly. First, for the lower order terms in (3.9), we notice that, by the definition of u_I in (3.5) and (3.7),

$$\begin{aligned} &\int_{\hat{K}} (u - u_I)_x y^j \psi_{xy^j}(x, 0) d\mathbf{x} \\ =& \int_{-1}^1 (u - u_I) y^j \psi_{xy^j}(x, 0)|_{x=-1}^{x=1} dy - \int_{\hat{K}} (u - u_I) y^j \psi_{x^2y^j}(x, 0) d\mathbf{x} \\ =&0 \qquad \text{when } j = 0, 1, \ldots, k-2. \end{aligned} \tag{3.10}$$

Please be aware that $\psi_{x^2y^j}(x,0) \in P_{k-1}(x)$. Hence, we need to deal with only the last two terms in (3.9).

For the last two terms in (3.9), in order to do integration by parts, we express polynomials y^{k-1} and y^k by derivatives of another polynomial.

$$s_k(y) = \frac{(y^2-1)^{k+1}}{(2k+2)!} = \frac{y^{2k+2}}{(2k+2)!} - \frac{(k+1)y^{2k}}{(2k+2)!} + \cdots = \frac{y^{2k+2}}{(2k+2)!} + \tilde{p}_{2k}(y), \tag{3.11}$$

$$s_k^{(j)}(\pm 1) = 0, \quad j = 0, 1, \cdots, k, \tag{3.12}$$

$$s_k^{(k+2)}(y) = \frac{1}{k!} y^k + p_{k-2}(y), \tag{3.13}$$

Here $\tilde{p}_{2k}(y)$ and $p_{k-2}(y)$ denote a polynomial of degree $2k$ and $(k-2)$, respectively. We note that, as in (3.10), the integral of $(u-u_I)_x$ against $p_{k-2}(y)$ is zero. Thus, surprisingly simple, we have

$$\begin{aligned}
&\int_{\hat{K}} (u-u_I)_x \psi_x(x,y)dxdy \\
&= \int_{\hat{K}} (u-u_I)_x (s_{k-1}^{(k+1)}(y)\psi_{xy^{k-1}}(x,0) + s_k^{(k+2)}(y)\psi_{xy^k}(x,0))dxdy \\
&= \int_{-1}^{1} \Big[(u-u_I)_x (s_{k-1}^{(k)}(y)\psi_{xy^{k-1}}(x,0) + s_k^{(k+1)}(y)\psi_{xy^k}(x,0))\Big]_{y=-1}^{y=1} dx \\
&\quad - \int_{\hat{K}} (u-u_I)_{xy} (s_{k-1}^{(k)}(y)\psi_{xy^{k-1}}(x,0) + s_k^{(k+1)}(y)\psi_{xy^k}(x,0))dxdy.
\end{aligned} \tag{3.14}$$

Let us consider the first boundary integral in (3.14), on the top edge of the square $\hat{K}$. By (3.4) and (3.5),

$$\begin{aligned}
&\int_{-1}^{1} (u-u_I)_x(x,1) s_{k-1}^{(k)}(1)\psi_{xy^{k-1}}(x,0)dx \\
&= \Big[(u-u_I)(x,1)s_{k-1}^{(k)}(1)\psi_{xy^{k-1}}(x,0)\Big]_{x=-1}^{1} \\
&\quad - s_{k-1}^{(k)}(1) \int_{-1}^{1} (u-u_I)(x,1)\psi_{x^2y^{k-1}}(x,0)dx = 0,
\end{aligned} \tag{3.15}$$

noting again that $\psi_{x^2y^{k-1}}(x,0)$ is a P_{k-1} polynomial in x only. The other boundary integral in (3.14) is also 0 as $\psi_{x^2y^k}(x,0) \in P_{k-1}$ too:

$$\begin{aligned}
&\int_{-1}^{1} (u-u_I)_x(x,1) s_k^{(k+1)}(1)\psi_{xy^k}(x,0)dx \\
&= \Big[(u-u_I)(x,1)s_k^{(k+1)}(1)\psi_{xy^k}(x,0)\Big]_{x=-1}^{1} \\
&\quad - s_k^{(k+1)}(1) \int_{-1}^{1} (u-u_I)(x,1)\psi_{x^2y^k}(x,0)dx = 0.
\end{aligned}$$

That is the boundary integrals in (3.14) are all zero. We repeat the integration by parts in this direction, while the boundary terms would be zero by (3.12) and (3.5). By the integration by parts k times more, (3.14) would be

$$\begin{aligned}
&\int_{\hat K}(u-u_I)_x\psi_x dxdy\\
=&-\int_{\hat K}(u-u_I)_{xy}(s_{k-1}^{(k)}\psi_{xy^{k-1}}(x,0)+s_k^{(k+1)}\psi_{xy^k}(x,0))dxdy\\
=&\int_{\hat K}(u-u_I)_{xy^2}(s_{k-1}^{(k-1)}\psi_{xy^{k-1}}(x,0)+s_k^{(k)}\psi_{xy^k}(x,0))dxdy\\
&-\int_{-1}^{1}(u-u_I)_{xy}(x,1)(s_{k-1}^{(k-1)}\psi_{xy^{k-1}}(x,0)+s_k^{(k)}\psi_{xy^k}(x,0))_{y=-1}^{y=1}dx\\
=&\int_{\hat K}(u-u_I)_{xy^2}(s_{k-1}^{(k-1)}\psi_{xy^{k-1}}(x,0)+s_k^{(k)}\psi_{xy^k}(x,0))dxdy\\
=&(-1)^{k+1}\int_{\hat K}(u-u_I)_{xy^{k+1}}(s_{k-1}\psi_{xy^{k-1}}(x,0)+s_k'\psi_{xy^k}(x,0))dxdy. \qquad (3.16)
\end{aligned}$$

We will perform the integration by parts one last time. But this time, we will treat the two terms in the last integral differently.

$$\begin{aligned}
\int_{\hat K}(u-u_I)_{xy^{k+1}}s_{k-1}\psi_{xy^{k-1}}(x,0)dxdy =& -\int_{\hat K}(u-u_I)_{x^2y^{k+1}}s_{k-1}\psi_{y^{k-1}}(x,0)dxdy\\
&+\int_{-1}^{1}\left[(u-u_I)_{xy^{k+1}}s_{k-1}\psi_{y^{k-1}}(x,0)\right]_{x=-1}^{x=1}dy,\\
\int_{\hat K}(u-u_I)_{xy^{k+1}}s_k'\psi_{xy^k}(x,0)dxdy =& -\int_{\hat K}(u-u_I)_{xy^{k+2}}s_k\psi_{xy^k}(x,0)dxdy.
\end{aligned}$$

For the second integral, the boundary term disappears by the condition (3.12). For the first integral, we note that the boundary integrals will be cancelled due to the opposite line integrals on two sides of the vertical edge $x=x_i$ or due to the boundary condition on ψ. We note also that the $(k+1)$st and $(k+2)$nd partial derivatives on u_I above are all zero. Hence, we get (3.8) by summing over the estimation on all rectangles $K\in\mathcal{T}_h$, plus a scaling and the Schwartz inequality,

$$\begin{aligned}
&\left|\int_\Omega(u-u_I)_x\psi_x d\mathbf{x}\right|\\
=&\left|\sum_K\int_K(u-u_I)_x\psi_x d\mathbf{x}\right| = \left|\sum_K\int_{\hat K}(u-u_I)_x\psi_x d\mathbf{x}\right|\\
=&\left|\sum_K(-1)^{k+2}\int_{\hat K}\left(u_{x^2y^{k+1}}s_{k-1}\psi_{y^{k-1}}+u_{xy^{k+2}}s_k\psi_{xy^k}\right)d\mathbf{x}\right|\\
\le&\sum_K C|u|_{H^{k+3}(\hat K)}|\psi|_{H^1(\hat K)} = C\sum_K h^{k+2}|u|_{H^{k+3}(K)}|\psi|_{H^1(K)}\\
\le& Ch^{k+2}|u|_{H^{k+3}(\Omega)}|\psi|_{H^1(\Omega)}.
\end{aligned}$$

We note that the semi H^1-norm is needed above to bound $\psi_{y^{k-1}}$. Thus $k>1$ is required. ∎

In the proof, we can see that the decrease of one degree polynomial in y does not change the super-approximation of $Q_{k+1,k}$ in x direction. After (3.16), if we skip the last step of integration by parts, we would get the following corollary. That is, we avoid $|\psi_{y^{k-1}}|_{L^2}$ when $k=1$ which cannot be bounded by $|\psi|_{H^1}$.

Corollary 3.1 *(one-order superconvergence) For any $Q_{k+1,k}$ function $\psi \in V_{h,1}$, defined in* (3.1)*, for any $u \in H^{k+2}(\Omega)$, and for all $k \geq 1$,*

$$|\int_\Omega (u-u_I)_x \psi_x d\mathbf{x}| \leq Ch^{k+1}\|u\|_{H^{k+2}}\|\psi\|_{H^1}. \tag{3.17}$$

∎

Symmetrically, switching x and y in Lemma 3.1, we prove the following lemma.

Lemma 3.2 *(two-order superconvergence) For any $Q_{k,k+1}$ function $\psi \in V_{h,2}$, defined in* (3.2)*, and for any $u \in H^{k+3}(\Omega)$, if $k > 1$,*

$$|\int_\Omega (u-u_I)_y \psi_y d\mathbf{x}| \leq Ch^{k+2}\|u\|_{H^{k+3}}\|\psi\|_{H^1}. \tag{3.18}$$

∎

For the same reasons in Corollary 3.1, we get the following corollary from Lemma 3.2.

Corollary 3.2 *(one-order superconvergence) For any $Q_{k,k+1}$ function $\psi \in V_{h,2}$, defined in* (3.2)*, for any $u \in H^{k+2}(\Omega)$, and for all $k \geq 1$,*

$$|\int_\Omega (u-u_I)_y \psi_y d\mathbf{x}| \leq Ch^{k+1}\|u\|_{H^{k+2}}\|\psi\|_{H^1}. \tag{3.19}$$

∎

Though the interpolation order is $(k+2)$ in above two lemmas, only the $(k+1)$ order in two corollaries can be achieved in computation due to the coupling of terms in mixed formulation. We prove the approximation properties in the lower polynomial direction next. Now, even for $k = 1$, we have a two-order superconvergence.

Lemma 3.3 *(two-order superconvergence) For any $Q_{k+1,k}$ function $\psi \in V_{h,1}$, defined in* (3.1)*, for any $u \in H^{k+3}(\Omega)$, and for all $k \geq 1$,*

$$|\int_\Omega (u-u_I)_y \psi_y d\mathbf{x}| \leq Ch^{k+2}\|u\|_{H^{k+3}}\|\psi\|_{H^1}. \tag{3.20}$$

Proof. Again, we first consider the estimation on the reference element $\hat{K} = [-1,1]^2$. Since the polynomial degree in y is too low, we do Taylor expansion in x direction, different from the last lemma.

$$\psi_y(x,y) = \psi_y(0,y) + x\psi_{xy}(0,y) + \cdots + \frac{x^k}{k!}\psi_{x^k y}(0,y) + \frac{x^{k+1}}{(k+1)!}\psi_{x^{k+1}y}(0,y).$$

Again, similar to (3.9), the integral of $(u-u_I)_y$ against x^j terms are zero if $j \leq k-1$,

$$\begin{aligned}
&\int_{\hat{K}} (u-u_I)_y x^j \psi_{x^j y}(0,y) dxdy \\
&= \int_{-1}^{1} \left[(u-u_I)x^j \psi_{x^j y}(0,y)\right]_{y=-1}^{y=1} dx - \int_{\hat{K}} (u-u_I)x^j \psi_{x^j y^2}(0,y) dxdy = 0,
\end{aligned}$$

noting that $x^j\psi_{x^jy^2}(0,y)\in Q_{k-1,k-2}$. Using the polynomial function $s_k(x)$ defined in (3.11) we have, cf. (3.14),

$$\begin{aligned}
&\int_{\hat K}(u-u_I)_y\psi_y dxdy\\
=&\int_{\hat K}(u-u_I)_y(s_k^{(k+2)}(x)\psi_{x^ky}(0,y)+s_{k+1}^{(k+3)}(x)\psi_{x^{k+1}y}(0,y))dxdy\\
=&\int_{-1}^{1}\Big[(u-u_I)_y(s_k^{(k+1)}(x)\psi_{x^ky}(0,y)+s_{k+1}^{(k+2)}(x)\psi_{x^{k+1}y}(0,y))\Big]_{x=-1}^{x=1}dy\\
&-\int_{\hat K}(u-u_I)_{xy}(s_k^{(k+1)}(x)\psi_{x^ky}(0,y)+s_{k+1}^{(k+2)}(x)\psi_{x^{k+1}y}(0,y))dxdy.
\end{aligned}$$

Here, for the first time integration by parts, the boundary integral disappeared by (3.4), $(u-u_I)(\pm1,\pm1)=0$. In the next $(k+1)$ times of integration by parts, the boundary integrals on $x=\pm1$ would be zero, directly by the boundary condition (3.12) of $s_k(x)$.

$$\int_{\hat K}(u-u_I)_y\psi_y dxdy=(-1)^{k+2}\int_{\hat K}(u-u_I)_{x^{k+2}y}(s_k\psi_{x^ky}(0,y)+s_{k+1}'\psi_{x^{k+1}y}(0,y))dxdy.$$

Thus,

$$\begin{aligned}
\left|\int_{\hat K}(u-u_I)_y\psi_y dxdy\right| &\le C\|u_{x^{k+2}y}\|_{L^2(\hat K)}\|\psi_y\|_{L^2(\hat K)}\\
&\le C|u|_{H^{k+3}(\hat K)}|\psi\|_{H^1(\hat K)}.
\end{aligned}$$

The rest proof repeats that of Lemma 3.1. ∎

As for above lemmas and corollaries, we can get the following corollary from Lemma 3.3

Corollary 3.3 *(two-order superconvergence) For any $Q_{k+1,k}$ function $\psi\in V_{h,1}$, defined in* (3.1)*, for any $u\in H^{k+2}(\Omega)$, and for all $k\ge1$,*

$$|\int_\Omega(u-u_I)_y\psi_y d\mathbf{x}|\le Ch^{k+1}\|u\|_{H^{k+2}}\|\psi\|_{H^1}. \tag{3.21}$$

∎

Corollary 3.4 *For any $Q_{k,k+1}$ function $\psi\in V_{h,2}$, defined in* (3.2)*, and for any $u\in H^{k+3}(\Omega)$, and for all $k\ge1$,*

$$|\int_\Omega(u-u_I)_y\psi_y d\mathbf{x}|\le Ch^{k+2}\|u\|_{H^{k+3}}\|\psi\|_{H^1}, \tag{3.22}$$

$$|\int_\Omega(u-u_I)_y\psi_y d\mathbf{x}|\le Ch^{k+1}\|u\|_{H^{k+2}}\|\psi\|_{H^1}. \tag{3.23}$$

∎

Now we study the superconvergence in the both bilinear forms.

Lemma 3.4 *For any $(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times P_h$, defined in* (2.3) *and* (2.4)*, and for any $\mathbf{u}\in H^3(\Omega)\cap H_0^1(\Omega)$,*

$$|a(\mathbf{u}-\mathbf{I}_h\mathbf{u},\mathbf{v}_h)|\le Ch^{k+2}\|\mathbf{u}\|_{H^{k+3}(\Omega)^2}\|\mathbf{v}_h\|_{H^1(\Omega)^2},\quad k>1, \tag{3.24}$$

$$|a(\mathbf{u}-\mathbf{I}_h\mathbf{u},\mathbf{v}_h)|\le Ch^{k+1}\|\mathbf{u}\|_{H^{k+2}(\Omega)^2}\|\mathbf{v}_h\|_{H^1(\Omega)^2},\quad k\ge1, \tag{3.25}$$

$$|b(\mathbf{u}-\mathbf{I}_h\mathbf{u},q_h)|\le Ch^{k+1}\|\mathbf{u}\|_{H^{k+2}(\Omega)^2}\|q_h\|_{L^2(\Omega)},\quad k\ge1, \tag{3.26}$$

where $\mathbf{I}_h\mathbf{u}$ is the interpolation of $\mathbf{u}$ defined by (3.3).

Proof. (3.24) is a combination of (3.8), (3.20), (3.22), and (3.18). (3.25) is a combination of (3.17), (3.21), (3.23), and (3.19).

For (3.26), we will lose one order of convergence. Let $q_h = \operatorname{div} \mathbf{w}_h$ for some $\mathbf{w}_h = (\phi, \psi) \in \mathbf{V}_h$. We have, denoting $\mathbf{u} = (u, v)$,

$$b(\mathbf{u} - \mathbf{I}_h \mathbf{u}, q_h) = \sum_K \int_K ((u - u_I)_x + (v - v_I)_y)(\phi_x + \psi_y) d\mathbf{x}.$$

Here we have two old integrals, $\int_K (u - u_I)_x \phi_x d\mathbf{x}$ and $\int_K (v - v_I)_y \psi_y d\mathbf{x}$, and two new integrals, $\int_K (u - u_I)_x \psi_y d\mathbf{x}$ and $\int_K (v - v_I)_y \phi_x d\mathbf{x}$. The approximation order can be one order higher for the two old integrals. For the two new integrals, by symmetry, we consider $\int_K (u - u_I)_x \psi_y d\mathbf{x}$. We use the following Taylor expansion on the reference element $\hat{K}$ in the y direction. We note that the Taylor expansion in x direction would lead to a too high order polynomial in y direction each term in (3.27) below.

$$\psi_y(x, y) = \psi_y(x, 0) + y\psi_{y^2}(x, 0) + \cdots + \frac{y^{k-1}}{(k-1)!}\psi_{y^k}(x, 0) + \frac{y^k}{k!}\psi_{y^{k+1}}(x, 0). \tag{3.27}$$

Here all $\psi_{y^j}(x, 0)$ are polynomials of degree k in x. That is, a generic term $y^j \psi_{y^{j+1}}(x, 0) \in Q_{k,j}$. This is the same as the generic term $y^j \psi_{xy^j}(x, 0)$ in the early Taylor expansion (3.9). Thus repeating the proof of Lemma 3.1, we get

$$\begin{aligned}
\int_{\hat{K}} (u - u_I)_x \psi_y d\mathbf{x} &= \int_{\hat{K}} (u - u_I)_x (s_{k-1}^{(k+1)} \psi_{y^k}(x, 0) + s_k^{(k+2)} \psi_{y^{k+1}}(x, 0)) dx dy \\
&= (-1)^{k+1} \int_{\hat{K}} u_{xy^{k+1}} (s_{k-1} \psi_{y^k}(x, 0) + s_k' \psi_{y^{k+1}}(x, 0)) dx dy.
\end{aligned}$$

For the second integral, we can do an integration by parts to raise one more order. But we are limited by the first integral above to get only

$$\left| \int_{\hat{K}} (u - u_I)_x \psi_y d\mathbf{x} \right| \le \|u\|_{H^{k+2}(\hat{K})} \|\psi\|_{H^1(\hat{K})}.$$

Similarly, we have the same bound for $\left|\int_{\hat{K}} (u - u_I)_y \psi_x d\mathbf{x}\right|$. (3.26) follows by the Schwartz inequality and the scaling of referencing mappings. ∎

Finally, we estimate the approximation to p.

Lemma 3.5 *For any function $\mathbf{v}_h \in \mathbf{V}_h$, defined in (2.3), and for any $p \in H^{k+1}(\Omega) \cap L_0^2(\Omega)$,*

$$\left| \int_\Omega \operatorname{div} \mathbf{v}_h (p - p_I) d\mathbf{x} \right| \le C h^{k+1} \|\mathbf{v}_h\|_{H^1} \|p\|_{H^{k+1}}, \tag{3.28}$$

where p_I is a special nodal interpolation of p in P_h, defined in (3.29) below.

Proof. We note that P_h are discontinuous Q_k functions, $P_h = \operatorname{div} \mathbf{V}_h$. We define an interpolation operator for P_h via that $\mathbf{I}_h$ for $\mathbf{V}_h$ defined in (3.3). For a $p \in H^2(\Omega) \cap L_0^2(\Omega)$, Arnold, Scott and Vogelius shown in [3] that there is a $\mathbf{w} \in H^3(\Omega)^2 \cap H_0^1(\Omega)^2$, such that

$$\operatorname{div} \mathbf{w} = p, \quad \text{and} \quad \|\mathbf{w}\|_{H^3} \le C \|p\|_{H^2}.$$

For simplicity, we assume the above lifting exists up to order $k+1$. We define

$$p_I = \operatorname{div} \mathbf{w}_I, \tag{3.29}$$

for $\mathbf{w}_I = \mathbf{I}_h \mathbf{w}$ defined by (3.3). In order to use (3.26), we use notations:

$$\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix}, \ \mathbf{w}_I = \begin{pmatrix} u_I \\ v_I \end{pmatrix}, \ \mathbf{v}_h = \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Repeating the proof in Lemma 3.4, we get

$$\begin{aligned}
|\int_\Omega \operatorname{div} \mathbf{v}_h (p - p_I) d\mathbf{x}| &= |\int_\Omega \operatorname{div} \mathbf{v}_h \operatorname{div}(\mathbf{w} - \mathbf{w}_I) d\mathbf{x}| \\
&= |\int_\Omega ((u-u_I)_x + (v-v_I)_y)(\phi_x + \psi_y) d\mathbf{x} \\
&\le Ch^{k+1} \left| \begin{pmatrix} u \\ v \end{pmatrix} \right|_{H^{k+2}} \left| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right|_{H^1} \\
&\le Ch^{k+1} \|p\|_{H^{k+1}} \|\mathbf{v}_h\|_{H^1}.
\end{aligned}$$

∎

We derive the main theorem.

Theorem 3.1 *The finite element solution $(\mathbf{u}_h, p_h)$ of (2.5) has the following superconvergence property, one order higher than the optimal order,*

$$\|\mathbf{u}_h - \mathbf{I}_h \mathbf{u}\|_{H^1} + \|p_h - p_I\|_{L^2} \le Ch^{k+1}(\|\mathbf{u}\|_{H^{k+2}} + \|p\|_{H^{k+1}}), \tag{3.30}$$

where the interpolations $(\mathbf{I}_h\mathbf{u}, p_I)$ are defined in (3.3) and (3.29).

Proof. By the inf-sup condition shown in [14, 25], it follows that, cf. [17], for all $(\mathbf{w}_h, r_h) \in \mathbf{V}_h \times P_h$,

$$\sup_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h} \frac{a(\mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{v}_h, r_h) + b(\mathbf{w}_h, q_h)}{\|\mathbf{v}_h\|_{H^1} + \|q_h\|_{L^2}} \ge C(\|\mathbf{w}_h\|_{H^1} + \|r_h\|_{L^2}). \tag{3.31}$$

By Corollary 3.4 and Lemma 3.5, we have

$$\begin{aligned}
&\|\mathbf{u}_h - \mathbf{I}_h \mathbf{u}\|_{H^1} + \|p_h - p_I\|_{L^2} \\
&\le C \sup_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h} \frac{a(\mathbf{u}_h - \mathbf{I}_h\mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h - p_I) + b(\mathbf{u}_h - \mathbf{I}_h \mathbf{u}, q_h)}{\|\mathbf{v}_h\|_{H^1} + \|q_h\|_{L^2}} \\
&= C \sup_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h} \frac{a(\mathbf{u} - \mathbf{I}_h\mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_I) + b(\mathbf{u} - \mathbf{I}_h \mathbf{u}, q_h)}{\|\mathbf{v}_h\|_{H^1} + \|q_h\|_{L^2}} \\
&\le Ch^{k+1}(\|\mathbf{u}\|_{H^{k+2}} + \|p\|_{H^{k+1}}).
\end{aligned}$$

Note that, due to the pointwise divergence free property, we have above that

$$b(\mathbf{u}_h - \mathbf{I}_h\mathbf{u}, q_h) = b(-\mathbf{I}_h\mathbf{u}, q_h) = b(\mathbf{u} - \mathbf{I}_h\mathbf{u}, q_h).$$

∎

Here, to be precise, we do not have a superconvergence for p in Theorem 3.1. As P_h are degree-k polynomials, the best order approximation to p in L^2-norm would be $(k+1)$. However, due to the mixed formulation, the convergence of p_h to p is limited to the optimal order convergence of $\mathbf{u}_h$, which is $(k-1)$ in H^1-norm as $\mathbf{u}_h$ has polynomial degree k only in y direction for its first component. In this sense, the superconvergence result (3.30) does lift the order of p_h by one. For $k > 1$, we may have two order superconvergence for the velocity. Such numerical examples are shown in [25] and in next section. That is, for some special functions $\mathbf{u}$, $\mathbf{I}_h\mathbf{u}$ might be also in the divergence-free subspace of $\mathbf{V}_h$. If so, we have a two-order superconvergence result.

Theorem 3.2 *(two-order superconvergence) For some solution* $\mathbf{u}$ *of* (2.1)*, if*

$$\mathbf{I}_h\mathbf{u} \in \mathbf{Z}_h := \{\mathbf{z}_h \in \mathbf{V}_h \mid \operatorname{div} \mathbf{z}_h = 0\},$$

where $\mathbf{I}_h$ *is defined in* (3.3)*, and if* $k > 1$*, then*

$$\|\mathbf{u}_h - \mathbf{I}_h\mathbf{u}\|_{H^1} \le Ch^{k+2}\|\mathbf{u}\|_{H^{k+3}}. \tag{3.32}$$

Proof. By (3.24), limited in to the divergence-free subspace,

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{I}_h\mathbf{u}\|_{H^1} &\le \sup_{\mathbf{w}_h \in \mathbf{Z}_h} \frac{a(\mathbf{u}_h - \mathbf{I}_h\mathbf{u}, \mathbf{w}_h)}{\|\mathbf{w}_h\|_{H^1}} \\ &= \sup_{\mathbf{w}_h \in \mathbf{Z}_h} \frac{a(\mathbf{u} - \mathbf{I}_h\mathbf{u}, \mathbf{w}_h)}{\|\mathbf{w}_h\|_{H^1}} \le Ch^{k+2}\|\mathbf{u}\|_{H^{k+3}}.\end{aligned}$$

∎

4 Numerical tests

In this section, we report some results of numerical experiments on the $Q_{k+1,k}$-$Q_{k,k+1}$element for the stationary Stokes equations (2.1) on the unit square $\Omega = [0,1]^2$. The grids $\mathcal{T}_h$ are depicted in Figure 2, i.e., each squares are refined into 4 sub-squares each level. The initial grid, level one grid, is simply the unit square.

We choose an exact solution for the Stokes equations (2.1):

$$\mathbf{u} = \mathbf{curl}\, g, \quad p = \Delta g. \tag{4.1}$$

Here

$$g = 2^8(x^3 - x^4)^2(y^3 - y^4)^2.$$

So we can compute the right hand side function $\mathbf{f}$ for (2.1) as

$$\mathbf{f} = -\Delta\, \mathbf{curl}\, g + \nabla\Delta g. \tag{4.2}$$

We note that, unlike [25, 14], we intentionally choose a non-symmetric solution so that no ultraconvergence would happen, which does not exist in general. The solution p is plotted in Figure 4.

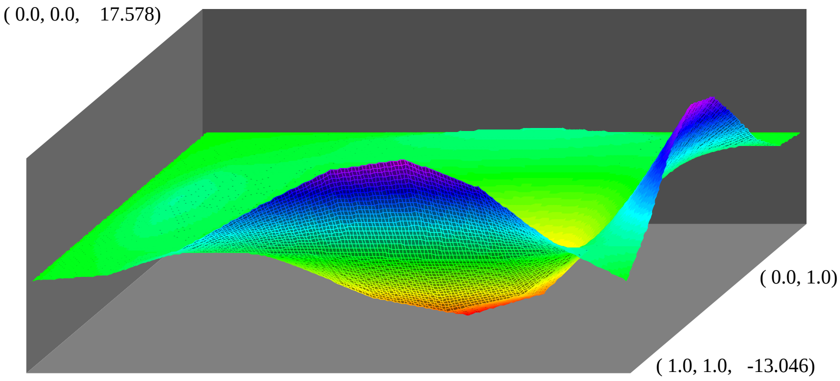

Figure 4: The solution p (the errors are shown in Figure 5.)

We compute the Stokes solution on refined grids, cf. Figure 2, by the divergence $Q_{k+1,k}$-$Q_{k,k+1}$element (2.3) and by the rotated Bernardi-Raugel element [5, 9, 17]:

$$\begin{aligned} \mathbf{V}_h^{BR} &= \left\{ \mathbf{v}_h \in C(\Omega)^2 \cap H_0^1(\Omega)^2 \mid \mathbf{v}_h|_K \in Q_{k+1,k} \times Q_{k,k+1} \ \forall K \in \mathcal{T}_h \right\}, \\ P_h^{BR} &= \left\{ q_h \in L_0^2(\Omega) \mid q_h|_K \in Q_{k-1} \right\}. \end{aligned} \tag{4.3}$$

Following the analysis in [14], the stability of the rotated Bernardi-Raugel element would be proved. For the rotated Bernardi-Raugel element, the system of finite element equations is solved by the Uzawa iterative method, cf. [9, 17, 13]. The stop criterion is the difference $|p_h^{(n)} - p_h^{(n-1)}| \le 10^{-6}$. We list the number of Uzawa iterations in the data tables by #Uz. Here the interpolation operators are standard Lagrange nodal interpolations [11].

Table 1: The errors $\mathbf{e}_h = \mathbf{u} - \mathbf{I}_h\mathbf{u}$ and $\epsilon_h = p - p_I$ for (4.1).

	$\lvert\mathbf{e}_h\rvert_{L^2}$	h^n	$\lvert\mathbf{e}_h\rvert_{H^1}$	h^n	$\lVert\epsilon_h\rVert_{L^2}$	h^n	
	$Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element, $k=1$						#it
2	0.264345		1.341770		5.965379		4
3	0.102329	1.4	0.795594	0.8	1.896372	1.7	4
4	0.026839	1.9	0.219469	1.9	0.481076	2.0	3
5	0.006773	2.0	0.055901	2.0	0.120363	2.0	3
6	0.001697	2.0	0.014035	2.0	0.030083	2.0	3
7	0.000424	2.0	0.003512	2.0	0.007520	2.0	3
	rotated Bernardi-Raugel element (4.3)						#Uz
2	0.570990		3.531380		7.497615		29
3	0.244967	1.2	3.028368	0.2	6.943183	0.1	65
4	0.074335	1.7	1.797533	0.8	3.300598	1.1	136
5	0.019849	1.9	0.946426	0.9	1.575390	1.1	297
6	0.005080	2.0	0.481087	1.0	0.762341	1.0	330
7	0.001281	2.0	0.241916	1.0	0.373990	1.0	204

For the $Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element, the pressure does not enter into computation, but is obtained as a byproduct, because $P_h = \operatorname{div} \mathbf{V}_h$. The resulting linear system of $Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element equations can be formulated as symmetric positive definite.

Then the iterated penalty method [13, 25] can be applied to obtain the divergence-free finite element solution for the velocity, and a byproduct $p_h = \operatorname{div} \mathbf{w}_h$ for the pressure. In our computation, the iterated penalty parameter is 2000. The stop criterion is the divergence $\| \operatorname{div} \mathbf{u}_h^{(n)} \|_0 \le 10^{-9}$. The number of iterated penalty iterations is also listed as #it in the data tables.

In Table 1, we list the errors in various norms for the $Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element and for the rotated Bernardi-Raugel element, for $k = 1$. It is clear that the order of convergence is 2, one order higher than that of latter. We note that the convergence order is only 2 for $Q_{2,1}$-$Q_{1,2}$ divergence-free elements in L^2-norm, i.e., no L^2-superconvergence. But we do see L^2-superconvergence for $k > 1$ next.

Table 2: The errors $\mathbf{e}_h = \mathbf{u} - \mathbf{I}_h\mathbf{u}$ and $\epsilon_h = p - p_I$ for (4.1).

	$\vert\mathbf{e}_h\vert_{L^2}$	h^n	$\vert\mathbf{e}_h\vert_{H^1}$	h^n	$\Vert\epsilon_h\Vert_{L^2}$	h^n	
	$Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element, $k = 2$						#it
1	0.322530	0.0	1.580066	0.0	3.546405	0.0	3
2	0.071851	2.2	0.699614	1.2	1.010498	1.8	4
3	0.005510	3.7	0.089611	3.0	0.131816	2.9	3
4	0.000355	4.0	0.010471	3.1	0.015587	3.1	3
5	0.000022	4.0	0.001280	3.0	0.001904	3.0	3
6	0.000001	4.0	0.000159	3.0	0.000236	3.0	3
7	0.000000	4.0	0.000020	3.0	0.000029	3.0	3
	rotated Bernardi-Raugel element (4.3)						#Uz
1	0.645475	0.0	4.250791	0.0	1.143688	0.0	27
2	0.191342	1.8	2.518701	0.8	5.499136	—	67
3	0.025892	2.9	0.673622	1.9	1.621194	1.8	100
4	0.003307	3.0	0.172036	2.0	0.441596	1.9	156
5	0.000419	3.0	0.043543	2.0	0.113029	2.0	266
6	0.000053	3.0	0.010954	2.0	0.028424	2.0	130
7	0.000007	3.0	0.002747	2.0	0.007117	2.0	101

In Table 2, we list the computation results for $k = 2$ elements. Again, the divergence-free element is one order higher than the rotated Bernardi-Raugel element. To show the difference in the two elements, we plot the errors by two elements on level 4 grid in Figure 5. One can see the advantage of the divergence-free element, which fully utilizes the approximation power of $\mathbf{u}_h$ by lifting the pressure polynomial degree. Of course, another advantage is the divergence-free solution after such a lift. We finally report the results for $k = 3$ in Table 3. All numerical results confirm the theory, and also show the sharpness of the superconvergence analysis.

Finally, we test the two-order superconvergence in Theorem 3.2. We choose a symmetric function as the exact solution of the Stokes equations (2.1):

$$\mathbf{u} = \mathbf{curl}\, g, \quad g = 2^8(x - x^2)^2(y - y^2)^2. \tag{4.4}$$

Comparing to the data in Table 3, we can see, in Table 4, that the velocity does converge with another order higher than the optimal order. This is predicted in (3.32). Here the order of convergence for the pressure is the same as that in Table 3. It indicates that the analysis in

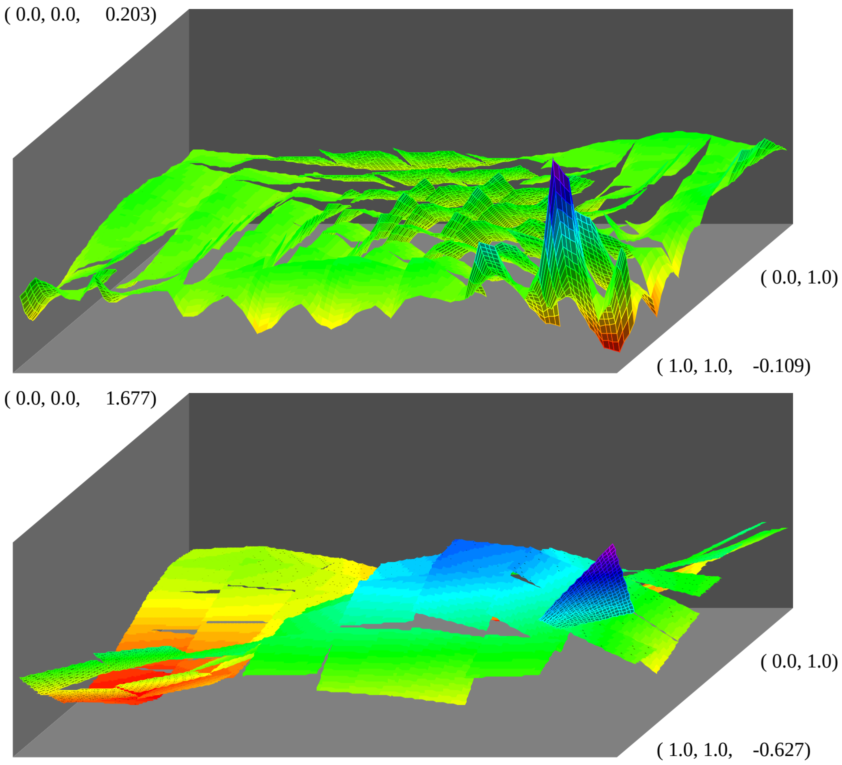

Figure 5: The errors of p_h for the divergence-free (top) and BR elements.

Theorem 3.1 is sharp. Here we have an order-two superconvergence in L^2-norm too, for the velocity. But this is not proved in this manuscript.

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Table 3: The errors $\mathbf{e}_h = \mathbf{u} - \mathbf{I}_h\mathbf{u}$ and $\epsilon_h = p - p_I$ for (4.1).

	$\vert\mathbf{e}_h\vert_{L^2}$	h^n	$\vert\mathbf{e}_h\vert_{H^1}$	h^n	$\Vert\epsilon_h\Vert_{L^2}$	h^n	
	$Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element, $k=3$						#it
1	0.123142	0.0	1.128619	0.0	1.642992	0.0	4
2	0.004515	4.8	0.065512	4.1	0.128938	3.7	3
3	0.000147	4.9	0.003911	4.1	0.008007	4.0	3
4	0.000004	5.0	0.000234	4.1	0.000494	4.0	3
5	0.000000	5.0	0.000014	4.0	0.000031	4.0	3
	rotated Bernardi-Raugel element (4.3)						#Uz
1	0.374364	0.0	3.512050	0.0	6.061521	0.0	57
2	0.021063	4.2	0.375407	3.2	0.736746	3.0	76
3	0.001597	3.7	0.058926	2.7	0.117000	2.7	123
4	0.000111	3.8	0.008169	2.9	0.013672	3.1	177
5	0.000007	3.9	0.001065	2.9	0.001666	3.0	102

Table 4: The errors $\mathbf{e}_h = \mathbf{u} - \mathbf{I}_h\mathbf{u}$ and $\epsilon_h = p - p_I$ for (4.4).

	$\vert\mathbf{e}_h\vert_{L^2}$	h^n	$\vert\mathbf{e}_h\vert_{H^1}$	h^n	$\Vert\epsilon_h\Vert_{L^2}$	h^n	
	$Q_{k+1,k}$-$Q_{k,k+1}$divergence-free element, $k=3$						#it
2	0.001196745		0.024927233		0.1147364	3.8	4
3	0.000045519	4.7	0.001383336	4.2	0.0069166	4.1	4
4	0.000000937	5.6	0.000051730	4.7	0.0004383	4.0	4
5	0.000000016	5.8	0.000001826	4.9	0.0000276	4.0	4
6	0.000000000	5.9	0.000000060	4.9	0.0000017	4.0	4

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