

# ON A ROBIN-ROBIN DOMAIN DECOMPOSITION METHOD WITH OPTIMAL CONVERGENCE RATE

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**Abstract.** In this paper, we shall solve a long-standing open problem: Is it possible that the convergence rate of the Lions' Robin-Robin nonoverlapping domain decomposition(DD) method is independent of the mesh size  $h$ ? We shall design a two-parameter Robin-Robin domain decomposition method. It is shown that the new DD method is optimal, which means the convergence rate is independent of the mesh size  $h$ . The traditional Robin-Robin domain decomposition method converges at a rate of  $1 - O(h^{1/2})$ , even under the optimal parameter. Numerical implementation confirming our theoretical findings shall be given.

**AMS subject classifications.** Primary 65N30; Secondary 65M60

**Key words.** Finite element, Robin-Robin domain decomposition method, convergence rate

**1. Introduction.** Domain decomposition (DD) methods have become increasingly important tools for solving PDEs with the development of parallel computers. In this paper, we shall study a class of nonoverlapping DD iterative procedure, which is based on using Robin-Robin boundary conditions as transmission conditions on the subdomain interfaces. The idea of employing Robin-Robin boundary conditions as interface transmission conditions was first proposed by P.L. Lions in [24]. Recently there have been several theoretical analysis and applications of this DD method: Despres[8](1991), Douglas and Huang [12, 13](1997, 1998), Deng [5, 7](1997, 2003), Du[14](2001), Gander *et al.*[20, 21](2001,2002, 2003),Guo and Hou[22](2003), Discacciati [9](2004), Flauraud and Nataf[16](2006), Gander [17, 18](2006, 2007), Qin and Xu[27, 28, 29](2006, 2008), Discacciati *et al.*[10](2007), Lui [25](2009), Chen *et al.*[2, 3](2010, 2011). We should say this list is far from being complete.

By comparison with other DD methods, Lions' DD method has several advantages. First, the iterative procedure is very simple, so the iterative procedure is much more highly parallel than others. Second, because we employ Robin boundary condition along the boundary of the subdomains, this DD method is specially suitable for solving Helmholtz and time-harmonic Maxwell equations. Actually there exist a lot of important works in this direction (cf.[8, 1, 21, 11] for details).

From the theoretical point of view, we are interested in analyzing the convergence rate of this DD method for the corresponding finite element discrete iterative procedure. It was proved in [1, 13, 12, 15, 7, 22] that the convergence rate of the DD method could be  $1 - O(h)$ . Afterwards it was first pointed out by Gander, Halpern and Nataf in [20] that the optimal choice of relaxation parameter was  $O(h^{-1/2})$  and the convergence rate  $1 - O(h^{1/2})$  could be achieved. Recently, Xu and Qin [30] used the Dirichlet-to-Neumann operator to obtain a rigorous convergence analysis for the Robin-Robin DD method in two subdomains and many subdomains cases. They further proved that the convergence result was asymptotically sharp.

The main purpose of this paper is to answer a long-standing open problem: Is

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it possible that the convergence rate of the Lions' Robin-Robin nonoverlapping DD method is independent of the mesh size  $h$ ? We shall design a two-parameter Robin-Robin domain decomposition method. It is shown that the error reduction rate of the new method is optimal, which means that the convergence rate is independent of the mesh size  $h$ .

Next, we introduce our new DD method through a simple model problem. We solve the following model problem in 2D, which is decomposed into two subproblems (cf. Figure 1.1):

$$\begin{cases} -\Delta u = f & \text{in } \Omega_1, \\ u = 0 & \text{on } \partial\Omega \cap \partial\Omega_1, \\ u - w = \frac{\partial u}{\partial \mathbf{n}} - \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ -\Delta w = f & \text{in } \Omega_2, \\ u = 0 & \text{on } \partial\Omega \cap \partial\Omega_2, \end{cases} \quad (1.1)$$

where  $\Gamma$  is an interface separating  $\Omega_1$  and  $\Omega_2$ , and  $\mathbf{n}$  is an outward normal vector of  $\Omega_1$  at  $\Gamma$ . The DD method can be applied to general elliptic PDEs, general domains and multiple subdomains.

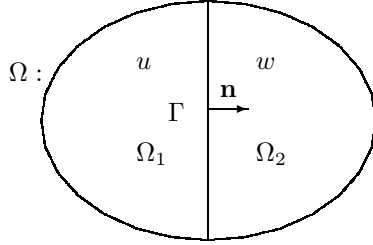


FIG. 1.1. A domain is decomposed into two subdomains.

The Dirichlet and Neumann interface conditions on  $\Gamma$  in (1.1) are combined to form two Robin interface conditions:

$$\gamma_1 u + \frac{\partial u}{\partial \mathbf{n}} = \gamma_1 w + \frac{\partial w}{\partial \mathbf{n}} = g_1 \quad \text{on } \Gamma, \quad (1.2)$$

$$\gamma_2 u - \frac{\partial u}{\partial \mathbf{n}} = \gamma_2 w - \frac{\partial w}{\partial \mathbf{n}} = g_2 \quad \text{on } \Gamma. \quad (1.3)$$

Here we allow  $\gamma_1, \gamma_2$  to be any positive constants. For example,  $\gamma_1$  is arbitrarily close to zero and  $\gamma_2$  is close to infinity (but the linear systems would become near singular.) Note that the traditional Robin interface conditions do not allow any of  $\gamma_1$  and  $\gamma_2$  to be small and large simultaneously, as there is only one parameter  $\gamma_2 = \gamma_1$ . By selecting two parameters, our Robin-Robin domain decomposition method should be better than all existing Dirichlet-Neumann, Neumann-Neumann and Robin-Robin domain decomposition methods.

Let  $V_i = H_0^1(\Omega)|_{\Omega_i}$ . By (1.2), we do an integration by parts on  $\Omega_1$  to get

$$\begin{aligned} \int_{\Gamma} g_1 v ds &= \int_{\Gamma} \left( \frac{\partial u}{\partial \mathbf{n}} + \gamma_1 u \right) v ds = \int_{\Omega_1} (\nabla u \cdot \nabla v + \Delta u v) d\mathbf{x} + \gamma_1 \int_{\Gamma} u v ds \\ &= \int_{\Omega_1} (\nabla u \cdot \nabla v - f v) d\mathbf{x} + \gamma_1 \int_{\Gamma} u v ds. \end{aligned}$$

Thus

$$a_1(u, v) + \gamma_1 \langle u, v \rangle = (f, v)_{\Omega_1} + \langle g_1, v \rangle \quad \forall v \in V_1,$$

where

$$\begin{aligned} a_i(u, v) &= \int_{\Omega_i} \nabla u \cdot \nabla v d\mathbf{x}, \quad i = 1, 2, \\ (f, v)_{\Omega_i} &= \int_{\Omega_i} f v d\mathbf{x}, \quad i = 1, 2, \\ \langle u, v \rangle &= \int_{\Gamma} u v ds. \end{aligned}$$

Similarly, by (1.3) and an integration by parts on  $\Omega_2$ , it follows (noting that  $\mathbf{n}$  is an inward normal vector to  $\Omega_2$ ) that

$$\begin{aligned} \int_{\Gamma} g_2 v ds &= \int_{\Gamma} (\gamma_2 w - \frac{\partial w}{\partial \mathbf{n}}) v ds = \int_{\Omega_2} (\nabla w \cdot \nabla v + \Delta w v) d\mathbf{x} + \gamma_2 \int_{\Gamma} w v ds \\ v &= \int_{\Omega_2} (\nabla w \cdot \nabla v - f v) d\mathbf{x} + \gamma_2 \int_{\Gamma} w v ds. \end{aligned}$$

This way, we get the second variational problem on  $\Omega_2$ :

$$a_2(w, v) + \gamma_2 \langle w, v \rangle = (f, v)_{\Omega_2} + \langle g_2, v \rangle \quad \forall v \in V_2.$$

DEFINITION 1.1. (*Two-parameter Robin-Robin domain decomposition method.*) Given  $g_1^0 (= 0)$  on  $\Gamma$ , a serial version domain decomposition iteration consists the following five steps ( $m = 0, 1, \dots$ ):

1. Solve on  $\Omega_1$  for  $u^m$ :

$$a_1(u^m, v) + \gamma_1 \langle u^m, v \rangle = (f, v)_{\Omega_1} + \langle g_1^m, v \rangle \quad \forall v \in V_1. \quad (1.4)$$

2. Update the interface condition on  $\Gamma$ :

$$g_2^m = -g_1^m + (\gamma_2 + \gamma_1) u^m. \quad (1.5)$$

3. Solve on  $\Omega_2$  for  $w^m$ :

$$a_2(w^m, v) + \gamma_2 \langle w^m, v \rangle = (f, v)_{\Omega_2} + \langle g_2^m, v \rangle \quad \forall v \in V_2. \quad (1.6)$$

4. Update the other interface condition on  $\Gamma$ :

$$\tilde{g}_1^m = -g_2^m + (\gamma_1 + \gamma_2) w^m. \quad (1.7)$$

5. Get the next iterate by a relaxation:

$$g_1^{m+1} = \theta g_1^m + (1 - \theta) \tilde{g}_1^m. \quad (1.8)$$

In section 2, we shall show that although the new Robin-Robin domain decomposition method cannot get the geometrical convergence rate at the continuous PDE level, but it does at the discrete level. In section 3, we shall give an explicit convergence rate of the DD method on uniform meshes. It is proved that the best convergence rate is  $1/7$  on the special grids. In section 4, we shall extend our method to more general quasi-uniform meshes. Using the Dirichlet-to-Neumann operator, we shall prove that the Robin-Robin DD method is optimal. Finally, in the last section, we shall present some numerical results to support our theory. It is seen from our numerical implementation that our new DD method is better than Dirichlet-Neumann DD method and traditional Robin-Robin DD method.

**2. Von Neumann analysis.** In this section, through a simple model problem, we shall show that for the new DD method it is impossible to get the geometrical convergence rate strictly less than one at the continuous level or the PDE level, but it is possible for us to prove it at the discrete level.

Let us assume that  $\Omega_1 = [-\pi, 0] \times [0, \pi]$  and  $\Omega_2 = [0, \pi] \times [0, \pi]$ , and it is enough for us to assume that  $f \equiv 0$  so that the true solutions of Equation (1.1) vanishes. Now if  $g_1 = \hat{g}_1 \sin ky$  on  $\Gamma$ , from Equations (1.1) and (1.2), the solution in  $\Omega_1$

$$u = \hat{u} \sinh(k(x+1)) \sin ky, \quad \text{where} \quad \hat{u} = \frac{\hat{g}_1}{\gamma_1 \sinh k + k \cosh k}. \quad (2.1)$$

Similarly, if  $g_2 = \hat{g}_2 \sin ky$  on  $\Gamma$ , from Equations (1.1) and (1.3), the solution in  $\Omega_2$

$$w = \hat{w} \sinh(k(x-1)) \sin ky, \quad \text{where} \quad \hat{w} = -\frac{\hat{g}_2}{\gamma_2 \sinh k + k \cosh k}. \quad (2.2)$$

In the two-parameter Robin-Robin DDM algorithm, if the initial error  $g_1^m = \hat{g}_1^m \sin ky$  on  $\Gamma$ , then

$$g_2^m = \hat{g}_2^m \sin ky, \quad \text{where} \quad \hat{g}_2^m = \hat{g}_1^m \left( \frac{\gamma_2 + \gamma_1}{\gamma_1 + k \coth k} - 1 \right).$$

And if  $g_2^m$  is solved, by (1.6) and (1.7),

$$\tilde{g}_1^m = \hat{g}_1^m \sin ky, \quad \text{where} \quad \hat{g}_1^m = \hat{g}_2^m \left( \frac{\gamma_2 + \gamma_1}{\gamma_2 + k \coth k} - 1 \right).$$

Finally, after the relaxation step (1.8),

$$g_1^{m+1} = \hat{g}_1^{m+1} \sin ky,$$

where

$$\hat{g}_1^{m+1} = \theta \hat{g}_1^m + (1 - \theta) \tilde{g}_1^m = \rho \hat{g}_1^m,$$

and the factor

$$\rho = \theta + (1 - \theta) \left( \frac{\gamma_2 + \gamma_1}{\gamma_2 + k \coth k} - 1 \right) \left( \frac{\gamma_2 + \gamma_1}{\gamma_1 + k \coth k} - 1 \right). \quad (2.3)$$

Now for the fixed parameters  $0 < \theta < 1$ ,  $\gamma_1 > 0$  and  $\gamma_2 > 0$ , if  $k$  tends to infinity, then

$$\rho \approx 1 - 2(1 - \theta) \frac{\gamma_1 + \gamma_2}{k}. \quad (2.4)$$

Therefore in the continuous case, it is impossible to get the convergence rate independent of the frequency(or the wave number)  $k$ . On the other hand, if  $k$  is bounded by  $1 \leq k \leq K$ , we may obtain the convergence rate  $\rho$ , which is independent of  $k$  (but dependent on  $K$ ), through choosing the three parameters  $\gamma_1$ ,  $\gamma_2$  and  $\theta$ .

LEMMA 2.1. *If  $a$  and  $b$  are two non-negative constants, then the function*

$$\rho(\theta) = \max \{ |\theta - (1 - \theta)a|, |\theta - (1 - \theta)b| \} \quad (2.5)$$

*attains the minimum value  $\frac{|b-a|}{2+a+b}$  at  $\theta_0 = \frac{a+b}{2+a+b}$ .*

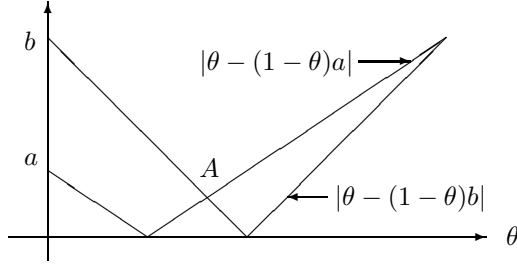


FIG. 2.1. Graphs of  $|\theta - (1 - \theta)a|$  and  $|\theta - (1 - \theta)b|$ ,  $b > a \geq 0$ .

*Proof.* Without loss of generality, we assume  $b \geq a$ . Both terms in (2.5) are piecewise linear functions. We plot them in Figure 2.1. The minimal value is attained at the point  $A$ , where two lines intersect:

$$\theta - (1 - \theta)a + \theta - (1 - \theta)b = 0.$$

That is  $\theta = \theta_0 = \frac{a+b}{2+a+b}$ , and

$$|\theta_0 - (1 - \theta_0)a| = |\theta_0 - (1 - \theta_0)b| = \frac{a + b}{2 + a + b}.$$

So we get the lemma.  $\square$

LEMMA 2.2. For any  $z \geq 0$ , the function

$$\omega(z) = \frac{\gamma_2 - z}{\gamma_2 + z} \cdot \frac{z - \gamma_1}{z + \gamma_1} \quad (2.6)$$

attains the maximum value at  $z_0 = \sqrt{\gamma_1 \gamma_2}$ :

$$\max_{z > 0} \omega(z) = \frac{(\eta - 1)^2}{(\eta + 1)^2}, \quad \text{where } \eta = \sqrt{\frac{\gamma_2}{\gamma_1}}. \quad (2.7)$$

*Proof.* The derivative of  $\omega(z)$  is

$$\omega'(z) = \frac{2(\gamma_1 + \gamma_2)(\gamma_1 \gamma_2 - z^2)}{(z + \gamma_1)^2 (z + \gamma_2)^2}. \quad (2.8)$$

So  $\omega(z)$  monotonically increases when  $z < z_0$  and monotonically decreases when  $z > z_0$ . In particular, then the minimum value of  $\omega(z)$  on an interval  $[z_1, z_2]$  is attained at one of the end points:

$$\min_{z \in [z_1, z_2]} \omega(z) = \min\{\omega(z_1), \omega(z_2)\}. \quad (2.9)$$

By (2.8),  $\omega(z)$  attains the only global maximum value at  $z_0 = \sqrt{\gamma_1 \gamma_2}$ :

$$\omega(z_0) = \frac{\gamma_2 - \sqrt{\gamma_1 \gamma_2}}{\sqrt{\gamma_1 \gamma_2} + \gamma_2} \frac{\sqrt{\gamma_1 \gamma_2} - \gamma_1}{\sqrt{\gamma_1 \gamma_2} + \gamma_1} = \frac{(\eta - 1)^2}{(\eta + 1)^2}.$$

The lemma is proved.  $\square$

Now from Equation (2.3),

$$\rho = \theta - (1 - \theta) \frac{\gamma_2 - k \coth k}{\gamma_2 + k \coth k} \cdot \frac{k \coth k - \gamma_1}{\gamma_1 + k \coth k} = \theta - (1 - \theta) \omega(k \coth k).$$

And if  $\gamma_1$  and  $\gamma_2$  are chosen that  $\gamma_1 < \coth 1$  and  $\gamma_2 > K \coth K$ , then  $\omega(k \coth k) > 0$ . So by Lemma 2.2,

$$|\rho| \leq \max\{|\theta|, |\theta - (1 - \theta)\omega(z_0)|\}.$$

And by Lemma 2.1, we may select  $\theta_0 = \frac{\omega(z_0)}{2 + \omega(z_0)}$  such that

$$|\rho| \leq |\theta_0| < \frac{1}{3}. \quad (2.10)$$

REMARK 2.1. If  $\eta > \frac{\sqrt{2}+1}{\sqrt{2}-1}$ , we may just set  $\theta = \frac{1}{3}$ , and  $|\rho|$  is also less than  $\frac{1}{3}$ . Moreover, this bound can be improved further if we carefully estimate the minimum value of  $\omega(z)$ .

REMARK 2.2. The constrain  $\gamma_1 < \coth 1$  can be relaxed. Actually, if  $\gamma_1 > \coth 1$ , then

$$|\rho| \leq \max\{|\theta + (1 - \theta)\zeta|, |\theta - (1 - \theta)\omega(z_0)|\},$$

where  $\zeta = \frac{\gamma_1 - \coth 1}{\gamma_1 + \coth 1}$ . Then we set  $\theta = 0$  if  $\omega(z_0) \leq \zeta$  and set  $\theta = \frac{\omega(z_0) - \zeta}{2 + \omega(z_0) - \zeta}$  if  $\omega(z_0) > \zeta$ , and

$$|\rho| \leq \begin{cases} \zeta, & \text{if } \omega(z_0) \leq \zeta, \\ \frac{\omega(z_0) + \zeta}{2 + \omega(z_0) - \zeta}, & \text{if } \omega(z_0) > \zeta. \end{cases}$$

Note that  $\omega(z_0) < 1$  and  $\frac{\omega(z_0) + \zeta}{2 + \omega(z_0) - \zeta} \leq \frac{1 + \zeta}{3 - \zeta}$  which is also independent of  $K$ .

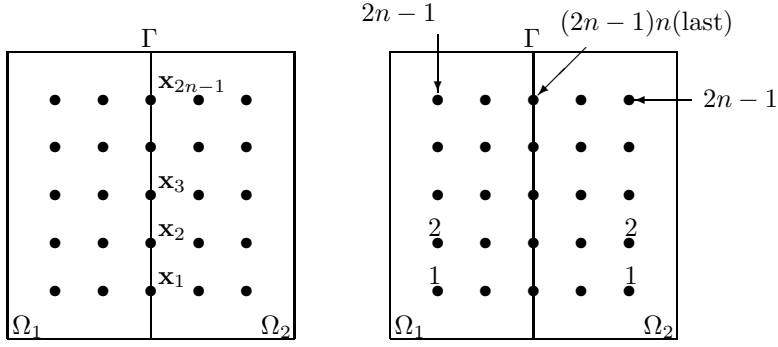
The Von Neumann analysis shows that the two-parameter Robin-Robin domain decomposition does have a constant rate of convergence, independent of the frequency number  $k$  or  $K$ . But the selection of the two parameters depends on  $K$ . The limit case indicates that the method deteriorates to, i.e.,  $\gamma_2 = \infty$ , a Robin-Dirichlet domain decomposition method.

**3. Convergence on uniform grids.** In this section, we analyze the two parameter Robin-Robin domain decomposition method on uniform grids. In this case, we give explicit eigenvalues of the iterative matrix, showing the optimal rate of convergence.

We post a uniform grid of size  $h = 1/(2n)$  on the domain  $\Omega = [0, 1]^2$ , the unit square, shown in Figure 3.1. We give two numberings of nodal values of the  $C^0$ - $P_1$  functions. One numbering is on the interface  $\Gamma$ . The other one is within each subdomain,  $\Omega_1$  and  $\Omega_2$ . When numbering the nodes in  $\Omega_2$ , we go from right to left so that the nodal index is symmetric to that on the left domain  $\Omega_1$ .

Let  $M_\Gamma$  and  $A_\Gamma$  be two tridiagonal  $(2n - 1) \times (2n - 1)$  matrices:

$$M_\Gamma = \frac{h}{6} \begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & & 1 & 4 \end{pmatrix}, \quad A_\Gamma = \frac{1}{2} \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & & -1 & 4 \end{pmatrix}.$$

FIG. 3.1. Nodal basis numberings, on  $\Gamma$  and on  $\Omega_1 \cap \Omega_2$ .

Here  $M_\Gamma$  is just the mass matrix of the inner product  $\langle \cdot, \cdot \rangle$ . Let  $R_h$  be the  $(2n-1) \times (2n-1)n$  matrix representing a restriction operator on  $\Gamma$ :

$$R_h = (0_{2n-1}, \dots, 0_{2n-1}, I_{2n-1}). \quad (3.1)$$

The stiffness matrix of the bilinear form  $a_1(\cdot, \cdot)$  (and  $a_2(\cdot, \cdot)$  too) is

$$A_h = A_0 - R_h^T A_\Gamma R_h,$$

where the matrix  $A_0$  is the stiffness matrix of size  $(2n-1)n$ , for the Laplace operator on a  $(2n) \times (n+1)$  uniform grid with zero Dirichlet boundary condition.  $A_0$  is same as the matrix of standard five-point finite difference matrix, which has the eigen-decomposition[6, 23]:

$$A_0 = (\Phi_n \otimes \Phi_{2n-1})^T (\Lambda_n \otimes I_{2n-1} + I_n \otimes \Lambda_{2n-1}) (\Phi_n \otimes \Phi_{2n-1}), \quad (3.2)$$

where  $\Lambda_m$  denotes an diagonal matrix whose  $(i, i)$ -th entry is

$$\lambda_i^{(m)} = 4 \sin^2 \frac{i\pi}{2(m+1)}, \quad (3.3)$$

and  $\Phi_m$  denotes an orthogonal matrix defined by

$$\Phi_m = \begin{pmatrix} \phi_1^{(m)} & \dots & \phi_m^{(m)} \end{pmatrix}, \quad \text{with } \phi_i^{(m)} = \sqrt{\frac{2}{m}} \begin{pmatrix} \sin \frac{i\pi}{m+1} \\ \sin \frac{2i\pi}{m+1} \\ \vdots \\ \sin \frac{mi\pi}{m+1} \end{pmatrix}. \quad (3.4)$$

Here in (3.2), a tensor product matrix  $C_{mk \times mk} = A_{m \times m} \otimes B_{k \times k}$  is defined with the  $(i, j)$ -th entry

$$C_{ij} = A_{i', j'} B_{i'', j''}, \quad \text{where } \begin{aligned} i &= (i' - 1)k + i'', \\ j &= (j' - 1)k + j''. \end{aligned}$$

In Definition 1.1, for (1.4), the error  $e_u^m = u - u^m$  satisfies the equation:

$$a_1(e_u^m, v) + \gamma_1 \langle e_u^m, v \rangle = \langle e_{g_1}^m, v \rangle \quad \forall v \in V_1.$$

Here  $e_{g_1}^m = g_1 - g_1^m$  is the error. In the matrix-vector form,

$$E_u^m = (A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1} R_h^T M_\Gamma E_{g_1}^m.$$

Here  $E_u^m$  is the vector representation of  $e_u^m$ . Therefore, by (1.6),

$$E_{g_2}^m = (-I + (\gamma_2 + \gamma_1) R_h (A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1} R_h^T M_\Gamma) E_{g_1}^m. \quad (3.5)$$

Symmetrically, by (1.7) and (1.8),

$$\tilde{E}_{g_1}^m = C_{\gamma_2} E_{g_2}^m. \quad (3.6)$$

Here, for simplicity, we denote the error reduction matrix by

$$C_{\gamma_k} = (-I + (\gamma_2 + \gamma_1) R_h (A_h + \gamma_k R_h^T M_\Gamma R_h)^{-1} R_h^T M_\Gamma), \quad k = 1, 2. \quad (3.7)$$

Finally, by (1.8), one Robin-Robin DD iteration reduces the initial error  $E_{g_1}^m$  to

$$E_{g_1}^{m+1} = [\theta I + (1 - \theta) C_{\gamma_2} C_{\gamma_1}] E_{g_1}^m. \quad (3.8)$$

We find the eigenvalue range of this error reduction matrix, via common eigenvectors of all matrices.

LEMMA 3.1. *The error reduction matrix (3.8) can be diagonalized by  $\Phi_{2n-1}$  defined in (3.4). That is,*

$$\Phi_{2n-1} [\theta I + (1 - \theta) C_{\gamma_2} C_{\gamma_1}] \Phi_{2n-1}^T = \text{diag}(\theta + (1 - \theta) c_j), \quad (3.9)$$

where in the  $j$ -th diagonal element,

$$c_j = \frac{\gamma_1 a_j - b_j}{\gamma_1 a_j + b_j} \cdot \frac{\gamma_2 a_j - b_j}{\gamma_2 a_j + b_j}. \quad (3.10)$$

Here in (3.10),

$$a_j = \lambda_{M_\Gamma, j} \tilde{\lambda}_j, \quad \lambda_{M_\Gamma, j} = h - \frac{h}{6} \lambda_j^{(2n-1)}, \quad (3.11)$$

$$b_j = 1 - \lambda_{A_\Gamma, j} \tilde{\lambda}_j, \quad \lambda_{A_\Gamma, j} = 1 + \frac{1}{2} \lambda_j^{(2n-1)}, \quad (3.12)$$

where  $\lambda_j^{(2n-1)}$  is defined in (3.3) and

$$\tilde{\lambda}_j = \frac{2}{n+1} \sum_{i=1}^n \sin^2 \frac{i n \pi}{n+1} (\lambda_i^{(n)} + \lambda_j^{(2n-1)})^{-1}. \quad (3.13)$$

*Proof.* In (3.5), by the Sherman-Morrison-Woodbury formula,

$$\begin{aligned} & (A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1} \\ &= (A_0 + R_h^T (-A_\Gamma + \gamma_1 M_\Gamma) R_h)^{-1} \\ &= A_0^{-1} - A_0^{-1} R_h^T ((-A_\Gamma + \gamma_1 M_\Gamma)^{-1} + R_h A_0^{-1} R_h^T)^{-1} R_h A_0^{-1}. \end{aligned}$$

Now letting  $B_0 = R_h A_0^{-1} R_h^T$ , we have

$$R_h (A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1} R_h^T = B_0 - B_0 ((-A_\Gamma + \gamma_1 M_\Gamma)^{-1} + B_0)^{-1} B_0.$$

By (3.1) and (3.2), note that  $(\Phi_n \otimes \Phi_{2n-1})R_h^T = \phi_n^{(n)} \otimes \Phi_{2n-1}$ , we can compute  $B_0$ :

$$\begin{aligned} B_0 &= (\phi_n^{(n)} \otimes \Phi_{2n-1})^T (\Lambda_n \otimes I_{2n-1} + I_n \otimes \Lambda_{2n-1}) (\phi_n^{(n)} \otimes \Phi_{2n-1}) \\ &= \sum_{i=1}^n (\phi_{n,i}^{(n)})^2 \Phi_{2n-1}^T (\lambda_i^{(2n-1)} I_{2n-1} + \Lambda_{2n-1})^{-1} \Phi_{2n-1} \\ &= \Phi_{2n-1}^T \left( \sum_{i=1}^n (\phi_{n,i}^{(n)})^2 (\lambda_i^{(2n-1)} I_{2n-1} + \Lambda_{2n-1})^{-1} \right) \Phi_{2n-1} \\ &= \Phi_{2n-1}^T \tilde{\Lambda}_0 \Phi_{2n-1}, \end{aligned}$$

where  $\phi_{n,i}^{(n)}$  is the  $i$ -th entry of vector  $\phi_n^{(n)}$  defined in (3.4), and  $\tilde{\Lambda}_0$  is a diagonal matrix, whose  $(j, j)$ -th entry is define in (3.13). The matrices on  $\Gamma$  are diagonalized as  $M_\Gamma = \Phi_{2n-1}^T \text{diag}(\lambda_{M_\Gamma, j}) \Phi_{2n-1}$  and  $A_\Gamma = \Phi_{2n-1}^T \text{diag}(\lambda_{A_\Gamma, j}) \Phi_{2n-1}$ , where  $\lambda_{M_\Gamma, j}$  and  $\lambda_{A_\Gamma, j}$  are defined in (3.11) and (3.12), respectively. Thus combining last two equalities, we get

$$\begin{aligned} &R_h (A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1} R_h^T \\ &= B_0 - B_0 (\Phi_{2n-1}^T (-\text{diag}(\lambda_{A_\Gamma, j}) + \gamma_1 \text{diag}(\lambda_{M_\Gamma, j}))^{-1} \Phi_{2n-1} + B_0)^{-1} B_0 \\ &= \Phi_{2n-1}^T \left[ \tilde{\Lambda}_0 - \tilde{\Lambda}_0^2 \{ (-\text{diag}(\lambda_{A_\Gamma, j}) + \gamma_1 \text{diag}(\lambda_{M_\Gamma, j}))^{-1} + \tilde{\Lambda}_0 \}^{-1} \right] \Phi_{2n-1} \\ &= \Phi_{2n-1}^T \left[ \tilde{\Lambda}_0^{-1} - \text{diag}(\lambda_{A_\Gamma, j}) + \gamma_1 \text{diag}(\lambda_{M_\Gamma, j}) \right]^{-1} \Phi_{2n-1}. \end{aligned}$$

By (3.7),

$$\begin{aligned} C_{\gamma_1} &= \Phi_{2n-1}^T (-I + (\gamma_2 + \gamma_1) \left[ (\tilde{\Lambda}_0^{-1} - \text{diag}(\lambda_{A_\Gamma, j})) \text{diag}(\lambda_{M_\Gamma, j}^{-1}) + \gamma_1 I \right]^{-1}) \Phi_{2n-1} \\ &= \Phi_{2n-1}^T \text{diag} \left( \frac{-1 + \gamma_2 \lambda_{M_\Gamma, j} \tilde{\lambda}_j + \lambda_{A_\Gamma, j} \tilde{\lambda}_j}{1 + \gamma_1 \lambda_{M_\Gamma, j} \tilde{\lambda}_j - \lambda_{A_\Gamma, j} \tilde{\lambda}_j} \right) \Phi_{2n-1} \\ &= \Phi_{2n-1}^T \text{diag} \left( \frac{\gamma_2 a_j - b_j}{\gamma_1 a_j + b_j} \right) \Phi_{2n-1}. \end{aligned}$$

In the same fashion, it follows that

$$C_{\gamma_2} = \Phi_{2n-1}^T \text{diag} \left( \frac{\gamma_1 a_j - b_j}{\gamma_2 a_j + b_j} \right) \Phi_{2n-1}.$$

Thus (3.9) follows.  $\square$

In next lemma, we estimate the eigenvalue  $c_j$  in the reduction matrix, (3.10).

LEMMA 3.2.  $(3a_j - b_j)$  is monotonically decreasing, i.e.,  $j = 1, \dots, 2n - 2$ ,

$$3a_j - b_j \geq 3a_{j+1} - b_{j+1}. \quad (3.14)$$

*Proof.* We rewrite the  $\tilde{\lambda}_j$  (in  $a_j$  and  $b_j$ ) in a symmetric form so that each  $i$ -term

is a decreasing function of  $j$  (the original term is not.)

$$\begin{aligned}\tilde{\lambda}_j &= \frac{2}{n+1} \frac{1}{2} \sum_{i=1}^n \frac{\sin^2(i\pi/(n+1))}{4\sin^2(j\pi/(4n)) + 4\sin^2(i\pi/(2n+2))} \\ &\quad + \frac{\sin^2((n+1-i)\pi/(n+1))}{4\sin^2(j\pi/(4n)) + 4\sin^2((n+1-i)\pi/(2n+2))} \\ &= \frac{1}{n+1} \sum_{i=1}^n \frac{\sin^2(i\pi/(n+1))(2\sin^2(j\pi/(4n)) + 1)}{4\sin^4(j\pi/(4n)) + 4\sin^2(j\pi/(4n)) + \sin^2(i\pi/(n+1))}.\end{aligned}$$

To shorten expression, we introduce two more notations

$$\xi_j = \sin^2 \frac{j\pi}{4n}, \quad (3.15)$$

$$\theta_j = (1 + 2\xi_j)(1 + 3h + 2\xi_j - 2h\xi_j). \quad (3.16)$$

By (3.11) and (3.12), we have

$$3a_j - b_j + 1 = \frac{1}{n+1} \sum_{i=1}^n \frac{\sin^2(i\pi/(n+1))\theta_j}{4\xi_j^2 + 4\xi_j + \sin^2(i\pi/(n+1))}. \quad (3.17)$$

We show that each term is a decreasing function of  $\xi_j$ . That is, each term

$$f_i(\xi) = \frac{(2\xi + 1)(2(1+h) - (1-h)(1-2\xi))}{4\xi^2 + 4\xi + \sin^2(i\pi/(n+1))}$$

is a decrease function of  $\xi$ , for  $\xi \in (0, 1)$ . By the quotient rule,

$$\begin{aligned}f'_i(\xi) &= \frac{(4(1+h) + 8(1-h)\xi)(4\xi^2 + 4\xi + \sin^2(i\pi/(n+1)))}{(4\xi^2 + 4\xi + \sin^2(i\pi/(n+1)))^2} \\ &\quad - \frac{((1+3h) + 4(1+h)\xi + 4(1-h)\xi^2)(8\xi + 4)}{(4\xi^2 + 4\xi + \sin^2(i\pi/(n+1)))^2}.\end{aligned}$$

The combined numerator is

$$-\left(4(1+h)\cos^2 \frac{i\pi}{n+1} + 8h\right) - \left(8(1-h)\cos^2 \frac{i\pi}{n+1} + 16h\right)\xi - (32h)\xi^2 < 0.$$

As each term  $f_i(\xi_j)$  is desecrating with respect to  $j$ , the sum is a desecrating function of  $j$ . We prove the lemma.  $\square$

We bound the biggest term  $(3a_1 - b_1)$  among all  $(3a_j - b_j)$  in order to bound the  $c_j$  in (3.10).

LEMMA 3.3. *For any  $n \geq 1$ ,  $h = 1/(2n)$ , the following bound holds,*

$$3a_1 - b_1 < -7h^2/16, \quad (3.18)$$

where  $a_1$  and  $b_1$  are defined in (3.11) and (3.12), respectively.

*Proof.* This will be proved in the Appendix. But we will prove a stronger lemma under a stronger condition,  $n \geq 11$ , next.  $\square$

LEMMA 3.4. *If  $n \geq 11$ , then (cf. (3.18))*

$$3a_1 - b_1 < -0.049h < -7h^2/16, \quad (3.19)$$

where  $a_1$  and  $b_1$  are defined in (3.11) and (3.12), respectively.

*Proof.* By (3.17), with the notations defined in (3.11), (3.12) and (3.13),

$$3a_1 - b_1 + 1 = (1 + 3h + \frac{1-h}{2}\lambda_1^{(2n-1)})\tilde{\lambda}_1. \quad (3.20)$$

We estimate an upper bound for

$$\begin{aligned} \tilde{\lambda}_1 &= \frac{2}{n+1} \sum_{i=1}^n \cos^2 \frac{i\pi}{2(n+1)} - \frac{2}{n+1} \sum_{i=1}^n \frac{\cos^2 \frac{i\pi}{2(n+1)} \sin^2 \frac{\pi}{4n}}{\sin^2 \frac{i\pi}{2(n+1)} + \sin^2 \frac{\pi}{4n}} \\ &= \frac{n}{n+1} - \frac{2 \sin^2 \frac{\pi}{4n}}{n+1} \sum_{i=1}^n \left( \frac{1 + \sin^2 \frac{\pi}{4n}}{\sin^2 \frac{i\pi}{2(n+1)} + \sin^2 \frac{\pi}{4n}} - 1 \right) \\ &= 1 - \frac{1 - 2n \sin^2 \frac{\pi}{4n}}{n+1} - \frac{2(1 + \sin^2 \frac{\pi}{4n})}{n+1} \sum_{i=1}^n \frac{\sin^2 \frac{\pi}{4n}}{\sin^2 \frac{i\pi}{2(n+1)} + \sin^2 \frac{\pi}{4n}}. \end{aligned} \quad (3.21)$$

As  $(\sin x/x)$  is a decreasing function of  $x$  on  $(0, \pi/2)$ , we have

$$\begin{aligned} \sum_{i=1}^n \frac{\sin^2 \frac{\pi}{4n}}{\sin^2 \frac{i\pi}{2(n+1)} + \sin^2 \frac{\pi}{4n}} &> \sum_{i=1}^n \frac{1}{\left(\frac{i\pi}{2(n+1)} / \frac{\pi}{4n}\right)^2 + 1} > \sum_{i=1}^n \frac{1}{1 + 4i^2} \\ &\geq \sum_{i=1}^{11} \frac{1}{1 + 4i^2} > 0.33462. \end{aligned}$$

Substituting the estimate into the expression of  $\tilde{\lambda}_1$ ,

$$\tilde{\lambda}_1 < 1 - \frac{1 + 2(1 + 2 \sin^2 \frac{\pi}{4n}) \cdot 0.33462 - 2n \sin^2 \frac{\pi}{4n}}{n+1} < 1 - \frac{1.55726}{n+1}.$$

By (3.20), if  $n \geq 11$ ,

$$\begin{aligned} 3a_1 - b_1 + 1 &< (1 + \frac{3}{2n} + \frac{\pi^2}{8n^2})(1 - \frac{1.55726}{n+1}) \\ &< 1 - 0.049h \leq 1 - 0.98h^2 < 1 - 7h^2/16. \end{aligned}$$

Now we prove the lemma.  $\square$

With the explicit eigenvalues of the reduction matrix and their bounds, we can easily choose a set of parameters  $\gamma_1$ ,  $\gamma_2$  and  $\theta$ , to get a constant rate of reduction, independent of mesh size  $h$ .

**THEOREM 3.5.** *Let  $\gamma_1 = 1$ ,  $\gamma_2 = 64h^{-1}$  and  $\theta = 3/7$  in Definition 1.1. The error reduction factor is bounded by  $1/7$ , independent of the grid size  $h$ ,*

$$\|e_{g_1}^{m+1}\|_{L^2(\Gamma)} \leq \frac{1}{7} \|e_{g_1}^m\|_{L^2(\Gamma)}.$$

*Proof.* We will apply Lemma 2.2. By (3.11) and (3.3),  $a_j > 0$ . By (3.14), (3.18) and (3.12),

$$\begin{aligned} 3a_j - b_j &\leq 3a_1 - b_1 \leq -7h^2/16, \\ b_j &\geq 3a_j + 7h^2/16 > 0. \end{aligned} \quad (3.22)$$

By (3.10),

$$c_j = \frac{1 - b_j/a_j}{1 + b_j/a_j} \cdot \frac{64h^{-1} - b_j/a_j}{64h^{-1} + b_j/a_j}.$$

We let  $z = b_j/a_j > 0$  in Lemma 2.2. The critical point is (may be outside the  $b_j/a_j$  range)

$$z_0 = \sqrt{\gamma_1 \gamma_2} = 8h^{-1/2}.$$

We find the two end points of possible  $z$ . First, by (3.13),

$$\tilde{\lambda}_j \geq \frac{2}{n+1} \sum_{i=1}^n \frac{1}{8} \sin^2 \frac{in\pi}{n+1} = \frac{n}{8(n+1)} > \frac{1}{8}.$$

Thus, by (3.11), (3.12) and (3.3),

$$\begin{aligned} a_j &\leq \left(h - \frac{h}{6} \cdot 0\right) \cdot 1 = h, \\ a_j &\geq \left(h - \frac{h}{6} \cdot 4\right) \cdot \frac{1}{8} = \frac{h}{12}, \\ b_j &\leq 1 - \left(1 + \frac{1}{2} \cdot 0\right) \cdot \frac{1}{8} = \frac{7}{8}. \end{aligned}$$

In the first inequality, we used (3.21) that  $\tilde{\lambda}_j < 1$ . We find one end point for  $z$ :

$$\frac{b_j}{a_j} \leq \frac{7/8}{h/12} = \frac{21}{2h} \equiv z_r.$$

For the other end point, by (3.22),

$$\frac{b_j}{a_j} \geq 3 + \frac{7h^2/16}{a_j} \geq 3 + \frac{7h^2/16}{h} = 3 + \frac{7h}{16} \equiv z_l.$$

By Lemma 2.2, the range of  $c_j$  is between its values at  $z = z_l, z_0, z_r$ . Note that  $z_l < z_0 < z_r$  here. At each point, we need to apply 2.2 again for  $h$  varying. But we can find some rough (but good enough) bounds at each point, directly.

$$\begin{aligned} \text{At } z = z_r: & \quad -0.718\dots = -\frac{107}{149} < c_j \leq -\frac{1070}{1639} = -0.65\dots \\ \text{At } z = z_l: & \quad -0.50098\dots = -\frac{2184975}{4361329} \leq c_j < -\frac{1}{2} = -0.5. \\ \text{At } z = z_0: & \quad -1 < c_j \leq \frac{32 - 129\sqrt{2}}{32 - 129\sqrt{2}} = -0.7015\dots \end{aligned}$$

Hence the value of  $c_j$  is always strictly between  $-1$  and  $-1/2$ . When  $\theta = 3/7$ , we get,

$$\theta + (1 - \theta)c_j > \frac{3}{7} + \frac{4}{7}(-1) = -\frac{1}{7}, \quad (3.23)$$

$$\theta + (1 - \theta)c_j < \frac{3}{7} + \frac{4}{7}\left(-\frac{1}{2}\right) = \frac{1}{7}. \quad (3.24)$$

This gives the error reduction factor.  $\square$

By (3.23) and (3.24), we can get the following result for a general relaxation parameter  $\theta$ .

**COROLLARY 3.6.** *Let  $\gamma_1 = 1$  and  $\gamma_2 = 64h^{-1}$  in Definition 1.1. The error reduction factor  $\rho$  on uniform grids is*

$$\rho = \begin{cases} 1 - 2\theta, & 0 \leq \theta \leq 3/7, \\ (3\theta - 1)/2, & 3/7 < \theta \leq 1. \end{cases}$$

That is,  $\|e_{g_1}^{m+1}\|_{L^2(\Gamma)} \leq \rho \|e_{g_1}^m\|_{L^2(\Gamma)}$ .

**4. Convergence on general grids.** In this section, we consider the convergence behavior of the Robin-Robin DD method on general quasi-uniform meshes. Define the errors as follows:

$$\varepsilon_1^m = g_1 - g_1^m, \quad \varepsilon_2^m = g_2 - g_2^m,$$

and

$$e_1^m = u - u^m, \quad e_2^m = w - w^m.$$

By the algorithm in Definition 1.1, for  $i = 1, 2$ ,

$$a_i(e_i^m, v) + \gamma_i \langle e_i^m, v \rangle = \langle \varepsilon_i^m, v \rangle, \quad \forall v \in V_i.$$

Let  $S_1$  and  $S_2$  be the standard Dirichlet-to-Neumann operators, defined in [26, 30]. The functions  $\varepsilon_i^n$  ( $i = 1$  or  $2$ ) restricted to interface  $\Gamma$  satisfies the relation

$$\varepsilon_i^m = (\gamma_i + S_i)e_i^m|_{\Gamma}. \quad (4.1)$$

Using the first interface update (1.5), we have

$$\varepsilon_2^m = -\varepsilon_1^m + (\gamma_1 + \gamma_2)e_1^m|_{\Gamma}. \quad (4.2)$$

For the second one, by (1.7) and (1.8),

$$\begin{aligned} \varepsilon_1^{m+1} &= \theta \varepsilon_1^m + (1 - \theta)[- \varepsilon_2^m + (\gamma_1 + \gamma_2)e_2^m|_{\Gamma}] \\ &= \theta \varepsilon_1^m + (1 - \theta)[-(\gamma_2 + S_2)e_2^m|_{\Gamma} + (\gamma_1 + \gamma_2)e_2^m|_{\Gamma}] \\ &= \theta \varepsilon_1^m + (1 - \theta)(\gamma_1 - S_2)e_2^m|_{\Gamma}. \end{aligned}$$

By (4.1), (4.2), we have

$$\begin{aligned} \varepsilon_1^{m+1} &= \theta \varepsilon_1^m + (1 - \theta)(\gamma_1 - S_2)(\gamma_2 + S_2)^{-1} \varepsilon_2^m \\ &= \theta \varepsilon_1^m + (1 - \theta)(\gamma_1 - S_2)(\gamma_2 + S_2)^{-1}(\gamma_2 - S_1)e_1^m|_{\Gamma} \\ &= [\theta + (1 - \theta)(\gamma_1 - S_2)(\gamma_2 + S_2)^{-1}(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}] \varepsilon_1^m. \end{aligned}$$

Define an operator

$$T := (\gamma_1 - S_2)(\gamma_2 + S_2)^{-1}(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}.$$

Then above iteration procedure can be expressed as

$$\varepsilon_1^{m+1} = [\theta + (1 - \theta)T] \varepsilon_1^m.$$

Next, we give the convergence analysis for the DD algorithm.

**4.1. Symmetric case:**  $S_1 = S_2(= S)$ . In this case, assume  $z$  is one of eigenvector of the operator  $T$ , and its corresponding eigenvalue is  $\lambda'$ . It is known that  $z$  is also the eigenvector of the operator  $S$ . We assume the corresponding eigenvalue is  $\lambda_s$ , then

$$\begin{aligned} [\theta + (1 - \theta)T]z &= [\theta + (1 - \theta) \frac{(\gamma_1 - \lambda_s)(\gamma_2 - \lambda_s)}{(\gamma_1 + \lambda_s)(\gamma_2 + \lambda_s)}]z \\ &= [\theta - (1 - \theta)\omega(\lambda_s)]z. \end{aligned}$$

It is known [30] that

$$\lambda_s \in [c_0, C_0 h^{-1}].$$

Now if we choose  $0 < \gamma_1 < c_0$  and  $\gamma_2 > C_0 h^{-1}$ , by Lemma 2.2, we get

$$0 < \omega(\lambda_s) \leq \frac{(\eta - 1)^2}{(\eta + 1)^2}, \quad \eta = \sqrt{\frac{\gamma_2}{\gamma_1}}. \quad (4.3)$$

Then we know that the spectrum of the operator  $\theta + (1 - \theta)T$  belongs to

$$[\theta - (1 - \theta) \frac{(\eta - 1)^2}{(\eta + 1)^2}, \theta].$$

By choosing the parameter  $\theta = 1/3$ , cf. Remark 2.1, we can make the convergence rate less than  $\frac{1}{3}$ , which is independent of the mesh size  $h$ .

**4.2. Nonsymmetric cases:**  $S_1 \approx S_2$ . In this case, there exist two positive constant  $0 < s \leq 1$  and  $t \geq 1$  such that for all  $v \in V_i|_\Gamma$  (cf.[26] for details):

$$s(S_1 v, v) \leq (S_2 v, v) \leq t(S_1 v, v). \quad (A1)$$

So for any constant  $\gamma > 0$ ,

$$s((\gamma + S_1)v, v) \leq ((\gamma + S_2)v, v) \leq t((\gamma + S_1)v, v). \quad (4.4)$$

REMARK 4.1. *The condition (A1) has another version if we replace  $v$  by  $S_1^{-\frac{1}{2}}v$ :*

$$s(v, v) \leq (S_2 S_1^{-\frac{1}{2}}v, S_1^{-\frac{1}{2}}v) \leq t(v, v). \quad (4.5)$$

*This inequality also means that the spectrum of  $S_1^{-\frac{1}{2}}S_2S_1^{-\frac{1}{2}}$  lies in  $[s, t]$ , then the spectrum of  $(S_1^{-\frac{1}{2}}S_2S_1^{-\frac{1}{2}})^{-1}$  lies in  $[t^{-1}, s^{-1}]$ , i.e.,*

$$\frac{1}{t}(v, v) \leq (S_2^{-1}S_1^{\frac{1}{2}}v, S_1^{\frac{1}{2}}v) \leq \frac{1}{s}(v, v). \quad (4.6)$$

*Then we have*

$$\frac{1}{t}(S_1^{-1}v, v) \leq (S_2^{-1}v, v) \leq \frac{1}{s}(S_1^{-1}v, v). \quad (4.7)$$

*Similarly, for any constant  $\gamma > 0$ ,*

$$\frac{1}{t}((\gamma + S_1)^{-1}v, v) \leq ((\gamma + S_2)^{-1}v, v) \leq \frac{1}{s}((\gamma + S_1)^{-1}v, v). \quad (4.8)$$

Note that  $S_i (i = 1, 2)$  are symmetric and positive definite (SPD). Let  $\underline{\lambda}_i$  be the minimal eigenvalue, and  $\bar{\lambda}_i$  be the maximal eigenvalue.

LEMMA 4.1. *If the assumption (A1) is satisfied, then*

$$s\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq t\bar{\lambda}_1, \quad s\underline{\lambda}_1 \leq \underline{\lambda}_2 \leq t\underline{\lambda}_1. \quad (4.9)$$

*Proof.* Let  $v$  be the normalized eigenvector of  $S_2$  corresponding to  $\bar{\lambda}_2$ , then by the assumption (A1),

$$\bar{\lambda}_2 = (S_2 v, v) \leq t(S_1 v, v) \leq t\bar{\lambda}_1. \quad (4.10)$$

The other three inequalities can be similarly proved.  $\square$

We also assume that the parameters are chosen to satisfy

$$\gamma_2 > \bar{\lambda}_i, \quad \text{and} \quad \gamma_1 < \underline{\lambda}_i, \quad \text{for } i = 1, 2. \quad (A2)$$

in this subsection. The operator

$$\tilde{T} = (\gamma_1 + S_1)^{-\frac{1}{2}} (\gamma_2 - S_1)^{\frac{1}{2}} (S_2 - \gamma_1) (\gamma_2 + S_2)^{-1} (\gamma_2 - S_1)^{\frac{1}{2}} (\gamma_1 + S_1)^{-\frac{1}{2}} \quad (4.11)$$

is symmetric and similar to  $-T$ .

The low bound (the minimal eigenvalue) of  $\tilde{T}$  can be estimated by the following lemma.

LEMMA 4.2. *If Assumption (A2) is satisfied, then  $\tilde{T}$  is SPD.*

*Proof.* Note that  $(S_2 - \gamma_1)(\gamma_2 + S_2)^{-1} = I - (\gamma_1 + \gamma_2)(\gamma_2 + S_2)^{-1}$ . Its minimal eigenvalue is

$$1 - (\gamma_1 + \gamma_2)(\gamma_2 + \underline{\lambda}_1)^{-1} = \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2},$$

which is positive by Assumption (A2). Similarly, the the minimal eigenvalue of  $(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}$  is  $(\gamma_2 - \bar{\lambda}_1)/(\gamma_1 + \bar{\lambda}_1)$ , which is also positive by Assumption (A2). Now for any  $v \in V_i|_\Gamma$ , we have, denoting  $\tilde{v} = (\gamma_2 - S_1)^{\frac{1}{2}} (\gamma_1 + S_1)^{-\frac{1}{2}} v$ ,

$$\begin{aligned} (\tilde{T}v, v) &= ((S_2 - \gamma_1)(\gamma_2 + S_2)^{-1} \tilde{v}, \tilde{v}) \\ &\geq \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2} (\tilde{v}, \tilde{v}) \\ &= \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2} ((\gamma_2 - S_1)(\gamma_1 + S_1)^{-1} v, v) \\ &= \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2} \cdot \frac{\gamma_2 - \bar{\lambda}_1}{\gamma_1 + \bar{\lambda}_1} (v, v). \end{aligned}$$

It means that the minimal eigenvalue of  $\tilde{T}$  is greater than  $\frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2} \frac{\gamma_2 - \bar{\lambda}_1}{\gamma_1 + \bar{\lambda}_1} > 0$ . That is to say, the symmetric operator  $\tilde{T}$  is also positive definite.  $\square$

Now we estimate the upper bound of  $\tilde{T}$ . Let us define

$$\tilde{T}_1 = (S_2 - \gamma_1)(\gamma_2 + S_2)^{-1} (\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}, \quad (4.12)$$

and

$$\tilde{T}_2 = (\gamma_1 + S_2)^{\frac{1}{2}}(\gamma_2 - S_2)^{-\frac{1}{2}}(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}(\gamma_2 - S_2)^{-\frac{1}{2}}(\gamma_1 + S_2)^{\frac{1}{2}}. \quad (4.13)$$

Both  $\tilde{T}_1$  and  $\tilde{T}_2$  are SPD. We also define

$$\bar{T}_2 = (\gamma_1 + S_2)^{\frac{1}{2}}(\gamma_2 - S_2)^{-\frac{1}{2}}(\gamma_1 + S_1)^{-\frac{1}{2}}(\gamma_2 - S_1)^{\frac{1}{2}}. \quad (4.14)$$

As the transpose of  $\bar{T}_2$  is

$$\bar{T}_2' = (\gamma_2 - S_1)^{\frac{1}{2}}(\gamma_1 + S_1)^{-\frac{1}{2}}(\gamma_2 - S_2)^{-\frac{1}{2}}(\gamma_1 + S_2)^{\frac{1}{2}}, \quad (4.15)$$

we have  $\tilde{T}_2 = \bar{T}_2 \bar{T}_2'$  and

$$\tilde{T} = \bar{T}_2 \tilde{T}_1 \bar{T}_2'. \quad (4.16)$$

Following the same analysis in the symmetric case under the assumption (A2), we know  $\|\tilde{T}_1\| < 1$ .

LEMMA 4.3. *If the assumption (A1) and (A2) hold, then, for the  $t$  defined in (A1),*

$$((\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}v, v) \leq t((\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}v, v). \quad (4.17)$$

*Proof.* For  $i = 1, 2$ , note that  $(\gamma_2 - S_i)(\gamma_1 + S_i)^{-1} = (\gamma_1 + \gamma_2)(\gamma_1 + S_i)^{-1} - I$ , then from the relationship (4.8),

$$\begin{aligned} ((\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}v, v) &= ((\gamma_1 + \gamma_2)(\gamma_1 + S_1)^{-1}v, v) - (v, v) \\ &\leq t((\gamma_1 + \gamma_2)(\gamma_1 + S_2)^{-1}v, v) - (v, v) \\ &= t((\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}v, v) + (t - 1)(v, v). \end{aligned}$$

The lemma is proved since  $t \geq 1$ .  $\square$

LEMMA 4.4. *If assumptions (A1) and (A2) hold, then*

$$\|\tilde{T}\| \leq t. \quad (4.18)$$

*Proof.* Since  $T_1$  is SPD and  $\|\tilde{T}_1\| \leq 1$ ,

$$(\tilde{T}v, v) \leq (\tilde{T}_2'v, \tilde{T}_2'v) = (\tilde{T}_2v, v).$$

Next using Lemma 4.3 and Remark 4.1,

$$(\tilde{T}_2v, v) \leq t(v, v).$$

So we prove the lemma.  $\square$

THEOREM 4.5. *If the assumptions (A1) and (A2) hold, then the convergence rate of the DD is independent of the grid size.*

*Proof.* The error equation is

$$\varepsilon_1^{n+1} = [\theta + (1 - \theta)T]\varepsilon_1^n = [\theta - (1 - \theta)(-T)]\varepsilon_1^n.$$

Since  $-T$  is similar to  $\tilde{T}$ , the spectral radius of  $-T$  equals to the one of  $\tilde{T}$ , from the low and upper bounds of  $\tilde{T}$ , the convergence rate

$$\rho = \max\{|\theta|, |\theta - (1 - \theta)t|\},$$

where  $t$  is defined in (A1). The constant  $t$  are independent of the grid size. By Lemma 2.1 and setting  $\theta = t/(2 + t)$ , then the error reduction rate is

$$\rho = \frac{t}{2 + t},$$

which is independent of the grid size  $h$ .  $\square$

**5. Numerical test.** For numerical test, we solve the Poisson equation (1.1) on the unit square  $[0, 1]$ . We use standard  $P_1$  conforming finite element on uniform criss grids. The exact solution is chosen

$$u(x, y) = 2^6(x^3 - x^4)(y - y^2).$$

We choose  $x = 1/2$  as the domain decomposition interface. First, we do the Robin-Robin iteration (Definition 1.1) for problems with different grid size. The parameters used are  $\gamma_1 = 1$ ,  $\gamma_2 = 64/h$  and  $\theta = 3/7$ . The iteration stops when  $|g_1^{m+1} - g_1^m|_{L^\infty} < 10^{-11}$ . The number of iteration, the error and the order of convergence for the finite element solution are listed in Table 5.1.

TABLE 5.1  
The errors and the iteration numbers, by Definition 1.1.

$h$	$\ u_I - u_h\ _{L^2}$	$h^n$	$ u_I - u_h _{H^1}$	$h^n$	#DD
1/4	0.0027120		0.203663		14
1/12	0.0000716	1.65	0.004456	1.74	14
1/20	0.0000098	1.93	0.000605	1.95	14
1/28	0.0000026	1.97	0.000159	1.98	14
1/36	0.0000009	1.99	0.000058	1.99	14
1/44	0.0000004	1.99	0.000026	1.99	14
1/52	0.0000002	1.99	0.000013	2.00	14

Next, we check our theoretic estimation in Theorem 3.5. In (3.23) and (3.24), if we vary  $\theta$  from  $0/7$  to  $6/7$ , we can get the following theoretic bounds:

$$\frac{7}{7}, \frac{5}{7}, \frac{3}{7}, \frac{1}{7}, \frac{5}{14}, \frac{8}{14}, \frac{11}{14}.$$

We compute the real bounds for these  $\theta$  on various meshes, and list them in Table 5.2. We note that, when  $\theta = 0/7 = 0$ , the method is reduced to the traditional Robin-Robin domain decomposition method which converges at a rate of  $1 - C\sqrt{h}$ . This can be seen in the first column of Table 5.2.

Finally, we compare the new Robin-Robin domain decomposition method with the traditional Dirichlet-Neumann domain decomposition method. We code directly the Dirichlet-Neumann domain decomposition method, defined as follows.

DEFINITION 5.1. (*The Dirichlet-Neumann domain decomposition method.*)

Given  $w^0 (= 0)$  on  $\Gamma$ , find  $u^m \in V_1$ ,  $u^m|_\Gamma = w^m$ :

$$a_1(u^m, v) = (f, v)_{\Omega_1} \quad \forall v \in V_1 \cap H_0^1(\Omega_1).$$

TABLE 5.2  
The reduction rate with different  $\theta$  in Definition 1.1.

$h \setminus \theta$	0	1/7	2/7	3/7	4/7	5/7	6/7
1/4	0.764	0.512	0.260	0.096	0.322	0.548	0.774
1/12	0.865	0.598	0.332	0.115	0.336	0.557	0.779
1/20	0.894	0.624	0.353	0.116	0.337	0.558	0.779
1/28	0.910	0.637	0.364	0.116	0.337	0.558	0.779
1/36	0.920	0.646	0.371	0.116	0.337	0.558	0.779
1/44	0.927	0.652	0.377	0.116	0.337	0.558	0.779
1/72	0.943	0.665	0.388	0.116	0.337	0.558	0.779
1/288	0.971	0.689	0.408	0.126	0.337	0.558	0.779
1/1152	0.985	0.702	0.418	0.134	0.337	0.558	0.779
Corollary 3.6	1.000	0.714	0.428	0.143	0.357	0.571	0.786

Find  $\tilde{w}^{m+1} \in V_2$ :

$$a_2(\tilde{w}^{m+1}, v) = (f, v)_{\Omega_2} - a_1(u^m, v) \quad \forall v \in V_2,$$

where  $v$  is extended into  $\Omega_1$  with 0 nodal values. Then

$$w^{m+1} = \theta w^m + (1 - \theta) \tilde{w}^{m+1}.$$

In Table 3, we list the number of Dirichlet-Neumann domain decomposition iterations for the above test problem, for various  $\theta$ . It seems that no matter how to choose  $\theta$ , the Dirichlet-Neumann domain decomposition method (21 iterations) is worse than the new Robin-Robin domain decomposition method (14 iterations).

TABLE 5.3  
The iteration number for Dirichlet-Neumann DD (Definition 5.1.)

$h \setminus \theta$	0	0.25	0.35	0.4	0.45	0.5	0.55	0.75
1/4	88	24	22	25	29	33	38	78
1/12	237	34	21	23	26	30	35	71
1/20	392	37	22	22	25	29	33	68
1/28	548	38	23	21	24	28	32	66
1/36	705	39	23	21	24	27	31	64

## 6. Appendix.

We give a proof to Lemma 3.3.

*Proof.* By (3.17), with the notations defined in (3.15) and (3.16),

$$\begin{aligned}
3a_1 - b_1 + 1 &= \frac{\theta_1}{n+1} \sum_{i=1}^n \frac{\sin^2(i\pi/(n+1))}{4 \sin^4 \frac{\pi}{4n} + 4 \sin^2 \frac{\pi}{4n} + \sin^2(i\pi/(n+1))} \\
&\approx \theta_1 \int_0^1 g_1(x) dx,
\end{aligned} \tag{6.1}$$

where  $\theta_1$  is defined in (3.16) and

$$g_1(x) = \frac{\sin^2(\pi x)}{4 \sin^4 \frac{\pi}{4n} + 4 \sin^2 \frac{\pi}{4n} + \sin^2(\pi x)}. \tag{6.2}$$

We note that the above sum is the trapezoidal approximation of the integral. In fact, the sum is less than the integral for all  $n$ , to be show later. First, we show

$$\theta_1 \int_0^1 g_1(x) dx < 1 - \frac{h^2}{2}.$$

The integral can be computed exactly. It leads us to

$$\theta_1(1 - 1/\sqrt{1 + 1/a}) < 1 - \frac{h^2}{2},$$

where  $(\xi_1$  is defined in (3.15))

$$a = 4\xi_1^4 + 4\xi_1^2. \quad (6.3)$$

By (3.16) and (6.3), we need to prove

$$\frac{1 + 3h + 2(1 - h)\xi_1}{2\xi_1 + 1 + \sqrt{4\xi_1^2 + 4\xi_1}} < 1 - \frac{h^2}{2}.$$

Or

$$(3 + h/2 - 2\xi_1 + \xi_1 h)^2 h^2 < (1 - h^2/2)^2 (4\xi_1^2 + 4\xi_1).$$

We apply the inequalities  $x^2 - x^4/3 < \sin^2 x < x^2$  to  $\xi_1 = \sin^2(\pi/(4n))$ . The above inequality is implied by

$$\begin{aligned} & \left( 3 + \frac{h}{2} - 2\left(\frac{\pi}{4n}\right)^2 - \frac{2}{3}\left(\frac{\pi}{4n}\right)^4 + \left(\frac{\pi}{4n}\right)^2 h \right)^2 h^2 \\ & < (1 - h^2/2)^2 4 \left( \left(\frac{\pi}{4n}\right)^2 - \frac{1}{3}\left(\frac{\pi}{4n}\right)^4 \right) \left( 1 + \left(\frac{\pi}{4n}\right)^2 - \frac{1}{3}\left(\frac{\pi}{4n}\right)^4 \right) \end{aligned}$$

Expanding all terms in the inequality, with  $h = 1/(2n)$ , it is rewritten, after multiplying both sides by  $n^{10}$ , as

$$\begin{aligned} & n^8 \frac{\pi^2 - 9}{4} - n^7 \frac{3}{8} + n^6 \frac{\pi^4 + 12\pi^2 - 3/2}{96} - n^5 \frac{\pi^2}{32} - n^4 \frac{\pi^6 + \pi^4}{1536} \\ & + n^3 \frac{5\pi^4}{3072} - n^1 \frac{\pi^6}{24576} + n^2 \frac{\pi^8 + 48\pi^6 - 12\pi^4}{147456} - n^0 \frac{\pi^8 + 3\pi^6}{294912} > 0. \end{aligned}$$

When  $n = 1$ , the above inequality is  $0.57... > 0$ . For  $n > 1$ , combining each negative term with some positive term, and replacing some  $n$  by 2 in those positive coefficient terms, we get

$$\begin{aligned} & n^7 \frac{\pi^2 - 9 - 3/4}{2} + n^5 \frac{(21/3)\pi^2 - 3/2}{48} + n^4 \frac{\pi^4 - (\pi^6 + \pi^4)/64}{24} \\ & + n^1 \frac{5\pi^4 - \pi^6/32}{768} + n^0 \frac{7\pi^8/8 + 381\pi^6/8 - 12\pi^4}{36864} > 0. \end{aligned} \quad (6.4)$$

Now, each coefficient is positive. Thus, the lemma is proved, provided (6.1) is an inequality, which is shown below.

The function  $g_1$  is an increasing function:

$$g_1'(x) = \frac{a\pi \sin(2\pi x)}{(a + \sin^2(\pi x))^2} > 0 \quad \forall x \in (0, 1/2).$$

For concaveness, we compute

$$\begin{aligned} g_1''(x) &= (a + \sin^2(\pi x))^{-3} \cdot 2\pi^2 \cdot g(x), \quad \text{where} \\ g(x) &= (a + \sin^2(\pi x)) \cos(2\pi x) - \sin^2(2\pi x). \end{aligned}$$

The difficult point is that  $g_1''(x)$  changes its sign near  $x = 0$ . We have to exclude a small interval near  $x = 0$ . We will show next that  $g_1''(x) < 0$  for  $x \in (1/(2n+2), 1/2)$ , and for all  $n \geq 1$ . This is to be done in two steps. The first step is to check the left end-point that  $g(1/(2n+2)) \leq 0$  for every  $n$ . The second step is to show  $g'(x) < 0$  for  $x \in (1/(2n+2), 1/2)$ .

By Taylor expansion, for all  $x > 0$ ,

$$\begin{aligned} x - \frac{x^3}{6} &< \sin x < x, \\ 1 - \frac{x^2}{2} &< \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \\ \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x > x^2 - \frac{x^4}{3}. \end{aligned}$$

At the end point,  $x_0 = 1/(2n+2)$ , when  $n = 1$ ,  $g(x_0) = -1 < 0$ . When  $n > 1$ , by (6.3), for  $x_0 = 1/(2n+2)$ ,

$$g(x_0) < \frac{x_0^2}{4} \left( \frac{(n+1)^4}{(4n)^4} + \frac{(n+1)^2}{(4n)^2} + 1 \right) \cdot \left( 1 - \frac{x_0^2}{2} + \frac{x_0^4}{24} \right) - x_0^2 \left( 1 - \frac{x_0^2}{3} \right).$$

When  $n > 1$ ,  $4n > 2n+2$ , and

$$\frac{1}{4} \left( \frac{(n+1)^4}{(4n)^4} + \frac{(n+1)^2}{(4n)^2} + 1 \right) < 1.$$

Also when  $n > 1$ ,  $x_0 \leq 1/6$  and

$$\left( 1 - \frac{x_0^2}{2} + \frac{x_0^4}{24} \right) < 1 - \frac{x_0^2}{2} + \frac{x_0^2}{6} = 1 - \frac{x_0^2}{3}.$$

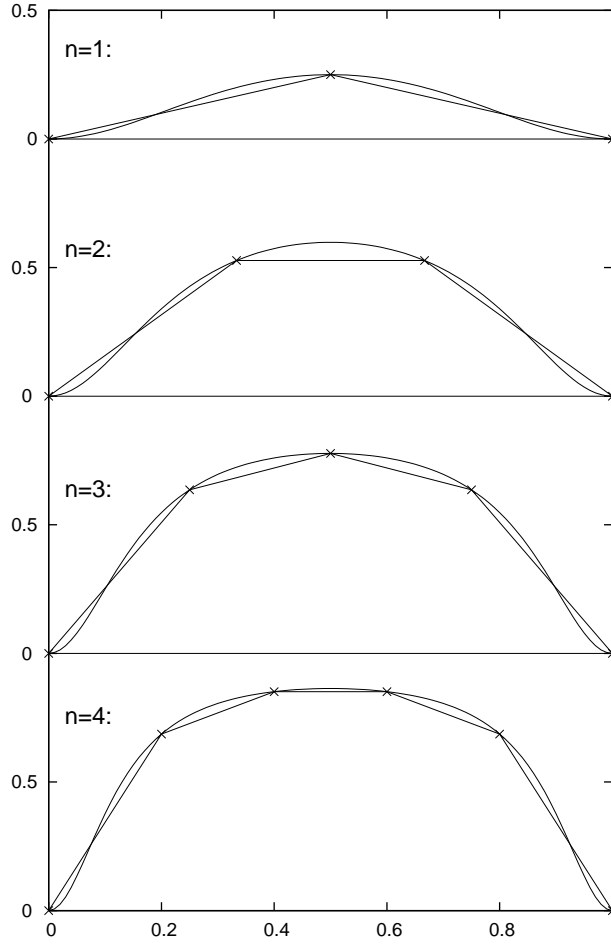
Thus  $g(x_0) < 0$  for all  $n$ . Next

$$\begin{aligned} g'(x) &= -\frac{3}{2}\pi \sin(4\pi x) - (a + \sin^2(\pi x))2\pi \sin(2\pi x) \\ &= -\left(\frac{1}{2} + \cos(2\pi x) + a\right)2\pi \sin(2\pi x). \end{aligned}$$

This shows  $g'(x) < 0$  at least for  $x \leq 1/3$ . Then  $g(x) < 0$  for  $x \in (1/(2n+2), 1/3)$ . But for  $x \in (1/4, 1/2)$ , both terms in  $g(x)$  are negative. We concluded that  $g(x) < 0$ , and consequently  $g_1''(x) < 0$  for  $x \in (1/(2n+2), 1/2)$ .

When using the trapezoidal rule in (6.1), because  $g_1(x)$  is an increasing, concave down function on  $(1/(2n+2), 1/2)$ , the area of each trapezoid is less than the area under the graph  $g_1(x)$ , cf. Figure 6.1, except the first and the last trapezoid. In fact, the area of the first (and last, by symmetry) trapezoid is also smaller than the integral on the interval there:

$$\begin{aligned} &|\Delta| - \int_0^{1/(n+1)} g_1(x) dx \\ &= \frac{1}{2} \cdot \frac{1}{n+1} g_1\left(\frac{1}{n+1}\right) - \frac{1}{n+1} + \frac{\arctan(\sqrt{1+1/a} \tan(\pi/(n+1)))}{\pi \sqrt{1+1/a}} \\ &< 0. \end{aligned}$$

FIG. 6.1. Trapezoidal approximation of integral  $\int g_1$ , cf. (6.1).

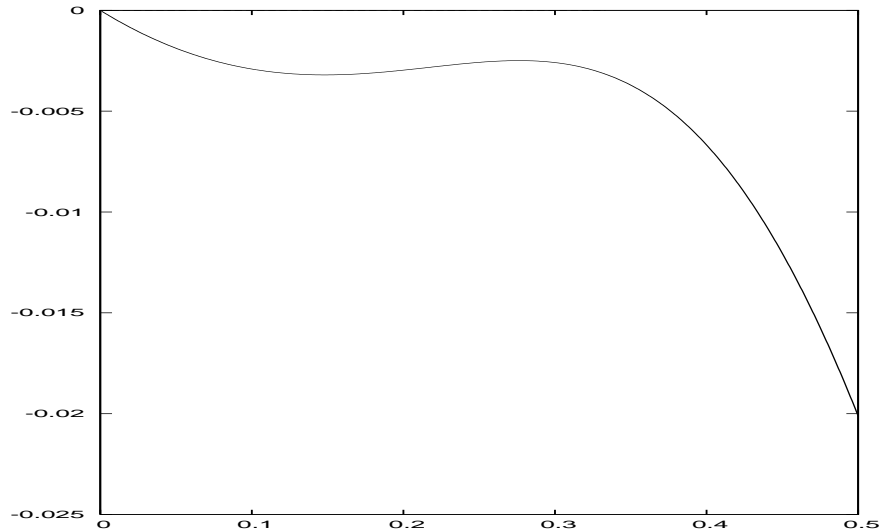
But for our need, we only prove

$$\theta_1(|\Delta| - \int_0^{1/(n+1)} g_1(x) dx) < \frac{1}{8}h^2.$$

Note that  $n$  is fixed inside the integral, but it changes together with the end point of the integral. We introduce a new function, with  $n = 1/x - 1$ ,

$$g_2(x) = \theta_1\left(\frac{x}{2}g_1(x) - x + \frac{\arctan(\sqrt{1+1/a}\tan(\pi x))}{\pi\sqrt{1+1/a}}\right) - \frac{x^2}{32(1-x)^2}. \quad (6.5)$$

The graph of  $g_2$  is shown in Figure (6.2) that  $g_2(x) < 0$  for all  $x \leq 1/2$ , i.e., for all  $n$ . Due to a lack of Taylor expansion of  $\arctan$ , we would introduce another function and differentiate the new function to get rid of  $\arctan$  completely. Then, we do a Taylor expansion as we did above for  $g(x)$ . After we group terms as in (6.4) to get all negative terms, we would show  $g_2(x) < 0$ . It is elementary, but very long. We omit the details.  $\square$

FIG. 6.2. Graphing  $g_2(x)$ , cf. (6.5).

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