Stability of the finite elements $9/(4c + 1)$ and $9/5c$ for stationary Stokes equations

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Abstract

In this paper we study the stability of two quadrilateral elements $9/(4c + 1)$ and $9/5c$ for stationary Stokes equations. These two elements are very similar. However their stabilities are quite different. It is shown that the element $9/(4c + 1)$ is stable. The stability of the element $9/5c$ is shown mesh-dependent. Numerical tests are provided.

Keywords: Mixed finite elements; Stokes equations; Macroelements; Reduced inf–sup condition

1. Introduction

In this paper we analyze the stability of two quadrilateral finite elements for the Stokes equations. One element is the so called $9/(4c + 1)$, introduced by Gresho et al. in [1] and [2], the other one is the element $9/5c$. Both elements employ 9-node “quadratic” approximation for the velocity and 4-node “bilinear” (continuous) approximation augmented by some functions on each quadrilateral for the pressure. The element $9/(4c + 1)$ is augmented by constants and $9/5c$ by bubbles on each quadrilateral. The element $9/(4c + 1)$ performs well numerically [1]. This element is better than $9/4c$ (no pressure augment) in the sense of preserving the incompressibility of incompressible flows. The $9/(4c + 1)$ and $9/5c$ elements look similar, with the same degrees of freedom, but they behave quite differently. Chapelle and Bathe tested the stability of both elements numerically in [3]. Their numerical results show that $9/(4c + 1)$ passes the inf–sup test and $9/5c$ does not. Therefore, the stability of an element requires more insight than a mere constraint count. In fact, out of the twelve types of mixed elements tested in [3], other than these two, $9/(4c + 1)$ and $9/5c$, all elements have analytical conclusions (see Table 1 of [3]). This paper is to provide a theoretic analysis of the two elements. The triangular counterpart of the element $9/(4c + 1)$ was analyzed by Thatcher and Silvester [4].

In this paper, we show that the element $9/(4c + 1)$ is stable on any quadrilateral mesh satisfying the standard regularity condition [5]. The stability of the element $9/5c$ is more complicated. It depends on the underlying meshes. For example, it is unstable on meshes of uniform squares, or of rectangles, or of parallelograms. On such meshes, it is shown that there is a spurious pressure mode. Further, it is shown that even after filtering such a spurious pressure mode, the inf–sup constant (see (17)) still deteriorates when refining such meshes. It is also shown that the element $9/5c$ is stable on a family of meshes with certain local patterns, i.e., macroelements. The major tool used in our analysis is the macroelement technique (cf. [6–10]). A macroelement is a polygonal region cover by some triangles or quadrilaterals of the original mesh. The macroelement technique relates the stability of an element to its local stability on each macroelement. There are many versions of macroelement technique. In this paper we use the

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version developed in [8]. We may find more information on the inf-sup condition of some other commonly used finite elements in [11].

This paper is organized as follows. In Section 2, we introduce the macroelement technique in the stability analysis of mixed finite element methods. In Section 3, we establish the stability and approximation property of the element 9/(4c + 1). Then, we devote Section 4 to the analysis of the stability of the element 9/5C, where we display some spurious pressure modes for certain meshes. Finally, we report some numerical tests on both elements in Section 5.

2. Macroelement technique

A major part of work in analyzing a mixed element method for the Stokes equation is to check the stability of the element. One technique is the use of macroelements to localize the stability condition. Namely the stability is tested by checking local stability and a relatively simple global stability. For many cases, we only need to check the local stability. Many variations of the macroelement technique have been introduced, cf. [6–10], for example. We adopt the version of [8].

We consider the Stokes equations
\[- \nabla u + \nabla p = f \quad \text{in } \Omega, \]
\[
\text{div } u = 0 \quad \text{in } \Omega, \]
\[
\text{div } u = 0 \quad \text{on } \partial \Omega, \]
where \( u = (u_1, u_2) \) is the velocity of the fluid, \( p \) is the pressure, \( f = (f_1, f_2) \) is the external force, and \( \Omega \) is a bounded polygonal domain in \( \mathbb{R}^2 \).

The weak formulation of (1) seeks \((u, p)\) in \( V \times P := H^1(\Omega) \times L^2(\Omega) \) such that
\[
\int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} p \text{div } v = \int_{\Omega} f \cdot v, \quad \forall v \in V,
\]
\[
\int_{\partial \Omega} \text{div } u = 0, \quad \forall q \in P. \tag{2}
\]

Here \( H^1(\Omega) \) and \( L^2(\Omega) \) are the usual Sobolev spaces:
\[
L^2(\Omega) := \left\{ q : \int_{\Omega} q^2 < \infty \right\},
\]
\[
H^1(\Omega) := \left\{ v : \int_{\Omega} |v|^2 \right\},
\]
\[
H^1_0(\Omega) := \left\{ v \in H^1(\Omega) : |v|_{\partial \Omega} = 0 \right\},
\]
\[
H^1(\Omega) := \left\{ v = (v_1, v_2) : |v_1|_{\partial \Omega}, \quad \forall v \in H^1(\Omega). \right\}
\]

To obtain a unique pressure \( p \) in (2), we need impose a side condition \( |\partial \Omega| = 0 \).

On the finite element level, we solve a discrete analogue of the system (2) in a finite element space \( V_h \times P_h \)
\[
\int_{\Omega} \nabla u_h \cdot \nabla v - \int_{\Omega} p_h \text{div } v = \int_{\Omega} f \cdot v, \quad \forall v \in V_h,
\]
\[
\int_{\partial \Omega} \text{div } u_h = 0, \quad \forall q \in P_h. \tag{3}
\]

Here the parameter \( h \) refers to the meshsize, and \((u_h, p_h)\) is the finite element solution.

It is well known that in order to get a stable element, the spaces \( V_h \) and \( P_h \) must be carefully balanced. A method behaves well if the following BB (or LBB, or inf-sup) condition is satisfied:
\[
\inf_{v \in V_h} \sup_{0 \neq q \in P_h} \frac{\int_{\Omega} \text{div } v}{|v|_{1, \Omega}} |p|_{0, \Omega} := \gamma_h \geq \gamma > 0, \tag{4}
\]
where \( \bar{F}_h = \{ q \in P_h : \int_{\Omega} q = 0 \} \). \( \gamma_h \) is called the inf-sup constant. A mixed finite element is said to be stable on a collection of meshes \( \gamma \) if (4) holds for each \( h \). The uniqueness and convergence properties of the finite element solution of (3) are established in the following theorem by Babuska [12] and Brezzi [13].

Theorem 2.1. Suppose the finite element spaces \( V_h \) and \( P_h \) satisfy (4). The system (3) has a unique solution \((u_h, p_h)\) satisfying
\[
|u - u_h|_{1, \Omega} + |p - p_h|_{0, \Omega} \leq C \inf_{v \in V_h, q \in P_h} (|u - v|_{1, \Omega} + |p - q|_{0, \Omega}), \tag{5}
\]
where \((u, p)\) is the solution of (2) and \( C \) is a positive constant depending on \( \gamma_h \).

Although (4) is not a necessary condition, it is the main means of checking the stability of an element. There are many techniques to verify the BB condition for an element, cf. [14] and [9]. In this paper we briefly introduce the macroelement technique of [8]. For more details of the macroelement technique, one may view [6–8,14,9,10] and the reference therein.

Let \( \Omega \) be partitioned into convex quadrilaterals. We denote the partition by \( \mathcal{H}_h \), with meshsize \( h \). The regularity condition (quasiform) for a collection of partitions \( \{ \mathcal{H}_h, h > 0 \} \) can be found in [5].

For any quadrilateral \( T \in \mathcal{H}_h \), with the nodes \( t_1, t_2, t_3, \) and \( t_4 \), there exists exactly one invertible, bilinear mapping \( F_T \in \mathcal{Q}_h(T) \) that maps \( T \) onto \( T \) such that \( F_T(t_i) = t_i \).

Here \( T \) is the reference element, a unit square at the origin, and \( \mathcal{Q}_h(T) = \{ \sum_{0 \leq |a| \leq 1} a_i x^i y^j : a_i \in \mathbb{R} \} \). Note that \( F_T \) maps the sides of \( T \) to the corresponding sides of \( T \) and, in fact, the restriction of \( F_T \) to the sides of \( T \) is affine. Further, let \( V_T \) and \( P_T \) be two polynomial spaces defined on \( T \). The finite element spaces for the velocity and pressure are defined as follows:
\[
V_h = \left\{ v \in H^1(\Omega) : v|_T = \tilde{v} \circ F^{-1}_T, \tilde{v} \in V_T, T \in \mathcal{H}_h \right\},
\]
\[
P_h = \left\{ q \in L^2(\Omega) : q|_T = \tilde{q} \circ F^{-1}_T, \tilde{q} \in P_T, T \in \mathcal{H}_h \right\}. \tag{6}
\]

A macroelement \( U \) is a polygonal region formed by some quadrilaterals from mesh \( \mathcal{H}_h \). A macroelement covering \( \mathcal{H}_h \) is a set of macroelements such that each quadrilateral of \( \mathcal{H}_h \) is covered by at least one macroelement in \( \mathcal{H}_h \). Two macroelements, \( U_1 = \bigcup_{i=1}^{n_1} T^{(1)}_i \) and \( U_2 = \bigcup_{i=1}^{n_2} T^{(2)}_i \), where \( T^{(j)}_i \) may not be from a same \( \mathcal{H}_h \), are said to be
equivalent if they can be mapped continuously onto each other, or more precisely, if there is a continuous mapping $G : U_1 \rightarrow U_2$ such that

1. $G(U_1) = U_2$.
2. $G(T^{(1)}_j) = T^{(2)}_j$, $1 \leq j \leq k$.
3. $G|_{T^{(1)}_j} = F_{T^{(2)}_j}^{-1} F_{T^{(1)}_j}$, $1 \leq j \leq k$.

We note that both Items 1 and 2 above are consequences of Item 3. An equivalence macroelement class is a set of all the macroelements which are equivalent to each other. For each macroelement $U$ of $\mathcal{M}_h$ we define the restricted spaces

$$V^U_h = (V|_{\mathcal{M}_h}) \cap \tilde{H}^1(U), \quad P^U_h = (P|_{\mathcal{M}_h}) \cap L^2(U).$$

Further, we define the space of “spurious” pressure modes on $U$ by

$$N^U_h = \left\{ q \in P^U_h \mid \int_U q \text{div } v = 0, \forall v \in V^U_h \right\}.$$  \hspace{1cm} (7)

For convenience, we denote the set of all edges of $\mathcal{M}_h$ by $\mathcal{E}_h$, and all the interior edges by $\mathcal{E}_h^i$.

**Theorem 2.2.** For quasiuniform meshes $\mathcal{M}_h$ ($h > 0$) if there is a macroelement covering $\mathcal{M}_h$ for each $\mathcal{M}_h$, a fixed set of equivalence classes $\mathcal{M}_i$, $i = 1, \ldots, m$, of macroelements of all the $\mathcal{M}_h$, and a fixed positive integer $l$ such that

-(C1) $\tilde{V}_p$ contains all bilinear polynomials defined on $\tilde{T}$;
-(C2) for each $U \in \mathcal{M}_h$, the space $N^U_h$ is one dimensional, consisting of constant functions on $U$;
-(C3) each $U \in \mathcal{M}_h$ belongs to one of the classes $\mathcal{M}_i$, $i = 1, \ldots, m$;
-(C4) each $T \in \mathcal{M}_h$ is covered by at least one and no more than $l$ macroelements of $\mathcal{M}_h$;
-(C5) each $e \in \mathcal{E}_h$ is contained in the interior of at least one and no more than $l$ macroelements of $\mathcal{M}_h$.

Then the BB condition (4) holds with $\gamma$ independent of $h$.

The proof of the above theorem can be found in [15].

### 3. Stability analysis of the element 9/(4c + 1)

In this section we use the macroelement technique to analyze the stability of the element 9/(4c + 1). We show that this element is stable on quasiuniform meshes. In order to apply the macroelement technique, we need to determine the macroelement covering $\mathcal{M}_h$ for each $\mathcal{M}_h$ such that the conditions (C2)–(C5) of the Theorem 2.2 are satisfied. The crucial step is to determine a macroelement type to form $\mathcal{M}_h$. We choose the type of two quadrilaterals sharing one common edge. Thus, we only have one equivalence macroelement class. Clearly, the conditions (C3)–(C5) hold automatically under the choice of this type of macroelements. The condition (C1) holds for both elements 9/(4c + 1) and 9/5c by definition. The only condition that needs to be verified is (C2).

Let us formally define the element 9/(4c + 1) now. On the reference $\bar{T}$ we define polynomial space

$$\bar{Q}_h(\bar{T}) = \left\{ \sum_{0 \leq j \leq k} a_j \bar{x}^j \mid a_j \in \mathbb{R} \right\},$$

where $k \geq 0$ is an integer. For each quadrilateral $T \in \mathcal{M}_h$ we define

$$Q_h(T) = \{ v \circ F^{-1}_T \mid v \in \bar{Q}_h(\bar{T}) \}.$$ 

Over a partition $\mathcal{M}_h$, we define finite element spaces

$$Q^1_h(\mathcal{M}_h) = \{ v \in H^1(\Omega) \mid v|_T \in Q_h(T), \forall T \in \mathcal{M}_h \},$$

$$Q^0_h(\mathcal{M}_h) = \{ v \in L^2(\Omega) \mid v|_T \in Q_h(T), \forall T \in \mathcal{M}_h \}.$$ 

Therefore, $Q^0_h(\mathcal{M}_h)$ denotes the space of piecewise constant functions defined on $\mathcal{M}_h$. The element 9/(4c + 1) is defined by

$$V_h = Q^1_h(\mathcal{M}_h) \cap \tilde{H}^1(\Omega), \quad P_h = Q^1_h(\mathcal{M}_h) + Q^0_h(\mathcal{M}_h).$$

The above definition is conformable to (6). The degrees of freedom of this element are depicted in **Fig. 1**. A filled (resp. emptied) circle indicates a continuous (resp. discontinuous) degree of freedom.

**Lemma 3.1.** Let $U$ be a macroelement of $\mathcal{M}_h$ such that $U$ consists of two quadrilaterals $T_1$ and $T_2$ which share one common edge (depicted in **Fig. 2**). On the macroelement $U$ we define

$$V^U_h = Q^1_h(U) \cap \tilde{H}^1(U), \quad P^U_h = Q^1_h(U) + Q^0_h(U).$$

Then $N^U_h$ defined in (7), is one-dimensional, consisting of constant functions.

**Proof.** For convenience, we write any function $v \in V^U_h$ in the form

$$v = \sum_{i=1}^{3} \left( e^{(i)}_1, e^{(i)}_2 \right) \phi_i,$$

where $\phi_i$ are the nodal variables as depicted in **Fig. 1**.
where $\phi_i$ is the nodal basis function of $Q^i_h(\Omega_h)$ at node $i$ (the three nodes are shown in the first picture of Fig. 2). Consequently, $(v^{(i)}_1, v^{(i)}_2)$ is the nodal value of $v$ at node $i$. Similarly, we write any function $q \in P^U_h$ as

$$q = \sum_{i=1}^{6} q^{(i)} \psi_i + q^{(7)} \chi^T + q^{(8)} \chi^T;$$

where $\psi_i$, $i = 1, \ldots, 6$, are nodal basis functions of $Q^i_h(\Omega_h)$, $q^{(i)}$, $i = 1, \ldots, 8$, real numbers, and $\chi^T$, $i = 1, 2$, two characteristic functions. Further, we assume the coordinates of all the vertices of $T_1$ and $T_2$ are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$ and $(x_5, y_5)$, $(x_6, y_6)$, $(x_5, y_5)$, respectively.

We need to show any function $q \in N^U_h$ is a constant on two elements, if $q$ satisfies

$$\int_U q \text{div} v = 0, \quad \forall v \in V^U_h.$$  
(8)

By the fact of $V^U_h \subset V^T_h$, Eq. (8) implies

$$\int_{T_1} q \text{div} v = 0, \quad \forall v \in V^T_h := Q^1_h(T_1) \cap H^1(T_1).$$

Namely,

$$\int_{T_1} (q^{(1)} \psi_1 + q^{(2)} \psi_2 + q^{(4)} \psi_4 + q^{(5)} \psi_5) \text{div} \left[ (v^{(1)}_1, v^{(1)}_2) \phi_1 \right] = 0,$$

$$\forall v^{(1)}_1, v^{(1)}_2 \in \mathbb{R},$$

or

$$\int_{T_1} (q^{(1)} \psi_1 + q^{(2)} \psi_2 + q^{(4)} \psi_4 + q^{(5)} \psi_5) \frac{\partial \phi_1}{\partial x} = 0,$$

$$\int_{T_1} (q^{(1)} \psi_1 + q^{(2)} \psi_2 + q^{(4)} \psi_4 + q^{(5)} \psi_5) \frac{\partial \phi_1}{\partial y} = 0.$$  
(9)

Here we used the facts $\int_{T_1} \frac{\partial \phi_1}{\partial x} = \int_{T_1} \frac{\partial \phi_1}{\partial y} = 0$. In order to evaluate the integrals in (9), we change variables to get integrals on $\tilde{T}$ via reference mapping $F_{T_1}$ and a calculation would derive that

$$\begin{align*}
(y_1 - y_3)(q^{(2)} - q^{(4)}) + (y_4 - y_2)(q^{(1)} - q^{(5)}) = 0, \\
(x_1 - x_3)(q^{(2)} - q^{(4)}) + (x_4 - x_2)(q^{(1)} - q^{(5)}) = 0.
\end{align*}$$  
(10)

The determinant of the above linear system, $|y_1 - y_3)(x_4 - x_2) - (y_4 - y_2)(x_1 - x_3)|$, is nonzero. In fact the determinant is the magnitude (with a possible minus sign) of the cross product of two diagonal vectors of $T_1$. By Fig. 3, the determinant is the area of the auxiliary parallelogram, which in turn is exactly twice the area of the original quadrilateral $T_1$. Therefore the system (10) has a unique, trivial solution

$$q^{(2)} - q^{(4)} = 0 \quad \text{and} \quad q^{(1)} - q^{(5)} = 0.$$  
(11)

Repeating the analysis above on $T_2$, we get

$$q^{(2)} = q^{(6)}, \quad q^{(3)} = q^{(5)}.$$  
(12)

A combination of (11) and (12) gives

$$q^{(2)} = q^{(4)} = q^{(6)}, \quad q^{(1)} = q^{(3)} = q^{(5)}.$$
Therefor for \( q \in N^U_h \),
\[
q = q^{(1)} \sum_{i=1}^{6} \psi_i + q^{(7)} \chi^U = C,
\]
on \( U \) for some constant \( C \). This proves the lemma. \( \square \)

By now, we have proven that all the conditions of Theorem 2.2 hold for the element 9/(4c + 1). Therefore, the element is stable on quasiuniform quadrilateral meshes. This element also provides optimally convergent finite element solutions because of the optimal approximation properties of the finite element space \( V_h \times P_h \). As a summary, we state a theorem as follows:

**Theorem 3.1.** Assume \( (u, p) \in H^1(\Omega) \times H^2(\Omega) \) solves (2) and \( (u_h, p_h) \in V_h \times P_h \) solves (3). For quasiuniform grids \( \mathcal{A}_h \), the 9/(4c + 1) element is stable. Moreover
\[
\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq C h^2 (\|u\|_{1,\Omega} + \|p\|_{2,\Omega}).
\]

4. Stability analysis of the element 9/5c

We will show that the stability of the element 9/5c is mesh-dependent. Namely, it could be stable for one mesh family but unstable for another. In this section, we will show that this element is unstable on meshes formed by small parallelograms of possibly different sizes. What we will show is that any nontrivial spurious pressure mode is a multiple of a function which is a bubble with height 1 on each small parallelogram. Therefore, the inf-sup constant is always zero on this type of meshes. On the other hand, we construct a family of meshes on which the element is stable.

We begin our discussion by a formal definition of the element 9/5c. On the reference square \( T \), we define a bubble function \( b_T = \tilde{b}(1 - \tilde{x})(1 - \tilde{y}) \). Similarly we define \( b_T = b_T \circ F_T^{-1} \) for any \( T \in \mathcal{A}_h \). We shall also define a space
\[
B(T) = \{ cb_T : c \in \mathbb{R} \},
\]
for any quadrilateral \( T \in \mathcal{A}_h \) and
\[
B(\mathcal{A}_h) = \{ q \in C_0(\Omega) | q|_T \in B(T), \forall T \in \mathcal{A}_h \},
\]
for the quadrilateral partition \( \mathcal{A}_h \). The definition of the element 9/5c reads
\[
V_h = Q_1^1(\mathcal{A}_h) \cap H^1(\Omega), \quad P_h = Q_1^1(\mathcal{A}_h) + B(\mathcal{A}_h).
\]
Therefore, the pressure space consists of only continuous functions. The element is depicted in Fig. 1 too.

In order to state our result more clearly, we define a global space
\[
N_q = \left\{ q \in P_h | \int_T q \text{div } v = 0, \forall v \in V_h \right\}.
\]
It is obvious that all the constant functions are in \( N_q \). Further, we denote the pressure mode \( q \), \( q|_T = b_T \) for each \( T \in \mathcal{A}_h \), by \( \delta \). For convenience, we call the meshes formed by parallelograms the parallelogram-meshes.

**Theorem 4.1.** On any parallelogram-mesh \( \mathcal{A}_h \), the element 9/5c has
\[
\dim N_q = 2.
\]
More precisely, \( N_q = \text{span}\{1, \delta\} \).

**Proof.** The proof is very similar to the proof of Lemma 3.1 in Section 3.

We consider an arbitrary quadrilateral partition \( \mathcal{A}_h \) first. Let \( U \) denote an arbitrary macroelement formed by two quadrilaterals \( T_1 \) and \( T_2 \) which share one common edge (depicted in Fig. 5).

Again, we write any function \( v \in V^U_h \) in the form
\[
v = \sum_{i=1}^{3} \left( v^{i(1)}_1, v^{i(2)}_2 \right) \phi_i,
\]
and \( q \in P^U_h \) in the form
\[
q = \sum_{i=1}^{6} q^{(i)} \psi_i + q^{(7)} \beta T_1 + q^{(8)} \beta T_2.
\]
Here \( \phi_i \) and \( \psi_i \) are nodal basis functions as we defined in the proof of Lemma 3.1, \( \beta T_1 \) equals \( b_{T_1} \) on \( T_1 \) and zero elsewhere.

Let \( q \in N_q \). According to the definition of \( N_q \) and \( N^U_q \), we have \( q|_U \in N^U_q \). Namely,
\[
\int_U q \text{div } v = 0, \forall v \in V^U_h, \quad U \in \mathfrak{U}_h.
\]
Again, we test (14) on element \( T_1 \) first:
\[
\int_{T_1} q \text{div } v = 0, \forall v \in V^T_{h_1}.
\]
Because \( \phi_1 = 16b_{T_1} \) and \( \int_{T_1} \text{div } \left( v^{(1)}_1, v^{(2)}_1 \right) \beta T_1 = 0 \), it follows that
\[
\int_{T_1} \left( q^{(1)} \psi_1 + q^{(2)} \psi_2 + q^{(4)} \psi_4 + q^{(5)} \psi_5 \right) \text{div } \left( v^{(1)}_1, v^{(2)}_1 \right) \beta T_1 = 0,
\]
\( \forall v^{(1)}_1, v^{(2)}_1 \in \mathbb{R} \).

The above system is exactly the same as the one we discussed in Lemma 3.1. Therefore, together with
\[
\int_{T_2} q \text{div } v = 0, \forall v \in V^T_{h_2},
\]
we get as we did earlier that
\[
q = q^{(1)}(\phi_1 + \phi_3 + \phi_5) + q^{(2)}(\phi_2 + \phi_4 + \phi_6) + q^{(7)} \beta T_1 + q^{(8)} \beta T_2.
\]
To determine the relation among \( q^{(1)}, q^{(2)}, q^{(7)} \), and \( q^{(8)} \) we evaluate (14) by the freedom of \( b \) at the interface, i.e.,
letting \( v = \left( v_1^{(2)}, v_2^{(2)} \right) \phi_2 \) with arbitrary \( v_1^{(2)}, v_2^{(2)} \in \mathbb{R} \). The integral leads to, cf. Fig. 5,

\[
180 \int_U q \text{div} v = \left[ 10(q^{(1)} - q^{(2)}) (y_3 - y_1 + y_6 - y_4) - q^{(7)} (y_1 - y_4) + q^{(8)} (y_3 - y_6) - 3(q^{(7)} - q^{(8)}) (y_2 - y_3) \right] v_1^{(2)}
+ \left[ 10(q^{(1)} - q^{(2)}) (x_3 - x_1 + x_6 - x_4) - q^{(7)} (x_1 - x_4) + q^{(8)} (x_3 - x_6) - 3(q^{(7)} - q^{(8)}) (x_2 - x_3) \right] v_2^{(2)} = 0,
\]

for any \( v_1^{(2)}, v_2^{(2)} \in \mathbb{R} \). If \( \mathcal{J}_h \) is a parallelogram-mesh, then all edges of elements \( T \) are parallel to two vectors \( A \) and \( B \). The edges may be of different lengths though. In this case, (15) can be written in vector form:

\[
C_1 10(q^{(1)} - q^{(2)}) A + C_4 (q^{(8)} - q^{(7)}) B = 0.
\]

This leads to a system like (13), i.e., vectors \( A \) and \( B \) are linearly independent. Therefore \( q^{(1)} = q^{(2)} \), \( q^{(3)} = q^{(8)} \) and

\[
q = q^{(1)} (\phi_1 + \phi_3 + \phi_5 + \phi_2 + \phi_4 + \phi_6) + q^{(7)} (\beta_{T_1} + \beta_{T_2}).
\]

(16)

Let \( \mathcal{U}_h \) be the collection of all the macroelements, formed by two parallelograms in \( \mathcal{J}_h \) sharing a common edge. \( \mathcal{U}_h \) is a macroelement covering of \( \mathcal{J}_h \). Since every macroelement of \( \mathcal{U}_h \) overlaps with some others and (16) holds on each macroelement, we conclude that

\[
q = C_1 + C_2 \delta,
\]

if \( q \in N_h \). This completes the proof of the theorem.

A common question one would ask, after Theorem 4.1, is if the inf-sup condition would hold after filtering the spurious pressure mode \( \delta \), defined in Theorem 4.1. The answer is no. It is answered in the last section of numerical tests of this paper, where we show that the reduced inf-sup constant goes to zero roughly at an order of the mesh size \( h \), when the uniform meshes on the unit square are tested. This is previously concluded by Chapelle and Bathe [3] too.

Now let us turn our attention to a stable mesh family. First, we consider a macroelement \( \tilde{U} \) formed by four squares \( T_1, T_2, T_3, \) and \( T_4 \). For simplicity, we take \( \tilde{U} = [0, 2] \times [0, 2] \). The degree of freedom of \( V_h \) and \( P_h \) is depicted in Fig. 6.

By moving the vertex \((0,0)\) of \( \tilde{U} \) to \((s,t)\), we obtain a new macroelement \( U \). We write function \( v \in V_h \) and \( q \in P_h \) in the form

\[
v = \sum_{i=1}^{9} \left( v_i^{(1)}, v_i^{(2)} \right) \phi_i,
q = \sum_{i=1}^{9} q_i^{(i)} \psi_i + q^{(10)} \beta_{T_1} + q^{(11)} \beta_{T_2} + q^{(12)} \beta_{T_3} + q^{(13)} \beta_{T_4}.
\]

Here \( \phi_i \) and \( \psi_i \), \( i = 1, \ldots, 9 \) are nodal basis functions, and \( \beta_{T_j}, j = 1, 2, 3, 4 \), are the bubble functions. Repeating the proof of Theorem 4.1 on \( T_2 \cup T_4 \) and \( T_3 \cup T_4 \), we would get

\[
q^{(2)} = q^{(3)} = \cdots = q^{(9)}, \quad q^{(11)} = q^{(12)} = q^{(13)}.
\]

Then testing (14) with the nodal freedom of \( v \) at node 1, it would derive \( q^{(1)} = q^{(2)} \). Therefore if \( q \in N_h \),

\[
q = q^{(1)} + q^{(10)} \beta_{T_1} + q^{(11)} (\beta_{T_2} + \beta_{T_3} + \beta_{T_4}).
\]

Finally, testing (14) with the nodal freedom of \( v \) at node 2, on the interface of \( T_1 \) and \( T_2 \), we get

\[
\frac{1}{180} \int_U q \text{div} \left( \left( v_1^{(2)}, v_2^{(2)} \right) \phi_2 \right) = \frac{1}{180} \left[ ([4 - t] q^{(10)} - 4q^{(11)} v_1^{(2)} + sq^{(10)} v_2^{(2)}) = 0, \right.
\]

for any \( v_1^{(2)} \) and \( v_2^{(2)} \) in \( \mathbb{R} \). Therefore \( q^{(10)} = q^{(11)} = 0 \) as long as \( (s,t) \neq (0,0) \). In this case, we conclude that \( \dim N_h = 1 \).

Next we extend the above \( U \) to a global mesh. Let \( O \) be a square (with size \( 4h \times 4h \)) partitioned in the way shown in the first graph of Fig. 7. Following the above arguments, we have \( \dim V_h = 1 \). Let \( \Omega \) be the unit square \([0,1] \times [0,1] \). Let \( \mathcal{Q}_h \) be a square partition of \( \Omega \) such that each square of \( \mathcal{Q}_h \) has size \( 4h \times 4h \). Further, we refine \( \mathcal{Q}_h \) by partitioning each square as a macroelement \( O \). The resulting mesh is denoted by \( \mathcal{J}_h \). The second graph of Fig. 7 is \( \mathcal{J}_h \) when \( h = 1/12 \). By the macroelement partition theorem, Theorem 2.2, the following theorem is proved.

![Fig. 6. The nodal variables of a four-square macroelement.](image)

![Fig. 7. A stable macroelement-type mesh.](image)
Theorem 4.2. Let $\Omega = [0, 1] \times [0, 1]$. If the mesh family is constructed as shown in Fig. 7, then the element 9/5c is stable on this mesh family.

Remark 4.1. Following the above idea, we can construct some other stable mesh families. By controlling the structure of a macroelement, we are able to eliminate the local “spurious pressure modes” from the discrete pressure space.

5. Numerical tests

We first compute the inf–sup constant $\gamma_h$ for both elements $9/(4c + 1)$ and $9/5c$. We then perform a simple test to compare the $9/(4c + 1)$ element and other elements. The inf–sup condition for $9/(4c + 1)$ and $9/5c$ elements is numerically tested by Chapelle and Bathe in [3]. We provide here an independent test. Our numerical results match exactly the conclusions made by Chapelle and Bathe in [3], and the numerical tests confirm the theory established in this paper.

The inf–sup constant is important not only because it tells the stability of an element but also because it directly affects the error estimates (the constant $C$ of (5) is inversely proportional to $\gamma_h^2$). In the tests, we consider the problem (1) on the unit square $\Omega = [0, 1] \times [0, 1]$. The partition $\mathcal{Q}_h$ of $\Omega$ is formed by small squares with size $h \times h$. The inf–sup constant of each method is computed for different meshsize $h$. Usually, unstable elements perform very poorly on rectangular meshes for quadrilateral elements and on diagonal meshes for triangular elements.

In order to understand the stability of an unstable element, we need to introduce the concept of reduced stability of an element, see [16, 10]. Let $M_h$ denote the $L^2$ orthogonal complement of $N_h$ in $P_h$. We define the reduced inf–sup constant $\tilde{\gamma}_h$ by

$$
\tilde{\gamma}_h = \inf_{\Theta \in M_h} \sup_{\psi \in \mathbf{V}_h} \frac{\int_{\Omega} \Theta \cdot \text{div} \psi}{\|\psi\|_{1, \Omega} \|\Theta\|_{0, \Omega}}.
$$

(17)

$\tilde{\gamma}_h > 0$ and $\tilde{\gamma}_h = \gamma_h$ if $\dim N_h = 1$. An element $\mathbf{V}_h \times P_h$ is called a reduced-stable element if the reduced inf–sup constant is bounded below by a positive number independent of $h$.

![Table 2](image)

<table>
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<th>$h$</th>
<th>$\gamma_h$</th>
<th>$\beta(\gamma_h) = Ch^\alpha$</th>
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The test results on $9/(4c + 1)$ are shown in Table 1. The inf–sup constant $\gamma_h$ is clearly bounded below by a positive number independent of $h$.

Table 2 contains the reduced inf–sup constant $\tilde{\gamma}_h$ for $9/5c$. It is very interesting to see that roughly

$$
\tilde{\gamma}_h = Ch.
$$

This fact makes the estimate (5) meaningless since the coefficient $C$ of (5) would become $1/h^2$. Therefore, the inf–sup condition fails even after one filters out the spurious pressure mode $\delta$, defined in Theorem 4.1. But as shown in Section 4, the condition holds on some macroelement types of meshes. We remark that the situation for the $9/5c$ elements is similar to that for the widely used $4/1 (= \mathbf{Q}_1/P_0)$ element. For the latter, on uniform meshes, the third singular value in the inf–sup stability condition, as well as a large number of singular values, goes to zero at order $h$ (see [9]).

Finally, we solve numerically Eq. (1) where

$$
\mathbf{u} = \text{curl} \, \mathbf{g}, \quad p = -g_{xx}, \quad g = 10(x - x^3)(y - y^3)^2,
$$

defined on the unit square $(0, 1) \times (0, 1)$. In the first part of Table 3, we listed the errors in various measures for the 9/1 element (which is the $\mathbf{Q}_2/P_0 = \mathbf{Q}_2^0(\mathcal{Q}) \cap H^1(\Omega)/Q_h^0(\mathcal{Q})$ mixed element) solutions. Because of low order approximation of the pressure, the second order for the velocity elements is wasted. Here $|\mathbf{e}_u|_{l^2}$ and $|\mathbf{e}_p|_{l^2}$ denote the maximal errors of finite element solutions to the velocity and to the pressure, respectively, at nodal points. To avoid the uncertainty of iterative methods, we use the Gaussian elimination to solve the resulting linear system.

In the second part of Table 3, we listed the errors in various measures for the $9/(4c + 1)$ mixed element (which is the $\mathbf{Q}_2/Q_1 = \mathbf{Q}_2^0(\mathcal{Q}) \cap H^1(\Omega)/Q_h^1(\mathcal{Q})$ mixed element) solutions. The approximation now is much better than that of 9/1 elements. Actually, the results are far better than that provided by the theory. This is because we use uniform square meshes. Then we would get the superconvergence because of local symmetry of meshes at each nodal point of $\mathbf{u}$ and each nodal point of $p_h$ (cf. [17]).

Part 3 of Table 3 lists the errors in various measures for the $9/(c4 + 1)$ mixed element solutions. The results for $9/(c4 + 1)$ and for $9/(4c + 1)$ are about the same, when the errors were measured in $H^1$ or $L^2$ norms. However, when comparing nodal errors, it seems that the $9/(c4 + 1)$
element did improve the $9/4c$ element. Because the pressure in $9/(4+1)$ element is discontinuous, we averaged the four values of $p_h$ at a vertex of meshes as its value to get $\bar{p}_h$, and we denoted the maximal error at vertices as $|e_p|_{\bar{p}_h}$. Since $p_h$ is continuous on each element (square), we also listed the nodal errors of $p_h$ at internal $6 \times 6$ Gaussian quadrature points of a Gaussian quadrature formula which we used in the computation. We denote the maximal nodal errors at such quadrature points as $|e_p|_{\bar{p}_h}$. In the $9/(4+1)$ element computation, we filtered out two extra pressure modes, one is the uniform constant 1 formed by $Q_1$ basis functions and the other one also the uniform constant 1 formed by $Q_0 (P_0)$ basis functions.

Acknowledgement

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References


Table 3

Convergence of mixed elements

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<th>$h$</th>
<th>$\dim V_h$</th>
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