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A family of 3D continuously differentiable finite elements on tetrahedral grids

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Abstract

A family of continuously differentiable piecewise polynomials of degree 9 and higher, on general tetrahedral grids, is constructed, by simplifying and extending the P_9 element of Ženišek. A mathematical justification and numerical tests are presented.

The current computing power is still limited for the computation with 3D C_1 finite elements in general. The construction here mainly serves the purposes of understanding and ensuring the approximation properties of C_1 finite elements spaces on tetrahedral grids. In particular, this construction indicates that the 3D divergence-free C_0-P_k elements have the full order of approximation for any degree $k \geqslant 8$.

MSC: 65M60; 65N30; 73V05

Keywords: Continuously differentiable finite element; High order element; 3D biharmonic equation; Stokes equations; Divergence-free element

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1. Introduction

The construction for continuously-differentiable (C_1) finite elements are relatively difficult in two and three space dimensions. Most such C_1-element designs were done in 1970s and earlier (cf. Ciarlet [6]). For example, we have the following well known C_1 elements, the Bogner–Fox–Schmit Q_3 rectangle (1965 [6]), the Argyris P_5-triangle (1968 [1]), the Bell reduced P_5-triangle (1969 [2]), the Morgan–Scott P_k-triangles ($k \geqslant 5$) (1975 [10]), the Hsieh–Clough–Tocher P_3-macrotriangles (1965 [6]), the reduced Hsieh–Clough–Tocher P_3-macrotriangles (1976 [11]), the Douglas–Dupont–Percell–Scott P_k-triangles (1979 [7]), the Powell–Sabin P_2-triangles (1977 [12]), the Fraeijs de Veubeke–Sander P_3 quadrilateral and its reduced version (1964 [13,8]), and the Ženišek P_9 tetrahedron (1973 [17]), It seems the only C_1 element on tetrahedral grids is constructed by Ženišek in [17].

The C_1 finite elements are not used as popularly as C_0 or weaker (nonconforming) finite elements. In general, the coding and computation of C_1 finite elements is relatively costing. In addition to the complexity in construction, this prevents probably further development in C_1 finite elements in last 30 years. Another reason might be that the current computer power is too small to compute C_1 finite elements in 3D (see Section 4). In this work, we modify (and simplify) the Ženišek P_9 tetrahedron [17] to define another C_1-P_9 finite element on tetrahedron. We further extend the degree of polynomial of the new C_1 element to all polynomials of degree $k \geqslant 9$. We may have to admit that such

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high-degree polynomial finite elements in 3D may not be of much practical use in the near future. Nevertheless, this work is of somewhat interest in mathematics as a gap in C_1 families of finite elements is filled, since currently only C_0 finite element families are listed in textbooks [6,5].

A main interest in constructing C_1 elements on tetrahedral grids is to understand the approximation properties of C_0 finite element spaces which contain the **curl** or the **grad** of some C_1-element spaces on the same grids. For example, Morgan and Scott constructed a basis for the C_1-P_k elements on triangular grids, for all $k \geqslant 5$ [10]. Subsequently, Scott and Vogelius proved the stability and the optimal order of approximation, for C_0-P_k divergence-free finite elements, i.e., the velocity is approximated by continuous piecewise polynomials of degree k and the pressure is approximated by discontinuous piecewise polynomials of degree $(k-1)$ for the Stokes equations, on triangular grids, for all $k \geqslant 4$ [14,15]. The 3D version of the question, what the smallest k is such that the C_0-P_l divergence-free finite element is stable for all $l \geqslant k$, was raised explicitly there by Scott and Vogelius. The problem remains open today. By the construction of C_1 finite elements of polynomial degree 9 and higher in this work, we will be able to solve partially this open problem of Scott and Vogelius that the magic number k in 3D is 8 or less. The work is to be reported elsewhere. We note that on special grids, the 3D Scott–Vogelius number k may be lower. For example, $k = 3$ for the Hsieh–Clough–Tocher macro-tetrahedral grids, cf. [18].

The rest of this manuscript is organized as follows. In Section 2, the C_1-P_k finite element family is defined. In Section 3, the finite element is shown to be continuously differentiable, and to have the optimal order of approximation property, both in interpolation and in Galerkin projection. Also in Section 3, a C_0-P_k family of Hermite-type finite elements is constructed, helping the analysis on the C_1-P_k element. In Section 4, some numerical tests are presented in supporting the analysis.

2. Construction of a C_1-P_k finite element

In this section, we define a family of finite elements. We then show the elements are well defined as the associated set of linear functions is uni-solvent.

A finite element is defined by a triple (P_K, K, Σ_K), where K is an element, P_K is a finite dimensional space of functions defined on K, and Σ is a dual basis for the linear functionals on P_K, cf. [6]. In this work, K is a tetrahedron, P_K is the space of polynomials P_k, $k \geqslant 9$. We need to define Σ_K. This is done in Definition 2.1, where $\Sigma_{\hat{K}}$ is defined for the reference element $(P_{\hat{K}}, \hat{K}, \Sigma_{\hat{K}})$. To illustrate the nodal value set of $\Sigma_{\hat{K}}$, we plot them in Figs. 1–4, where only the nodal values/derivatives on the front face-triangle are shown.

We note that, when $k = 9$, the construction of the C_1 element in Definition 2.1 is similar to the P_9 element constructed by Ženišek in 1973 [17]. However, we simplify Ženišek's P_9 construction by replacing the 2nd-order mixed derivatives at internal points of face-triangles by 1st-order normal derivatives at appropriate face-points, and by replacing the derivatives at internal points of the tetrahedron by the function values at some points.

Definition 2.1. Let the reference unit tetrahedron $\hat{K}$ be defined by

$$\hat{K} = \{(x, y, z) \mid 0 \leqslant x,\ 0 \leqslant y,\ 0 \leqslant z,\ (x + y + z) \leqslant 1\}.$$

For each edge on $\hat{K}$, denoted by $\mathbf{n}$ and $\mathbf{m}$, we choose arbitrarily two unit vectors orthogonal to the edge. Also $\mathbf{n}$ denotes a normal vector for each face triangle of $\hat{K}$. The set of linear functionals, $\Sigma_{\hat{K}} = \{f\}$, is defined by the following nodal values and derivatives, $f(u)$, for any $u \in P_k(\hat{K})$.

1. (Fig. 1) The nodal values and derivatives up to order 4 at 4 vertices p_i: $\{D_j u(p_i),\ j = 0, 1, 2, 3, 4\}$. Here D_j denotes an order j derivative. For example

   $$D_1 u(p_1) = \{u_x(p_1), u_y(p_1), u_z(p_1)\}.$$

   The total number of functionals at the 4 vertices is

   $$d_v = 4\left(\frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \frac{3\cdot 4}{2} + \frac{4\cdot 5}{2} + \frac{5\cdot 6}{2}\right) = 140. \tag{2.1}$$

2. (Fig. 2) The nodal values $\{u(q_i)\}$ at $(k-9)$ equally-distributed internal points, 2 first order normal derivatives $\{u_{\mathbf{n}}(q_i), u_{\mathbf{m}}(q_i)\}$ at $(k-8)$ equally-distributed internal points, and 3 second order normal derivatives

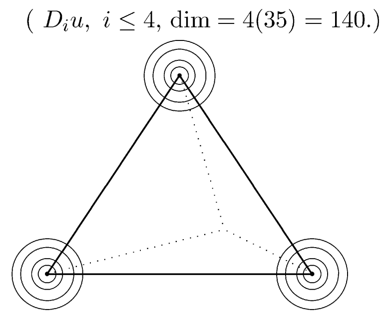

Fig. 1. Degrees of freedom at 4 vertices, for C_1-P_k polynomials.

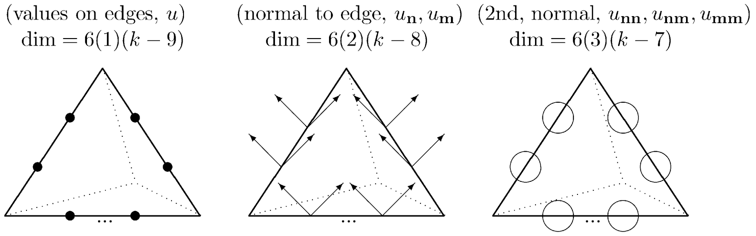

Fig. 2. Degrees of freedom at 6 edges, for C_1-P_k polynomials.

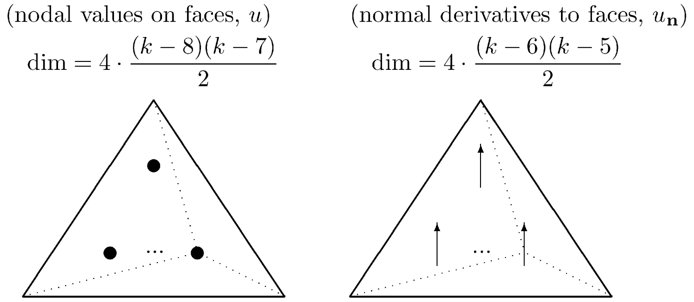

Fig. 3. Degrees of freedom on 4 face triangles, for C_1-P_k polynomials.

$\{u_{\mathbf{nn}}(q_i), u_{\mathbf{nm}}(q_i), u_{\mathbf{mm}}(q_i)\}$ at $(k-7)$ equally-distributed internal points, on each of the 6 edges. The total number of functionals on the 6 edges is

$$d_e = 6\big((k-9) + 2(k-8) + 3(k-7)\big) = 36k - 276. \tag{2.2}$$

3. (Fig. 3) The nodal values $\{u(r_i)\}$ at the standardly-distributed internal points for a 2D C_0-P_{k-9} element (cf. [6]), and first order normal derivatives $\{u_{\mathbf{n}}(r_i)\}$ at the standardly-distributed internal points for a 2D C_0-P_{k-7} element, on each of the 4 face triangles. The total number of functionals on the 4 triangles is

$$d_f = 4\left(\frac{(k-8)(k-7)}{2} + \frac{(k-6)(k-5)}{2}\right) = 4k^2 - 52k + 172. \tag{2.3}$$

4. (Fig. 4) The nodal values $\{u(s_i)\}$ at the standardly-distributed internal points for a 3D C_0-P_{k-8} element, inside the tetrahedron. The total number of functionals inside the tetrahedron is

$$d_t = \frac{(k-7)(k-6)(k-5)}{6} = \frac{1}{6}k^3 - 3k^2 + \frac{107}{6}k - 35. \tag{2.4}$$

We first show that the number of functionals in Σ is the same as the dimension of the space of 3D P_k polynomials. For the space of 3D P_k polynomials, the dimension is

$$\dim P_k = \frac{(k+1)(k+2)(k+3)}{6} = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1. \tag{2.5}$$

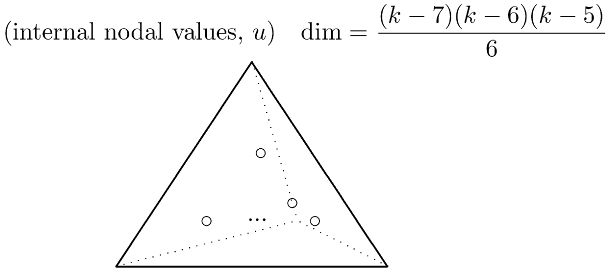

Fig. 4. Degrees of freedom inside a tetrahedron, for C_1-P_k polynomials.

Let us add the numbers of linear functionals in the four parts in Definition 2.1:

$$\begin{aligned} d_v + d_e + d_f + d_t &= 140 + 36k - 276 + 4k^2 - 52k + 172 + \frac{1}{6}k^3 - 3k^2 + \frac{107}{6}k - 35 \\ &= \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1 = \dim P_k. \end{aligned} \tag{2.6}$$

Theorem 2.1. *The set of functional $\Sigma_{\hat{K}}$ is a dual basis for P_k, i.e., for any $v \in P_k$ such that $f(v) = 0$ for all $f \in \Sigma_{\hat{K}}$, then $v = 0$.*

Proof. Let $v \in P_{\hat{K}} = P_k$. The analysis consists of the following 8 steps.

1. Restricting v on a face triangle $z = 0$ of the reference tetrahedra $\hat{K}$, let

$$v_1(x, y) = v(x, y, 0). \tag{2.7}$$

v_1 is a polynomial of degree k in (x, y). Further, let $v_2 = v_1(x, 0)$ be the restriction of v_1 on an edge $y = 0$ of the above triangle. $v_2(x)$ is a polynomial of degree k in x. Since $D_i(p_j) = 0$ at the four vertices $\{p_j\}$ of $\hat{K}$, by Definition 2.1.1, it follows that

$$v_2(0) = v_2'(0) = v_2''(0) = v_2'''(0) = v_2^{(4)}(0) = v_2(1) = v_2'(1) = v_2''(1) = v_2'''(1) = v_2^{(4)}(1) = 0.$$

So $v_2(x) = x^5(1-x)^5 p_{k-10}(x)$ where $p_{k-10}(x)$ is a degree $(k-10)$ polynomial. Since $v(q_i) = 0$ on $(k-9)$ internal points q_i of each edge of $\hat{K}$ by Definition 2.1.2, we have

$$0 = v_2\left(\frac{i}{k-8}\right) = p_{k-10}\left(\frac{i}{k-8}\right), \quad 1 \leqslant i \leqslant (k-9). \tag{2.8}$$

Since p_{k-10} is a polynomial of degree $(k-10)$ and it has at most $(k-10)$ roots, we conclude that $p_{k-10} \equiv 0$ as (2.8) says p_{k-10} has $(k-9)$ roots, and consequently $v_2(x) \equiv 0$.

2. Since $v_1(x, y)$ is identically zero on the edge $y = 0$, we can factor out this factor, i.e., for some polynomial $v_3(x, y)$ of degree $(k-1)$,

$$v_1(x, y) = y v_3(x, y). \tag{2.9}$$

Next, restricting $e\frac{\partial}{\partial y}v$ on the edge $y = 0$, we obtain

$$v_4(x) = v_3(x, 0) = \frac{\partial}{\partial y} v(x, 0, 0)$$

by (2.9). Hence $v_4(x)$ is a polynomial of degree $(k-1)$. Because at the four vertices, by Definition 2.1.1,

$$\frac{\partial^{i+1} v}{\partial y \partial^i x}(p_j) = 0, \quad 0 \leqslant i \leqslant 3,$$

it follows that

$$\frac{d^i v_4}{d^i x}(0) = \frac{d^i v_4}{d^i x}(1) = 0, \quad 0 \leqslant i \leqslant 3.$$

We can rewrite $v_4(x)$ as

$$v_4(x) = x^4(1-x)^4 p_{k-9}(x)$$

where $p_{k-9}(x)$ is a degree $(k-9)$ polynomial. $p_{k-9}(x)$ has $(k-8)$ zero-points. This is because at $(k-8)$ internal points of the edge $y=0$ (on the face triangle $z=0$), by Definition 2.1.2, the normal derivatives of v are zero $v_{\mathbf{n}}(q_i) = v_{\mathbf{m}}(q_i) = 0$, i.e.,

$$p_{k-9}\left(\frac{i}{k-7}\right) = \frac{\partial v}{\partial y}\left(\frac{i}{k-7}, 0, 0\right) = 0, \quad 1 \leqslant i \leqslant (k-8).$$

We note that as v is identically zero on the edge and $v_{\mathbf{t}}$ is zero on the edge, where $\mathbf{t}$ is a tangent vector to the edge. Since $v_{\mathbf{n}}$ and $v_{\mathbf{m}}$ are zero too at the nodal point, the gradient vector is 0, and so is $\partial v/\partial y$ there. Of course, if one of $\mathbf{n}$ or $\mathbf{m}$ happens in the direction of y, the above argument is not needed. Therefore $p_{k-9}(x) \equiv 0$ and $v_4(x) \equiv 0$. By (2.9), we can factor out another y from v_1:

$$v_1(x, y) = y^2 p_{k-2}(x, y).$$

By symmetry, we can factor out similarly at all three edges of triangle $z=0$:

$$v_1(x, y) = y^2x^2(1-x-y)^2 v_5(x, y) = v(x, y, 0), \tag{2.10}$$

where $v_5(x, y)$ is a polynomial of degree $k-6$.

3. For the degree $(k-6)$ polynomial $v_5(x, y)$ in (2.10), by Definition 2.1.1, we have

$$\frac{\partial^{i+j}}{\partial^j y \partial^i x} v_5(0, 0) = \frac{\partial^{i+j+m+n}}{\partial^{j+n} y \partial^{i+m} x} v(0, 0, 0) = 0, \quad i+j = 0, 1, 2,$$

i.e., $v_5(x, y)$ has all derivatives up to order 2 vanished at 3 vertices of the triangle $z=0$. Since the second order mixed-normal derivatives of v, $\{v_{\mathbf{mm}}, v_{\mathbf{mn}}, v_{\mathbf{nn}}\}$, are zero at $(k-7)$ internal points of the edge $y=0$, by Definition 2.1.2, we have

$$v_5\left(\frac{i}{k-6}, 0\right) = C\frac{\partial^2}{\partial y^2} v\left(\frac{i}{k-6}, 0, 0\right) = 0, \quad 1 \leqslant i \leqslant (k-7).$$

Here we note again that the three 2nd-order normal derivatives are zero, which ensures all 2nd-order normal derivative zero, in particular, $\partial^2 v/\partial y^2 = 0$. When restricting v_5 on the edge $y=0$, it has $(2+(k-7))$ zero points, one more than its polynomial degree. Therefore $v_5(x, y) = y p_{k-7}(x, y)$ for some polynomial p_{k-7} of degree $(k-7)$. By symmetry, we conclude that we can factor v_5 further as $v_5(x, y) = xy(1-x-y) p_{k-9}(x, y)$. Consequently we write that

$$v_1(x, y) = y^3x^3(1-x-y)^3 v_6(x, y) = v(x, y, 0), \tag{2.11}$$

where $v_6(x, y)$ is a polynomial of degree $(k-9)$.

4. By Definition 2.1.3, v has $(k-8)(k-7)/2$ zero points (distributed by the rule for the standard C_0-P_{k-9} finite element interpolation points) interior to each face. We get that, by (2.11),

$$v_6\left(\frac{i}{k-6}, \frac{j}{k-6}\right) = Cv\left(\frac{i}{k-6}, \frac{j}{k-6}, 0\right) = 0, \quad i > 0, j > 0, (i+j) < (k-6).$$

By the standard 2D Lagrange finite element theory, cf. [6], it follows that $v_6(x, y) \equiv 0$, as it is a degree $(k-9)$ polynomial. Therefore, by (2.7) and (2.11),

$$v_1(x, y) = v(x, y, 0) \equiv 0,$$

and we can factor out a factor z from v:

$$v(x, y, z) = z v_7(x, y, z), \tag{2.12}$$

where $v_7(x, y, z)$ is a polynomial of degree $(k-1)$.

5. We next consider the restriction, on $z=0$, of the partial derivative v_z, by (2.12),

$$\frac{\partial}{\partial z} v(x, y, 0) = v_7(x, y, z).$$

As all derivatives up to order 4 are zero for v at the vertices, by Definition 2.1.1, we have

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} v_6(p_i) = 0, \quad m+n = 0, 1, 2, 3, \tag{2.13}$$

where p_i is one of the three vertices of face triangle $z = 0$. On the edge $y = 0$, as all normal derivatives of v, $v_{\mathbf{n}} = v_{\mathbf{m}} = 0$, are zero at $(k-8)$ internal points,

$$v_7\left(\frac{i}{k-7}, 0, 0\right) = \frac{\partial}{\partial z} v\left(\frac{i}{k-7}, 0, 0\right) = 0, \quad 1 \leqslant i \leqslant (k-8).$$

Since the degree $(k-1)$ polynomial $v_7(x, 0, 0)$ has $8 + (k-8) = k$ zero points on edge $y = 0$, it must be identically 0. This would allow us to factor y from $v_7(x, y, 0)$:

$$v_7(x, y, 0) = y v_8(x, y) = \frac{\partial}{\partial z} v(x, y, 0), \tag{2.14}$$

where v_8 is a degree $(k-2)$ polynomial in x and y.

6. As all three mixed, 2nd-order, normal derivatives of v on edge $y = 0$ are zero at $(k-7)$ internal points, we get from Definition 2.1.2 and (2.14) that

$$v_8\left(\frac{i}{k-6}, 0\right) = \frac{\partial^2}{\partial y \partial z} v\left(\frac{i}{k-6}, 0, 0\right) = 0, \quad 1 \leqslant i \leqslant (k-7).$$

In addition, by (2.13), we have also (at vertices)

$$D_i v_8(0, 0) = D_i v_8(1, 0) = 0, \quad i = 0, 1, 2.$$

Therefore the degree $(k-2)$ polynomial $v_8(x, 0)$ has $(k-7) + 6 = (k-1)$ zero points on edge $y = 0$, and it must be identically zero. So we can factor out another factor y from $v_8(x, y)$:

$$v_7(x, y, 0) = y v_8(x, y) = y^2 p_{k-3}(x, y)$$

for some polynomial $p_{k-3}(x, y)$ of degree $(k-3)$. By symmetry,

$$v_7(x, y, 0) = x^2 y^2 (1 - x - y)^2 v_9(x, y), \tag{2.15}$$

where $v_9(x, y)$ is a degree $(k-7)$ polynomial.

7. Since $\partial v / \partial z$ has $(k-6)(k-5)/2$ internal zero points inside the bottom triangle $z = 0$, by Definition 2.1.3, we get

$$v_9\left(\frac{i}{k-4}, \frac{j}{k-4}\right) = C \frac{\partial v}{\partial z}\left(\frac{i}{k-4}, \frac{j}{k-4}, 0\right) = 0, \quad i > 0, j > 0, (i+j) < (k-4).$$

As $v_9(x, y)$ is a degree $(k-7)$ polynomial, having $(k-6)(k-5)/2$ zero-points at the standard finite element interpolation points inside the bottom face triangle $z = 0$, by [6], it is identically zero. By (2.15), $v_7(x, y, z)$ is identically zero on the face $z = 0$ and it can factor out a factor z. By (2.12),

$$v(x, y, z) = z^2 p_{k-2}(x, y, z),$$

for some degree $(k-2)$ polynomial $p_{k-2}(x, y, z)$. By symmetry, we can do such factoring on each face of tetrahedron $\hat{K}$ to get

$$v(x, y, z) = x^2 y^2 z^2 (1 - x - y - z)^2 v_{10}(x, y, z), \tag{2.16}$$

for some degree $(k-8)$ polynomial $v_{10}(x, y, z)$.

8. Finally, we determine $v_{10}(x, y, z)$ by the internal zero points of v. By Definition 2.1.4, v has $(k-7)(k-6)(k-5)/6$ internal zeros inside $\hat{K}$, i.e.,

$$v_{10}\left(\frac{i}{k-5}, \frac{j}{k-5}, \frac{l}{k-5}\right) = C v\left(\frac{i}{k-5}, \frac{j}{k-5}, \frac{l}{k-5}\right) = 0, \quad i > 0, j > 0, l > 0, (i+j+l) < (k-5).$$

By the standard Lagrange finite element theory ([6]), the degree $(k-8)$ polynomial $v_{10}(x, y, z)$ must be identically zero. By (2.16),

$$v(x, y, z) \equiv 0.$$

The proof is completed. □

Corollary 2.1. *Let K be a non-degenerate tetrahedron. The following set of functional Σ_K is a dual basis for P_k,*

$\Sigma_K = \{f \mid$ (a) *derivatives of order 0 to* 4 *at* 4 *vertices*,

(b) $(k-9)$ *values inside each of* 6 *edges,*

(c) 2 *normal derivatives at* $(k-8)$ *nodes inside each of* 6 *edges,*

(d) 3 2*nd-order normal derivatives at* $(k-7)$ *nodes inside each of* 6 *edges*,

(e) *nodal values at* $(k-8)(k-7)/2$ *evenly distributed points inside each of* 4 *face triangles,*

(f) $(k-6)(k-5)/2$ *normal derivatives at evenly distributed points inside each of* 4 *face triangles,*

(g) $(k-7)(k-6)(k-5)/6$ *values at evenly distributed points inside a tetrahedron*$\}$.

Proof. The proof for Theorem 2.1 remains the same, where we did not use any special property of the reference $\hat{K}$ except the simpler notations. □

We note that the nodal points inside edges, inside face-triangles, and inside tetrahedra, do not need to be evenly distributed in (2.17), as long as they satisfy certain uni-solvent conditions. But they have to be the same on the neighboring tetrahedra. For simplicity and practicality, one would assume them evenly distributed as specified by standard finite elements (cf. [6]).

3. Continuously differentiability and approximation

In this section, we define the C_1-P_k finite element space on tetrahedral grids, and show the continuously differentiability. We then define a C_0-P_k affine family of finite elements, similar to (2.17). Via the element, we will show the C_1-P_k finite element space is an almost-affine family (cf. [6]), i.e., having the optimal order of approximation. We then apply the finite element to solve a model of biharmonic equation in 3D.

Let Ω be a 3D polyhedral domain, subdivided by a family of quasiuniform tetrahedral grids $\Omega_h = \{K\}$, i.e.,

$$\max_{K\in\Omega_h} h_K \leqslant Ch, \quad \frac{\max_{K\in\Omega_h} h_K}{\min_{K\in\Omega_h} h_K} \leqslant C, \quad \max_{K\in\Omega_h} \frac{h_K}{\rho_K} \leqslant C, \tag{3.1}$$

where h_K is the diameter of tetrahedron K and ρ_K is the diameter of the largest ball inscribed in K. On each edge of $K \in \Omega_h$, we select two normal vectors arbitrarily, but the same for all K sharing the edge. We define the C_1-P_k finite element space, on a grid Ω_h, by

$$V_k = \bigl\{v \mid v|_K \in P_k\ \forall K \in \Omega_h,\ f(v|_{K_1}) = f(v|_{K_2})\ \forall f \in (\Sigma_{K_1} \cap \Sigma_{K_2})\bigr\}. \tag{3.2}$$

Theorem 3.1. *Let Ω_h be a quasiuniform grid* (3.1) *on Ω. The finite element space V_k defined by* (2.17) *and* (3.2) *is continuously differentiable*:

$$V_k \subset C_1(\Omega).$$

Proof. Given a $v \in V_k$. On each element K, it is a P_k polynomial, and continuously differentiable. We are left to show, on each inter-element triangle,

$$T = \overline{K_1} \cap \overline{K_2} \subset \bigl\{(x,y,z) \mid L(x,y,z) := ax + by + cz - d = 0\bigr\}, \tag{3.3}$$

v is continuously differentiable. Let

$$v_0(x,y,z) = v(x,y,z) - v_1(x,y,z), \qquad (x,y,z) \in \overline{K_1} \cup \overline{K_2},$$

where v_1 is a P_k polynomial, the restriction of v on K_1. Then v_0 is identically zero on K_1. All the nodal values of v_0 on the interface triangle T are zero in (2.17). Repeating the steps in the proof for Theorem 2.1 and for Corollary 2.1, we can factor out two factors $L(x,y,z)$ (defined in (3.3)) from the P_k polynomial (on K_2):

$$v_0(x,y,z) = L(x,y,z)^2 p_{k-2}(x,y,z) = (ax + by + cz - d)^2 p_{k-2}(x,y,z),$$

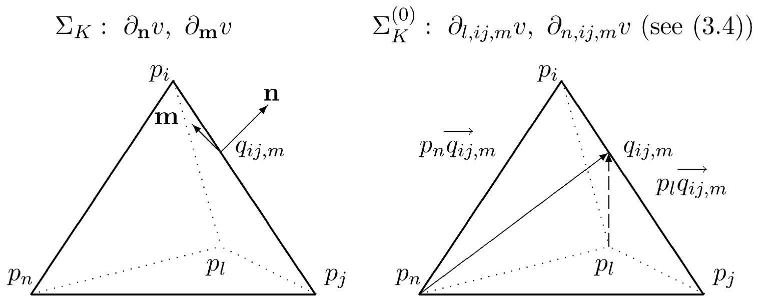

Fig. 5. Normal and scaled directional derivatives on an edge.

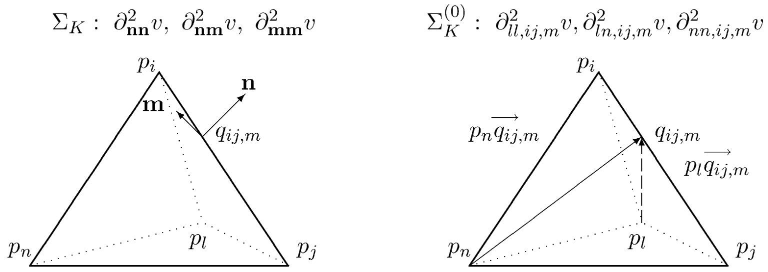

Fig. 6. Three 2nd order normal and scaled directional derivatives (3.5) on an edge.

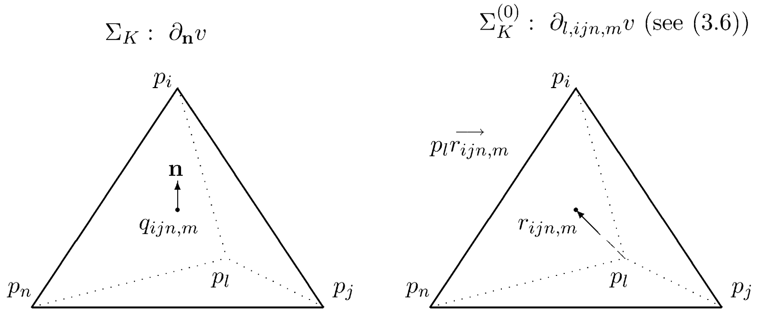

Fig. 7. Normal derivatives and scaled directional derivatives on a face triangle.

where $p_{k-2}(x, y, z)$ is degree $(k-2)$ polynomial. Therefore v_0 is C_1 over the two neighboring tetrahedra, and so is v. □

Next we define a C_0-P_k element, by replacing the 2 normal vectors at an internal edge-point by the vectors connecting the point with the two other vertices (of the tetrahedron) which are not on the edge, cf. Figs. 5–6, and by replacing the normal vector at an internal face-point by the vector connecting the point with the other vertex (of the tetrahedron) which is not on the face-triangle, cf. Fig. 7, in (2.17). The purpose of introducing this C_0 element is to utilize the standard finite element theory, as this element is of an affine family, i.e., invariant under the affine reference mapping. We note that as normal derivatives are used in the definition of Σ_K in (2.17), the C_1-P_k finite element is not of an affine family (cf. [6] for the definition). To make notations clear, we use the following conventions (see Figs. 5–7):

- The other two vertices not on an edge $p_i p_j$ of a tetrahedron K are p_l and p_n;
- An internal edge-point of Σ_K, on an edge $p_i p_j$, is denoted by $q_{ij,m}$;
- A scaled directional derivative of v at $q_{ij,m}$ is denoted by

$$\partial_{l,ij,m} v = \big(\nabla v(q_{ij,m})\big) \cdot (\overrightarrow{p_l q_{ij,m}}); \tag{3.4}$$

- A scaled 2nd-order directional derivative of v at $q_{ij,m}$ is denoted by

$$\partial^2_{ln,ij,m} v = \big(\nabla\big[\nabla v(q_{ij,m})\big] \cdot (\overrightarrow{p_l q_{ij,m}})\big) \cdot (\overrightarrow{p_n q_{ij,m}}); \tag{3.5}$$

- The other vertex not on a triangle $p_i p_j p_l$ of a tetrahedron K is p_n;
- An internal face-point of Σ_K, on a triangle $p_i p_j p_l$, is denoted by $r_{ijl,m}$;
- A scaled directional derivative of v at $r_{ijl,m}$ is denoted by

$$\partial_{n,ijl,m} v = \big(\nabla v(r_{ijl,m})\big) \cdot (\overrightarrow{p_n r_{ijl,m}}). \tag{3.6}$$

We next define a C_0-P_k Hermite type finite element by replacing the normal derivatives in Σ_K by scaled directional derivatives, cf. Figs 5–7. Let $K \in \Omega_h$, and $v \in P_K = P_k$.

$$\begin{aligned}
\Sigma_K^{(0)} = \{f \mid\ &\text{(a) derivatives of order 0 to 4 at 4 vertices of } K,\\
&\text{(b) } (k-9) \text{ values inside each of 6 edges},\\
&\text{(c) two scaled } \textit{directional} \text{ derivatives, } \partial_{l,ij,m}v,\ \partial_{n,ij,m}v, \text{ cf. (3.4)},\\
&\quad\ \text{at } (k-8) \text{ nodes } \{q_{ij,m}\} \text{ inside each of 6 edges},\\
&\text{(d) three 2nd-order scaled } \textit{directional} \text{ derivatives,} \partial^2_{ll,ij,m}v, \partial^2_{ln,ij,m}v, \text{ and } \partial^2_{nn,ij,m}v,\\
&\quad\ \text{defined in (3.5) at } (k-7) \text{ nodes inside each of 6 edges},\\
&\text{(e) nodal values at } (k-8)(k-7)/2 \text{ evenly distributed points inside each of 4 face triangles},\\
&\text{(f) } (k-6)(k-5)/2 \text{ scaled } \textit{directional} \text{ derivatives, } \partial_{n,ijl,m}v \text{ at evenly distributed points}\\
&\quad\ \{r_{ijl,m}\}, \text{ inside each of 4 face triangles}\\
&\text{(g) } (k-7)(k-6)(k-5)/6 \text{ values at evenly distributed points inside a tetrahedron}\}.
\end{aligned} \tag{3.7}$$

Theorem 3.2. *For a non-degenerate tetrahedron K, $\Sigma_K^{(0)}$, defined in* (3.7)*, is uni-solvent on P_k.*

Proof. The proof could be done by repeating the proof for Theorem 2.1, but the notations would be very complicated. Instead, the proof here is based on the result of Theorem 2.1 and Corollary 2.1. Let v be a P_k polynomial on K. Let $f(v) = 0$ for all $f \in \Sigma_K^{(0)}$.

Since the nodal freedoms, other than normal derivatives, of $\Sigma_K^{(0)}$ and Σ_K are the same, we conclude that v is identically zero on the boundary of K. Therefore all first and second order *tangential derivatives* of v on ∂K are zero.

At an internal edge-point $q_{ij,m}$ where the derivatives are used in $\Sigma_K^{(0)}$ and Σ_K, we have three first order directional derivatives, in the tangential direction $\overrightarrow{p_i p_j}$, and the two directions connecting the point with the other two vertices, $\overrightarrow{p_n q_{ij,m}}$ and $\overrightarrow{p_l q_{ij,m}}$, all zero. These three vectors are linearly independent because K is non-degenerate. Therefore all first order derivatives at $q_{ij,m}$, including the two normal derivatives $\partial_{\mathbf{n}} v$ and $\partial_{\mathbf{m}} v$ used in Σ_K, are zero.

Next, the normal derivative $\partial_{\mathbf{n}} v$ at each internal face-point used in Σ_K is zero too. This is because two tangential derivatives on the face triangle, and a third directional derivative, are all zero (see Fig. 7).

For the second order normal derivatives on edges, used in the definition of Σ_K, we have a similar argument. Because v is identically zero on a face triangle, the 2nd order face-tangential derivatives of v are zero, including the one at an internal edge-point in the two directions, $\overrightarrow{p_l q_{ij,m}}$ and $\overrightarrow{p_i p_j}$. Since all following six 2nd-order (not scaled, cf. Fig. 6) derivatives at an edge-point $q_{ij,m}$ are zero,

$$\begin{aligned}
&\partial^2_{\overrightarrow{p_i p_j},\overrightarrow{p_i p_j}} v = 0, &&\partial^2_{\overrightarrow{p_i p_j},\overrightarrow{p_l q_{ij,m}}} v = 0, &&\partial^2_{\overrightarrow{p_l q_{ij,m}},\overrightarrow{p_l q_{ij,m}}} v = 0,\\
&\partial^2_{\overrightarrow{p_l q_{ij,m}},\overrightarrow{p_n q_{ij,m}}} v = 0, &&\partial^2_{\overrightarrow{p_i p_j},\overrightarrow{p_n q_{ij,m}}} v = 0, &&\partial^2_{\overrightarrow{p_n q_{ij,m}},\overrightarrow{p_n q_{ij,m}}} v = 0.
\end{aligned}$$

Via a change of variables, i.e., a linear transformation, we conclude that the three 2nd-order normal derivatives used in Σ_K are all zero too:

$$\partial^2_{\mathbf{nn}} v(q_{ij,m}) = 0, \quad \partial^2_{\mathbf{nm}} v(q_{ij,m}) = 0, \quad \partial^2_{\mathbf{mm}} v(q_{ij,m}) = 0.$$

In conclusion, we have shown that

$$f(v)=0 \quad \forall f\in\Sigma_K^{(0)} \quad \Longrightarrow \quad f(v)=0 \quad \forall f\in\Sigma_K.$$

By Corollary 2.1, $v\equiv 0$. □

We next define a Hermite type C_0-P_k finite element space.

$$V_k^{(0)}=\bigl\{v \mid v|_K\in P_k\ \forall K\in\Omega_h,\ f(v|_{K_1})=f(v|_{K_2})\ \forall f\in\bigl(\Sigma_{K_1}^{(0)}\cap\Sigma_{K_2}^{(0)}\bigr)\bigr\}. \tag{3.8}$$

Since $\Sigma_K^{(0)}$ and Σ_K are the same except normal derivatives, shown in Theorem 3.2,

$$V_k^{(0)}\subset C_0(\Omega).$$

The nodal interpolation operators are defined by

$$\begin{aligned}&\Pi_k: C_4(\Omega)\to V_k, \quad f(\Pi_k v)=f(v) \quad \forall f\in\Sigma_k,\\ &\Pi_k^{(0)}: C_4(\Omega)\to V_k^{(0)}, \quad f\bigl(\Pi_k^{(0)}v\bigr)=f(v) \quad \forall f\in\Sigma_k^{(0)}.\end{aligned}$$

The next two theorems are the same as corresponding theorems in the Ciarlet book [6], where the approximation of interpolation of the C_1-P_5 Argyris triangles is established via the C_0-P_5 Hermit triangles. We remark that Theorems 3.3 and 3.4 may not require the $C_4(\Omega)$ condition, if generalized (boundary-averaging) interpolation operators are used instead. We refer to [16] and [9].

Theorem 3.3. *The C_0-P_k $(k\geqslant 9)$ finite element family $V_k^{(0)}$ has the optimal order of approximation property that, for $0\leqslant\alpha\leqslant 2$,*

$$\bigl\|u-\Pi_k^{(0)}u\bigr\|_{H^\alpha(\Omega)}\leqslant Ch^{\min(\beta,k+1)-\alpha}|u|_{H^\beta(\Omega)} \quad \forall u\in C_4(\Omega)\cap H^\beta(\Omega),$$

and that, for $3\leqslant\alpha\leqslant\min\{\beta,k+1\}$,

$$\Bigl\{\sum_{K\in\Omega_h}\bigl\|u-\Pi_k^{(0)}u\bigr\|_{H^\alpha(K)}^2\Bigr\}^{1/2}\leqslant Ch^{\min(\beta,k+1)-\alpha}|u|_{H^\beta(\Omega)} \quad \forall u\in C_4(\Omega)\cap H^\beta(\Omega).$$

Proof. Since $\Pi_k^{(0)}$ preserves P_k polynomials locally and $V_k^{(0)}$ is an affine family of finite element spaces, the proof is standard, cf. Theorems 3.1.2 and 3.1.3 in Ciarlet [6]. □

Since the letters m and l are to be used as indexes of basis functions, we use Greek letters α and β for the Sobolev space indexes in these two theorems. The next theorem is a 3D version of Theorem 6.6.1 of [6].

Theorem 3.4. *The C_1-P_k $(k\geqslant 9)$ finite element family V_k has the optimal order of approximation property that, for all $u\in C_4(\Omega)\cap H^\beta(\Omega)$,*

$$\begin{aligned}&\|u-\Pi_k u\|_{H^\alpha(\Omega)}\leqslant Ch^{\min(\beta,k+1)-\alpha}|u|_{H^\beta(\Omega)}, \quad \text{if } 0\leqslant\alpha\leqslant 2,\\ &\Bigl\{\sum_{K\in\Omega_h}\|u-\Pi_k u\|_{H^\alpha(K)}^2\Bigr\}^{1/2}\leqslant Ch^{\min(\beta,k+1)-\alpha}|u|_{H^\beta(\Omega)}, \quad \text{if } 3\leqslant\alpha\leqslant\min\{\beta,k+1\}.\end{aligned}$$

Proof. V_k is not a regular affine family of finite element spaces. It is almost-affine, by the definition of Ciarlet [6]. The auxiliary space $V_k^{(0)}$ is used in the analysis.

Let $u\in C_4(\Omega)\cap H^\beta(\Omega)$. On a single element $K\in\Omega_h$,

$$u-\Pi_k u=\bigl(u-\Pi_k^{(0)}u\bigr)+\bigl(\Pi_k^{(0)}u-\Pi_k u\bigr). \tag{3.9}$$

The first term in (3.9) is analyzed in Theorem 3.3. For simplicity, we let the second term be

$$w=\bigl(\Pi_k^{(0)}u-\Pi_k u\bigr)\in P_k(K).$$

Since the interpolations are same at vertices of K, and same for the function values at the interior points of edges, faces and the tetrahedron K itself, we separate w into 3 parts:

$$w = \sum_{i=1}^{3} w_i = \sum_{i=1}^{3} \sum_{m} f_{K,m,i}(w)\phi_{K,m,i}, \tag{3.10}$$

where $\phi_{K,l,i}$, $i = 1, 2, 3$, are the basis functions of $V_k^{(0)}$ associated with the three types of directional derivatives (see Figs. 5–7) on K. For a first order directional derivative on an edge (cf. Fig. 5), it is a linear combination of the gradient vector at the point:

$$\partial_{l,ij,m} w = \nabla w(q_{ij,m}) \cdot \overrightarrow{p_l q_{ij,m}} = \partial_{\mathbf{n}} w(q_{ij,m})\, \overrightarrow{p_l q_{ij,m}} \cdot \mathbf{n} + \partial_{\mathbf{m}} w(q_{ij,m})\, \overrightarrow{p_l q_{ij,m}} \cdot \mathbf{m} + \partial_{\mathbf{t}} w(q_{ij,m})\, \overrightarrow{p_l q_{ij,m}} \cdot \mathbf{t},$$

where $\mathbf{n}$ and $\mathbf{m}$ are two selected unit normal vectors on the edge for Σ_K, and $\mathbf{t}$ is a unit tangent vector on the edge $p_i p_j$. Therefore

$$\begin{aligned}\partial_{l,ij,m} w = {}& \partial_{\mathbf{n}}\big(\Pi_k^{(0)} u - u\big)(q_{ij,m})\, \overrightarrow{p_l q_{ij,m}} \cdot \mathbf{n} + \partial_{\mathbf{m}} w(q_{ij,m})\big(\Pi_k^{(0)} u - u\big)(q_{ij,m})\, \overrightarrow{p_l q_{ij,m}} \cdot \mathbf{m} \\ & + \partial_{\mathbf{t}} w(q_{ij,m})\big(\Pi_k^{(0)} u - u\big)(q_{ij,m})\, \overrightarrow{p_l q_{ij,m}} \cdot \mathbf{t}.\end{aligned}$$

We note that $\partial_{\mathbf{t}} w(q_{ij,m}) = 0$ as $w = \Pi_k^{(0)} u - \Pi_k u$ is identically zero on the edge. But for convenience, we keep the $\partial_{\mathbf{t}}$ term and denote the three vectors $\mathbf{n}$, $\mathbf{m}$ and $\mathbf{t}$ by $\mathbf{n}_n$, $n = 1, 2, 3$, respectively. So

$$\begin{aligned} w_1 &= \sum_m f_{K,m,1}(w)\phi_{K,m,1} \\ &= \sum_m \phi_{K,m,1} \sum_{n=1}^{3} \partial_{\mathbf{n}_n}\big(\Pi_k^{(0)} u - u\big)(q_{ij,m})\, \overrightarrow{p_l q_{ij,m}} \cdot \mathbf{n}_n.\end{aligned}$$

By the regularity assumption of tetrahedra in Ω_h, cf. Theorem 6.1.1 in Ciarlet [6],

$$\begin{aligned} &|\overrightarrow{p_l q_{ij,m}} \cdot \mathbf{n}_n| \leqslant Ch, \\ &|\phi_{K,m,1}|_{H^\alpha(K)} \leqslant Ch^{3/2-\alpha} |\hat{q}_{\hat{K},m,1}|_{H^\alpha(\hat{K})}, \\ &\big|\partial_{\mathbf{n}_n} w(q_{ij,m})\big| \leqslant \sqrt{3}\, \big|\Pi_k^{(0)} u - u\big|_{W^1_\infty(K)} \leqslant Ch^{\beta-5/2} |u|_{H^\beta(K)}.\end{aligned}$$

Combining the three estimates, we get the optimal order bound for w_1 in (3.10):

$$|w_1|_{H^\alpha(K)} \leqslant Ch^{\beta-\alpha} |u|_{H^\alpha(K)}. \tag{3.11}$$

Next, for the term w_2 in (3.10) containing the second order scaled directional derivatives on the edges of K (cf. Fig. 6), we estimate it similarly.

$$\begin{aligned} w_2 &= \sum_m f_{K,m,2}(w)\phi_{K,m,2} \\ &= \sum_m \phi_{K,m,2} \sum_{n_1,n_2=1}^{3} \partial^2_{\mathbf{n}_{n_1},\mathbf{n}_{n_2}}\big(\Pi_k^{(0)} u - u\big)(q_{ij,m})(\overrightarrow{p_l q_{ij,m}} \cdot \mathbf{n}_{n_1})(\overrightarrow{p_n q_{ij,m}} \cdot \mathbf{n}_{n_2}),\end{aligned}$$

where $\mathbf{n}_{n_1}$ are the three orthonormal vectors on the edge $p_i p_j$, at point $q_{ij,m}$, for $n_1 = 1, 2, 3$. Then

$$\begin{aligned} &|\overrightarrow{p_l q_{ij,m}} \cdot \mathbf{n}_{n_1}| \leqslant Ch, \\ &|\overrightarrow{p_n q_{ij,m}} \cdot \mathbf{n}_{n_2}| \leqslant Ch, \\ &|\phi_{K,m,2}|_{H^\alpha(K)} \leqslant Ch^{3/2-\alpha} |\hat{q}_{\hat{K},m,1}|_{H^\alpha(\hat{K})}, \\ &\big|\partial^2_{\mathbf{n}_{n_1},\mathbf{n}_{n_2}} w(q_{ij,m})\big| \leqslant 3\big|\Pi_k^{(0)} u - u\big|_{W^2_\infty(K)} \leqslant Ch^{\beta-7/2} |u|_{H^\beta(K)}.\end{aligned}$$

Hence,

$$|w_2|_{H^\alpha(K)} \leqslant Ch^{\beta-\alpha} |u|_{H^\alpha(K)}. \tag{3.12}$$

Finally, for the term w_3 in (3.10) containing the scaled directional derivatives on the faces of K (cf. Fig. 7), the estimation is the same as that for w_1. That is

$$|w_3|_{H^\alpha(K)} \leqslant Ch^{\beta-\alpha}|u|_{H^\alpha(K)}, \tag{3.13}$$

because (cf. Fig. 7)

$$\begin{aligned} w_3 &= \sum_m f_{K,m,3}(w)\phi_{K,m,3} \\ &= \sum_m \phi_{K,m,3} \sum_{n_1=1}^{3} \partial_{\mathbf{n}_{n_1}} \big(\Pi_k^{(0)} u - u\big)(r_{ijl,m})\ p_n\overrightarrow{r_{ijl,m}} \cdot \mathbf{n}_n, \end{aligned}$$

where $\mathbf{n}_n$ stands for the normal vector on the face triangle and two selected tangential vectors on the face.

The proof is completed by a triangle inequality applied to (3.9). □

We introduce next a 3D model of biharmonic equation, its variational form and its finite element approximation. We then establish the best order convergence of the finite element solution. We seek the solution of the following biharmonic equation with homogeneous boundary conditions,

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega \subset R^3, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Via integration by parts, we introduce the variational problem, finding $u \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v = \int_\Omega f v \quad \forall v \in H_0^2(\Omega).$$

Let $V_{h,0} = V_k \cap H_0^2(\Omega)$. The finite element solution u_h is defined by

$$a(u_h, v_h) = (f, v_h) \quad \forall v \in V_{h,0}, \tag{3.15}$$

where $a(u, v) = \int \Delta u \Delta v$ and $(f, v) = \int f v$.

Theorem 3.5. *The finite element solutions from* (3.15) *converge at the optimal order, i.e.,*

$$\|u - u_h\|_{H^2(\Omega)} \leqslant Ch^{\beta-2}\|u\|_{H^\beta(\Omega)}, \quad 2 < \beta \leqslant (k+1).$$

Proof. The proof follows the Cea's lemma (cf. [6]) and the approximation property, Theorem 3.4. □

4. Numerical test

In this section, we numerically check that the C_1-P_k element is well defined. We do a few numerical tests on the domain of unit cube, which has a tetrahedral grid shown in Fig. 8.

First, we generate the local and global bases for V_h defined in (3.2), corresponding to Σ_K. We solve local linear systems and express the basis functions by two methods. In one method, we express the basis function by monomials,

$$\phi = \sum_{i+j+l \leqslant k} c_{ijk} x^i y^j z^l. \tag{4.1}$$

In the other method, we express the basis function by the Lagrange C_0 nodal basis functions. The second method does not improve the round-off error significantly. We report the output data for the first method (4.1) only.

To check the correctness of the computation for generating the basis functions, in Fig. 9, we plot two global nodal basis functions. On the left side, it is the C_1-P_{10} basis function whose nodal value is 1 at $(1/2, 1/2, 1/2)$. On the right side of Fig. 9, we plot the global C_1-P_9 basis function which has value 1 for one of its two normal derivatives at the middle of the diagonal edge (see Fig. 8), $(1/2, 1/2, 1/2)$.

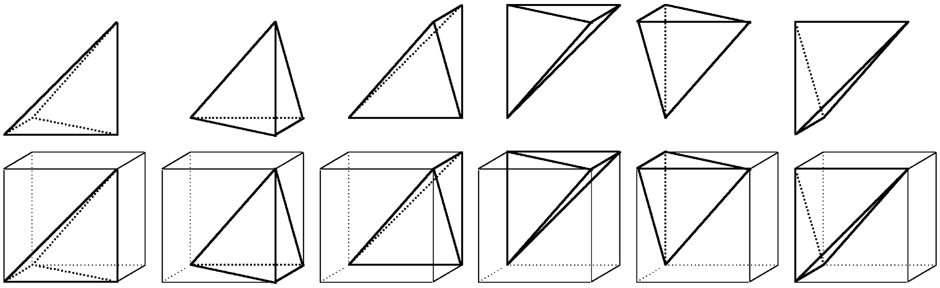

Fig. 8. A tetrahedral grid on the unit cube.

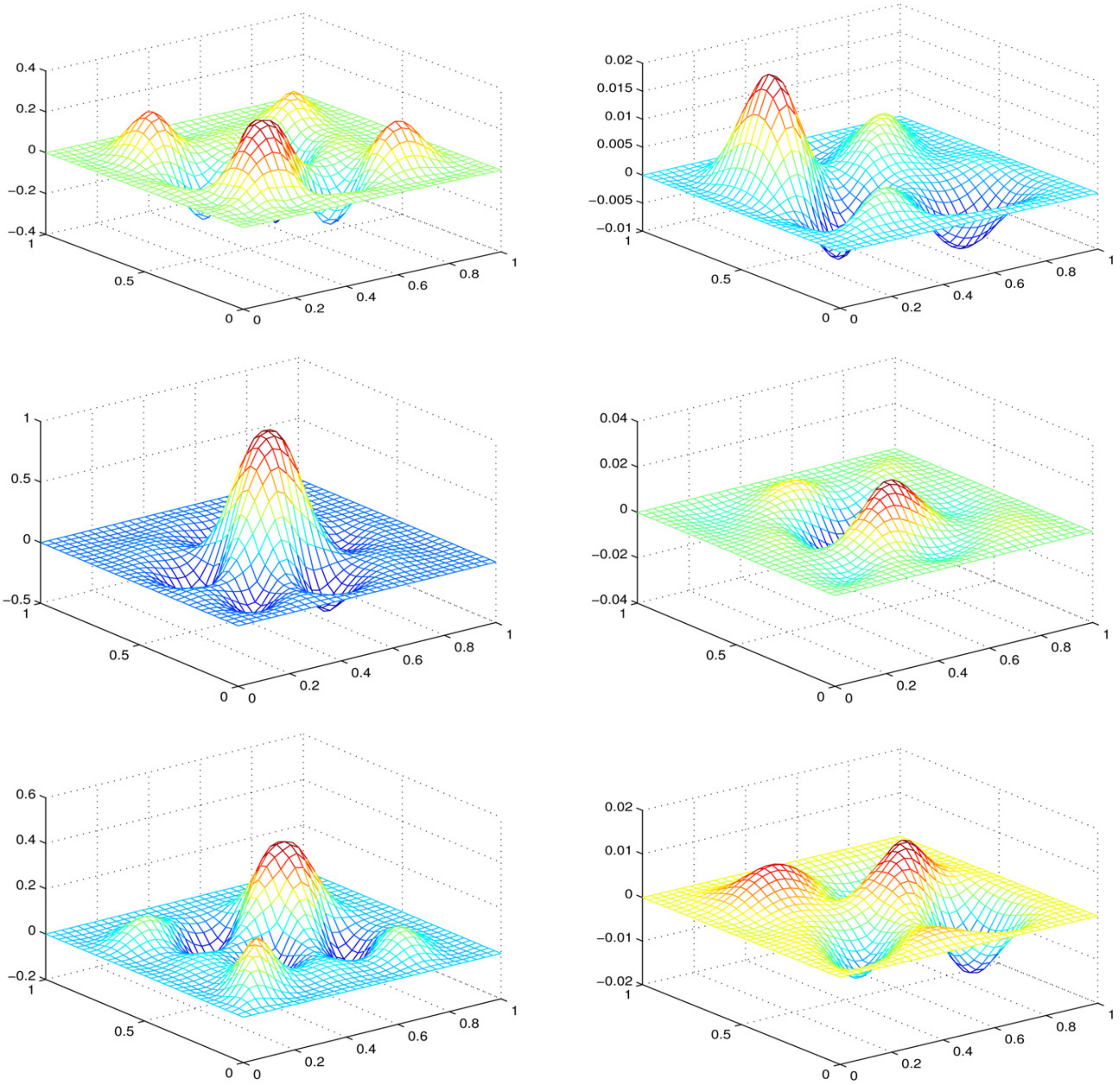

Fig. 9. The plots of a P_{10} (left) and a P_9 basis function ϕ at the cuts $z = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ (bottom).

Table 1
The interpolation errors for v_1 in (4.2)

Degree k	$\|v_1 - \Pi_k v_1\|_{L^\infty(\Omega)}$	$\mathrm{Cond}(A_{\mathrm{local}}, 2)$
9	0.941257465	1.6828×10^{11}
10	0.000000548	5.5419×10^{12}
11	0.000030431	1.7719×10^{14}
12	0.000546004	5.1170×10^{15}
13	0.008998926	1.3926×10^{17}
14	0.436249056	3.5941×10^{18}

Table 2
The errors of the finite element solutions for (4.3)

Degree k	$\|u - u_h\|_{L^\infty(\Omega)}$	$\|\Delta(u - u_h)\|_{L^2(\Omega)}$	$\frac{\|\Delta(u-u_h)\|_{L^2(\Omega)}}{\|\Delta u\|_{L^2(\Omega)}}$
9	0.91577142935	6.59061863	0.51004
10	0.00300771334	0.67166077	0.05197

Table 3
The errors of the finite element solutions for (4.5)

Degree k	$\|u - u_h\|_{L^\infty(\Omega)}$	$\|\Delta(u - u_h)\|_{L^2(\Omega)}$	$\frac{\|\Delta(u-u_h)\|_{L^2(\Omega)}}{\|\Delta u\|_{L^2(\Omega)}}$
9	0.00002775	0.03185083	0.00263929040
10	0.00536660	0.06399795	0.00530313128

Next we test the accuracy of the computer-generated basis functions, and of the nodal-value interpolation operators. We choose a P_{10} function on the cube,

$$v_1(x, y, z) = 2^{10}\big(x(1-x)y(1-y)\big)^2 z(1-z). \tag{4.2}$$

In Table 1, we list the interpolation errors, on the grid shown in Fig. 8. It is supposed that the error for $\Pi_k v_1$ is zero if $k \geqslant 10$. The computation is done by Matlab, which has an accuracy of 2.22×10^{-16}. So the computation for generating nodal basis functions for V_k loses about 10 significant digits for C_1-P_{10} finite elements. Here we remark that $\|v_1\|_{L^\infty(\Omega)} = 1$. As for degree 14 C_1 finite elements, we lost all 16 significant digits! Which simply indicates that we could not compute 3D C_1-P_{14} finite element this way, by (4.1). For the linear system to determine the coefficients of basis functions on the uniform tetrahedron (see Fig. 8), the condition number (of coefficient matrix A_{local}) is huge, as listed in Table 1, column 3, for various polynomial degrees. One way to make the 3D C_1-P_k finite element computable might be to construct nodal basis functions directly, as done in [3] and [4] where the basis for C_1-P_5 Argyris triangles and the basis for C_1-P_3 Hsieh–Clough–Tocher triangles are constructed. We leave such constructions as future research.

We then check the finite element solutions on the grid shown in Fig. 8. We solve the following biharmonic equation with non-homogeneous boundary conditions,

$$\begin{cases} \Delta^2 u = f\ (= \Delta^2 v_1) & \text{in } \Omega \subset R^3, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = g\ (= \frac{\partial}{\partial n} v_1) & \text{on} \partial\Omega. \end{cases} \tag{4.3}$$

Here the exact solution is $u = v_1$, defined in (4.2). Since the exact solution is a P_{10} polynomial, the finite element solutions of the C_1-P_k method *should* be exactly the same as v_1 itself, for all $k \geqslant 10$, if there is no round-off error caused by computer. In fact, again, as seen above, the round-off error is huge. In Table 2, we list the norms of the errors of the finite element solutions, on the grid shown in Fig. 8. By Table 2, we can see that the relative error in the energy norm $\|\Delta \cdot\|_{L^2(\Omega)}$ is 0.05197, not close to zero.

Finally, we test the finite element solutions for another exact solution of a lower degree polynomial:

$$v_2(x, y, z) = 2^6 x(1-x)y(1-y)z(1-z). \tag{4.4}$$

Then the biharmonic test problem is

$$\begin{cases} \Delta^2 u = f\ (= \Delta^2 v_2) & \text{in } \Omega \subset R^3, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = g\ (= \frac{\partial}{\partial n} v_2) & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Here the exact solution is $u = v_2$, defined in (4.4).

This time, if no computer round-off error, both C_1-P_9 and C_1-P_{10} finite element methods should give the exact solution, since the exact solution is a degree 6 polynomial. Again, our computer is too far from enough accuracy for such high-order elements, unless some special implementation methods are discovered. The errors are listed in Table 3.

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