Derivative superconvergence of rectangular finite elements for the Reissner–Mindlin plate

Zhimin Zhang,*, Shangyou Zhang

*Department of Mathematics, Texas Tech University, Lubbock, TX 79409, USA
Department of Mathematical Sciences University of Delaware, Newark, DE 19716, USA

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Abstract

The finite element method with rectangular meshes for the Reissner–Mindlin plate is analyzed. For a plate with thickness bounded below by a positive constant (moderately thick plate), derivative superconvergence of the finite element solution at the Gaussian points is justified and optimal error estimates for both rotation and displacement are established.

1. Introduction

The finite element method for the Reissner–Mindlin plate has been the focus of many investigations. There is a rich literature on this topic (see e.g. [1–3] and the references therein). It is well known that the standard finite element for the Reissner–Mindlin plate degenerates when the thickness of the plate goes to zero. This is called the locking phenomenon. Many methods have been suggested to overcome this difficulty. So far, most work regarding the finite element method for the Reissner–Mindlin plate has concentrated on treating the locking phenomenon (cf., e.g. [4–12]).

The present work is in a different direction. A square plate with thickness bounded below by a positive constant is considered, derivative superconvergence for a family of rectangular finite elements is established for a discrete norm and a semi-discrete norm. As a byproduct of the superconvergence analysis, the optimal convergence rate for the displacement and rotation is obtained.

For the superconvergence analysis, some regularity results of the exact solution are needed. As in [11], the Reissner–Mindlin plate with periodic boundary conditions is chosen as the model problem in the current work. This makes the regularity discussion simpler and enables us concentrate on the superconvergence analysis.

2. Regularity

Consider an isotropic plate occupying the region $[-\pi, \pi] \times [-t/2, t/2]$ with Young's modulus $E$ and Poisson ratio $\nu \in [0, 0.5)$. Denote by $\phi = (\phi_1, \phi_2, \phi_3)$ the rotation of the cross section normal to the midplane, by $w$ the transverse displacement of the midplane. On $\Omega = (-\pi, \pi)^2$, the Reissner–Mindlin plate model for $u = (\phi, w)$ is

* Corresponding author.
\[- \frac{D t^2}{2} [(1 - \nu) \Delta \phi_i + (1 + \nu) \nabla \cdot \phi_i] - \lambda (\nabla w_i - \phi_i) = 0 , \quad (2.1)\]

\[- \frac{\lambda}{t^2} \nabla \cdot (\nabla w_i - \phi_i) = g , \quad (2.2)\]

\[ \langle w_i, 1 \rangle = \int_{\Omega} w_i \, dx_1 \, dx_2 = 0 , \quad (2.3)\]

with the periodic boundary conditions

\[ u_i(x_1, n) - u_i(x_1, -n) , \quad u_i(n, x_2) = u_i(-n, x_2) , \quad |x_1|, |x_2| \leq \pi . \quad (2.4)\]

Here

\[ D = \frac{E}{12(1 - \nu^2)} , \quad \lambda = \frac{kE}{2(1 + \nu)} , \]

where \( k > 0 \) is the shear correction factor, \( g \) is the scaled loading function which is independent of \( t \) and satisfies the compatibility condition

\[ \langle g, 1 \rangle = 0 . \quad (2.5)\]

Let \( H^s(\Omega) , s = 0, 1, 2, \ldots \) be the usual Sobolev spaces, and let \( H^s_{+}(\Omega) \) be the corresponding space of functions with \( s \) periodic derivatives in both \( x_1 \) and \( x_2 \). By the interpolation theory [13], \( H^s(\Omega) , H^s_{+}(\Omega) \) may be defined for all real \( s \). Denote by \( \| \cdot \| \) and \( \| \| \), the semi-norm and norm, respectively. Similarly, let \( C^r(\Omega) \) denote the space of functions with \( r \) periodic continuous derivatives.

The weak form of (2.1)-(2.4) is

\[(P_t) \text{ For } t \in (0, 1) , \text{ given } g \in H^{-1}_{+}(\Omega) , \text{ find } u = (\phi, w) \in H^1_{+}(\Omega) \text{ such that } \langle w_i, 1 \rangle = 0 \text{ and }\]

\[ a(u, v) = a(\phi, \psi) + \frac{\lambda}{t^2} \langle \nabla w_i - \phi_i, \nabla z - \psi \rangle = \langle g, z \rangle , \quad \forall v = (\psi, z) \in H^1_{+}(\Omega) , \quad (2.6)\]

where

\[ a(\phi, \psi) = \frac{D}{2} \int_{\Omega} \left[ (1 - \nu)(\nabla \phi_1 \cdot \nabla \psi_1 + \nabla \phi_2 \cdot \nabla \psi_2) + (1 + \nu)\nabla \cdot \phi \nabla \cdot \psi \right] \, dx_1 \, dx_2 . \quad (2.7)\]

The following are some norms for \( v \in H^1_{+}(\Omega) \),

1. The energy norm corresponding to \( (P_t) \),

\[ \| v \|_{E,t}^2 = a(v, v) ; \]

2. For \( v \in H^s_{+}(\Omega) \),

\[ \| v \|_{s,t}^2 = \| (\psi, z) \|_{s,t}^2 = \| \psi_1 \|^2_s + \| \psi_2 \|^2_s + \| z \|^2_s , \]

\[ \| v \|_{s,t}^2 = \| (\psi, z) \|_{s,t}^2 = \| \psi_1 \|^2_s + \| \psi_2 \|^2_s + \| z \|^2_s , \]

3. For \( v \in H^{s+1}_{+}(\Omega) \),

\[ \| v \|_{s+1,t} = \| v \|_{s+1,t} + \frac{\lambda}{t^2} \| \nabla z - \psi \|_{s-1} . \]

We are able to verify that there exists a constant \( C \) such that

\[ C^{-1} \| v \|_{1,t} \leq \| v \|_{E,t} \leq Ct^{-1} \| v \|_{1,t} . \quad (2.8)\]

In other words, two norms are equivalent for any fixed \( t \).
THEOREM 2.1. Let \( g \in H^s_0(\Omega) \) with \( s \geq 1 \) satisfy (2.5). Then there exists a unique sequence of solutions \( \{u_t\} = \{(\phi_t, w_t)\} \) to (2.1)–(2.4) for \( t \in (0, 1] \) such that,

\[
\|\phi_t\|_{s,1} \leq C\|g\|_{s,2}, \quad \|w_t\|_{s} \leq C(\|g\|_{s-4} + t\|g\|_{s-2}), \quad \|u_t\|_{s,1} \leq C\|g\|_{s-3}.
\]

Here \( C \) is a constant independent of \( g \) and \( t \).

PROOF. For any \( f \in H^s_0(\Omega) \), it can be represented in terms of its Fourier series as

\[
f(x) = \sum_{k \in \mathbb{Z}^2} f^k e^{ik \cdot x}, \quad f^k = \frac{1}{4\pi^2} \langle f, e^{-ik \cdot x} \rangle, \quad k = (k_1, k_2), \quad x = (x_1, x_2).
\]

Denote \( Z^\lambda_0 - Z^\lambda_0 \{(0, 0)\} \). Then (2.1), (2.2) can be written as

\[
\sum_{k \in Z^\lambda_0 \backslash \{(0, 0)\}} \begin{bmatrix} D(2k_1^2 + (1 - \nu)k_2^2) + \frac{\lambda}{t^2} & \frac{D}{2} (1 + \nu)k_1k_2 + i\frac{\lambda}{t^2}k_1 \\ \frac{D}{2} (1 + \nu)k_1k_2 - i\frac{\lambda}{t^2}k_1 & \frac{D}{2} (2k_2^2 + (1 - \nu)k_1^2) + \frac{\lambda}{t^2} \end{bmatrix} \begin{bmatrix} \phi_t \\ \phi_{t\omega} \\ \phi_{t1} \\ \phi_{t2} \\ w_{t1} \\ w_{t2} \\ \end{bmatrix} = e^{ik \cdot x} \begin{bmatrix} 0 \\ 0 \\ \end{bmatrix},
\]

since \( w_t^\omega = 0, g^\omega = 0 \) by virtue of (2.3) and (2.5). It is easy to verify that

\[
u_t = \left( \begin{array}{c} \phi_{t1} \\ \phi_{t2} \\ \end{array} \right) = \sum_{k \in Z^\lambda_0 \backslash \{(0, 0)\}} \frac{1}{D|k|^6} \left( \begin{array}{c} ik_1 \\ ik_2 \\ \end{array} \right) g^k e^{ik \cdot x}
\]

is the solution of (2.1)–(2.4), and obviously

\[
\|w_t\|_{s} \leq C(\|g\|_{s-4} + t\|g\|_{s-2}),
\]

\[
\|u_t\|_{s+1,3} = \|\phi_{t1}\|_{s+1} + \|\phi_{t2}\|_{s+1} + \|w_t\|_{s} \leq C_1\|g\|_{s-2}^2.
\]

Further,

\[
\frac{\lambda}{t^2} (\nabla w_t - \phi_t) = \frac{\lambda}{t^2} \sum_{k \in Z^\lambda_0 \backslash \{(0, 0)\}} \frac{1}{D|k|^4} \left[ \left( 1 + \frac{Dt^2}{\lambda} |k|^2 \right) \left( \begin{array}{c} ik_1 \\ ik_2 \\ \end{array} \right) - \left( \begin{array}{c} ik_1 \\ ik_2 \\ \end{array} \right) g^k e^{ik \cdot x} \right] = \sum_{k \in Z^\lambda_0 \backslash \{(0, 0)\}} \frac{i}{|k|^2} \left( \begin{array}{c} k_1 \\ k_2 \\ \end{array} \right) g^k e^{ik \cdot x}.
\]

Hence

\[
\frac{\lambda}{t^2} \|\nabla w_t - \phi_t\|_{s-1} \leq C_2\|g\|_{s-2}.
\]

The conclusion follows by combining (2.9) with (2.10). \( \square \)

REMARK. This is a stronger result than Theorem 2.1 of [11] because of the regularity result for the shear stress term.

Without loss of generality, we assume that \((P)\) is equivalent to (2.1)–(2.4).
3. The finite element method

Let $\Delta_\xi$ and $\Delta_\eta$ be partitions of $I = (-\pi, \pi)$ such that

$$
\Delta_\xi: -\pi - x_0 < x_1 < \cdots < x_M = \pi, \quad \Delta_\eta: -\pi - y_0 < y_1 < \cdots < y_N = \pi.
$$

Denote

$$
I_i = (x_{i-1}, x_i), \quad h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, M;
$$
$$
J_j = (y_{j-1}, y_j), \quad \bar{h}_j = y_j - y_{j-1}, \quad j = 1, 2, \ldots, N;
$$
$$
h = \max (h_i, \bar{h}_j).
$$

Thus, $\Omega$ is partitioned into rectangular elements $R_{ij} = I_i \times J_j$ and this partition is denoted as $\mathcal{T}_h$. Assume that the partition is regular, i.e. there exist $C_i > 0$ independent of $h$ such that

$$
C_i^{-1} h^2 \leq |R_{ij}| = h_i \bar{h}_j \leq C_i h^2, \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N. \quad (3.1)
$$

Introduce the notation

$$
P(r) = \left\{ \sum_{0 \leq m + n \leq r} c_{mn} \xi^m \eta^n \right\}, \quad Q(r, s) = \left\{ \sum_{m=0}^{r} \sum_{n=0}^{s} c_{mn} \xi^m \eta^n \right\},
$$

and define for $s \geq -1$

$$
M^s_\eta(\Delta_\xi) = \{ g \in C^s_\eta(I), g|_{I_i} \in P(r), \quad i = 1, 2, \ldots, M \}.
$$

When $s = -1$, it is considered as lack of continuity at the nodal points $x_i$'s. In this case,

$$
M^{-1}_\eta(\Delta_\xi) = M^{-1}_\eta(\Delta_\xi) = \{ g|_{I_i} \in P(r) \quad i = 1, 2, \ldots, M \}.
$$

$M^s_\eta(\Delta_\xi)$ is defined similarly. Then the following tensor product spaces can be defined.

$$
Q^h = Q^h(\Delta) = M^{0,0}_\eta(\Delta_\xi) \otimes M^{0,0}_\eta(\Delta_\eta),
$$
$$
\Gamma^{h,-1} = \Gamma^{h,-1}(\Delta) = M^{-1,-1}(\Delta_\xi) \otimes M^{-1,-1}(\Delta_\eta) \otimes M^{-1,-1}(\Delta_\xi) \otimes M^{-1,-1}(\Delta_\eta),
$$

where $\Delta = \Delta_\xi \otimes \Delta_\eta$. Note that $Q^h$ is the space of piecewise polynomials of degree $r$ in both variables that are continuous across the element boundaries and satisfy the periodic boundary conditions, i.e.

$$
Q^h = \{ u \in H^1_\omega(\Omega), \partial u \in Q(r, r) \}.
$$

Here

$$
\tilde{v}(\xi, \eta) = v(x(\xi, \eta), y(\xi, \eta)).
$$

The affine mapping

$$
x(\xi, \eta) = \frac{x_{i-1} + x_i + h_i \xi}{2}, \quad y(\xi, \eta) = \frac{y_{j-1} + y_j + \bar{h}_j \eta}{2}, \quad (3.2)
$$

maps the reference element $\tilde{K} = [-1, 1] \times [-1, 1]$ onto element $R_{ij}$. Obviously, $\nabla Q^h \subset \Gamma^{h,-1}$.

The standard finite element problems for $(P_r)$ read:

$$
(S^h) \text{ Find } u^h_r = (\phi^h_r, w^h_r) \in (Q^h_r)^2 \times Q^h_\omega \text{ such that } \langle w^h_r, 1 \rangle = 0 \quad \text{and}
$$

$$
a(\phi^h_r, \psi) + \left\langle \nabla w^h_r - \phi^h_r, \nabla \psi - \psi \right\rangle = \langle g, z \rangle, \quad \forall \psi = (\psi, z) \in (Q^h_r)^2 \times Q^h_\omega. \quad (3.3)
$$

By the standard analysis and using (2.8), the error estimate for the above method is

$$
\| u - u^h_r \|_{1, 1} \leq \frac{C}{l} h^l \| g \|_{1, -1}. \quad (3.4)
$$
4. Superconvergence analysis

The superconvergence techniques used in the literature (see [14–16]) are generalized to treat the Reissner-Mindlin plate in this section.

Let $L_r(\xi)$ be the Legendre polynomial of order $r$ on $[-1, 1]$. Then $L_r(\xi)$ has $r$ zeroes $g_k^{(r)} \in (-1, 1)$, $k = 1, \ldots, r$, and $L'_r(\xi)$ has $r - 1$ zeroes $l_k^{(r)} \in (-1, 1)$, $k = 1, \ldots, r - 1$. Denote $l_0^{(r)} = -1$, $l_r^{(r)} = 1$, then

\[ l_0^{(r)} < g_1^{(r)} < l_1^{(r)} < \cdots < g_r^{(r)} < l_r^{(r)} < \cdots < g_{r-1}^{(r)} < l_{r-1}^{(r)} \]

We label the Gaussian points on $(-1, 1)$ as $g_k^{(r)}$, $k = 1, \ldots, r$, and the Lobatto points on $[-1, 1]$ as $l_k^{(r)}$, $k = 0, 1, \ldots, r$. Some technical lemmas are introduced first.

**Lemma 4.1.** Given $u = (\Phi, w) \in H^{r+1}_w \times H^{r+2}_w(\Omega)$, there exists an $u_i = (\Phi_i, w_i) \in (Q^h)^2 \times Q^h$ such that for any $\chi = (\chi_1, \chi_2) \in \Gamma_n^{-1}$,

\[ |(\chi, \nabla(w - w_i) - (\Phi - \Phi_i))| \leq C h^{r+1} \|\chi\|_{\Gamma_n} ||u||_{H^{r+1}, r+2}. \]

**Proof.** Choose $\Phi_i \in (Q^h)^2$ as the usual interpolation polynomial (equal distancing Lagrange interpolation) of $\Phi$, and let $w_i \in Q^h$ interpolate $w$ at the Lobatto points. For any one term in

\[ (\chi, \nabla(w - w_i)) = \sum_{j=1}^{M} \sum_{j=1}^{N} \int_{R_0} \left[ \chi_1 \frac{\partial}{\partial \xi} (w - w_i) + \chi_2 \frac{\partial}{\partial \eta} (w - w_i) \right] d\xi d\eta \]

we have

\[
\begin{align*}
&= h \int_{-1}^{1} \int_{-1}^{1} \chi_1 \frac{\partial}{\partial \xi} (w - w_i) d\xi d\eta + h \int_{-1}^{1} \int_{-1}^{1} \chi_2 \frac{\partial}{\partial \eta} (w - w_i) d\xi d\eta \\
&= h E_1(\hat{w}) + h E_2(\hat{w}).
\end{align*}
\]

For any $\chi \in \Gamma_n^{-1}$, $E_1(\hat{w})$ and $E_2(\hat{w})$ are linear functionals on $H^{r+1}(\hat{K})$. We shall prove that

\[ E_1(\hat{w}) = 0, \quad E_2(\hat{w}) = 0, \quad \forall \hat{w} \in P(r+1). \]

Since $\hat{w} \in Q(r, r)$, it is necessary to verify (4.3) for terms $\xi^{r+1}$ and $\eta^{r+1}$. Observe that

\[ \frac{\partial}{\partial \xi}(\eta^{r+1}) = 0, \quad \frac{\partial}{\partial \eta}(\xi^{r+1}) = 0, \]

we only need to verify

\[ E_1(\xi^{r+1}) = 0, \quad E_2(\eta^{r+1}) = 0. \]

Since $\hat{w}$ is the Lobatto interpolation of $\hat{w}$,

\[ \xi^{r+1} - (\xi^{r+1})_{l} = \frac{1}{r\alpha_r} (\xi^{r+1})_{l} = \frac{1}{r\alpha_r} (\xi^{r+1})_{l} = \frac{r+1}{\alpha_r} L_r(\xi). \]

Here $\alpha_r$ is the coefficient for the term $\xi^r$ in the Legendre polynomial $L_r(\xi)$. Hence, by the property of the Legendre polynomial,

\[ \frac{\partial}{\partial \xi}(\xi^{r+1} - (\xi^{r+1})_{l}) = \frac{1}{r\alpha_r} \frac{\partial}{\partial \xi} [(\xi^{r+1})_{l}] = \frac{r+1}{\alpha_r} L_r(\xi). \]

So

\[ E_1(\xi^{r+1}) = \frac{r+1}{\alpha_r} \int_{-1}^{1} \int_{-1}^{1} \hat{\chi}_1 L_r(\xi) d\xi d\eta = 0, \]
since \( \chi_i \in Q(r-1, r) \). The argument for \( E_2(\eta^{r+1}) = 0 \) is the same. By the Bramble–Hilbert Lemma,

\[
|E_i(\hat{w})| \leq C \| \chi \|_{0, R} \| \hat{w} \|_{r+2, R} \leq Ch' \| \chi \|_{0, R} \| w \|_{r+2, R}, \quad i = 1, 2.
\]  

(4.4)

Combining (4.2), (4.4) with (4.1), we have

\[
|\langle \chi, \nabla(w - w_i) \rangle| \leq \sum_{i=1}^{M} \sum_{j=1}^{N} Ch'^{-1} \| \chi \|_{0, R} \| w \|_{r+2, R} = Ch'^{-1} \| \chi \|_{0} \| w \|_{r+2}.
\]

(4.5)

For the other term, by a standard interpolation result,

\[
|\langle \chi, \Phi - \phi_i \rangle| \leq Ch'^{-1} \| \chi \|_{0} \| \Phi \|_{r+1}.
\]

(4.6)

The conclusion follows from (4.5) and (4.6). \( \Box \)

**Lemma 4.2.** Let \( \Phi = (\phi_1, \phi_2) \in H_n^{r+2}(\Omega)^2 \) and let \( \Phi_i \) be its Lobatto interpolant. Then

\[
|a(\Phi - \Phi_i, \Psi)| \leq Ch'^{-1} \| \Phi \|_{r+2} \| \Psi \|_1, \quad \forall \Psi = (\psi_1, \psi_2) \in (Q_n^2)^2.
\]

**Proof.** Set \( \mu = \Phi - \Phi_i \), then \( a(\mu, \Psi) \) contains terms

\[
\frac{\partial \mu_1}{\partial x}, \frac{\partial \psi_1}{\partial x}, \frac{\partial \mu_1}{\partial y}, \frac{\partial \psi_1}{\partial y}, \frac{\partial \mu_2}{\partial x}, \frac{\partial \psi_2}{\partial x}, \frac{\partial \mu_2}{\partial y}, \frac{\partial \psi_2}{\partial y}, \frac{\partial \mu_1}{\partial x}, \frac{\partial \psi_1}{\partial x}, \frac{\partial \mu_2}{\partial y}, \frac{\partial \psi_2}{\partial y}.
\]

(4.7)

First we consider

\[
\int_{\Omega} \frac{\partial \mu_1}{\partial x} \frac{\partial \psi_1}{\partial x} \, dx \, dy = \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{R_i} \frac{\partial \mu_1}{\partial x} \frac{\partial \psi_1}{\partial x} \, dx \, dy
\]

\[
= \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{h_i}{h_j} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \mu_1}{\partial \xi} \frac{\partial \psi_1}{\partial \xi} \, d\xi \, d\eta = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{h_i}{h_j} E(\Phi_i).
\]

\( E(\Phi_i) \) is a linear functional on \( H_n^{r+2}(\Omega) \). It vanishes for \( \Phi_i \in Q(r, r) \) since \( \mu_1 = 0 \), and vanishes for \( \Phi_i = \eta^{r+1} \) since \( \partial \mu_1 / \partial \xi = 0 \). If \( \Phi_i = \eta^{r+1} \), as in the proof for Lemma 4.1, we have

\[
(\xi^{r+1})_\xi = \xi^{r+1} - \frac{1}{r+1} (\xi^2 - 1)L'_\xi(\xi),
\]

(4.8)

\[
\frac{\partial \mu_1}{\partial \xi} = \frac{1}{r+1} \frac{\partial}{\partial \xi} \left[ (\xi^2 - 1)L'_\xi(\xi) \right] = \frac{r+1}{\alpha_r} L'_\xi(\xi),
\]

and

\[
E(\Phi_i) = \frac{1}{\alpha_r} \int_{-1}^{1} \int_{-1}^{1} L'_\xi(\xi) \frac{\partial \psi_1}{\partial \xi} \, d\xi \, d\eta = 0,
\]

because \( \partial \psi / \partial \xi \in Q(r-1, r) \). By the Bramble–Hilbert Lemma,

\[
|E(\psi_i)| \leq C \| \psi_i \|_{r+2, R} \| \Phi_i \|_{r+2, R} \leq Ch'^{-1} \| \psi_i \|_{r+2, R} \| \Phi_i \|_{r+2, R}.
\]

(4.9)

By Cauchy's inequality,

\[
\int_{\Omega} \frac{\partial \mu_1}{\partial x} \frac{\partial \psi_1}{\partial x} \, dx \, dy \leq Ch'^{-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \| \Phi_i \|_{r+2, R} \| \psi_i \|_{r+2, R} \leq Ch'^{-1} \| \Phi_i \|_{r+2} \| \psi_i \|_{r+2}.
\]

(4.10)

Next denote functionals
\[ F_1(\hat{\phi}_1) = \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \hat{u}_1}{\partial \xi} \frac{\partial \hat{\psi}_2}{\partial \eta} \, d\xi \, d\eta. \]

\[ F_2(\hat{\phi}_1) = -\int_{-1}^{1} \frac{\partial \hat{\mu}_1}{\partial \xi} (\xi, 1) [\hat{\psi}_2(\xi, 1) - \hat{\psi}_2(0, 1)] \, d\xi - \int_{-1}^{1} \frac{\partial \hat{\mu}_1}{\partial \xi} (\xi, -1) [\hat{\psi}_2(\xi, -1) - \hat{\psi}_2(0, -1)] \, d\xi. \]

Then

\[ \int_{\Omega} \frac{\partial \hat{\mu}_1}{\partial x} \frac{\partial \hat{\psi}_2}{\partial y} \, dx \, dy = \sum_{i=1}^{M} \sum_{j=1}^{N} [F_1(\hat{\phi}_1) - F_2(\hat{\phi}_1)] = \sum_{i=1}^{M} \sum_{j=1}^{N} F(\hat{\phi}_1). \]

We could subtract the sum \( \Sigma_{i=1}^{M} \Sigma_{j=1}^{N} F_2(\hat{\phi}_1) \) because it is zero. In fact, terms appear as coupled integrals over a common side of two adjacent elements taken in opposite directions with integrants which are the same, and the boundary terms are canceled by the periodic condition.

\( F(\hat{\phi}_1) \) is a continuous linear functional on \( H^{r+\alpha}(\hat{\Omega}) \). It vanishes for \( Q(r, r) \) functions and \( \eta^{r+\alpha} \). For \( \hat{\phi}_1 + \xi^{r+\alpha} \), using (4.8),

\[ F(\hat{\phi}_1) = \left( \int_{-1}^{1} L_r(\xi) \int_{-1}^{1} \frac{\partial \hat{\psi}_2}{\partial \eta} \, d\eta \right) \left( \int_{-1}^{1} L_r(\xi) [\hat{\psi}_2(\xi, 1) - \hat{\psi}_2(\xi, -1)] \, d\xi \right) = 0. \]

Again, by the Bramble–Hilbert Lemma and the Cauchy inequality, we have

\[ \left| \int_{\Omega} \frac{\partial \hat{\mu}_1}{\partial x} \frac{\partial \hat{\psi}_2}{\partial y} \, dx \, dy \right| \leq C h^{r+1} \| \hat{\phi}_1 \|_{r+\alpha} \| \hat{\psi}_2 \|_{r+\alpha}. \quad (4.11) \]

The other terms of (4.7) can be estimated similarly as (4.10) or (4.11) and the conclusion follows.

The derivative superconvergence will be established in a semi-discrete norm and a discrete norm defined in the following.

Define for \( w \in H^{1}_h(\Omega) + Q^h \),

\[ |w|^2 = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{1}{2} A_i A_j \| \hat{w}_i \|_{L^2} \left( \int_{-1}^{1} \left[ \frac{\partial \hat{w}_i}{\partial \xi} (g_i^{(r)}, \eta) \right]^2 \, d\eta \right) \left( \int_{-1}^{1} \left[ \frac{\partial \hat{w}_j}{\partial \eta} (g_j^{(r)}, \xi) \right]^2 \, d\xi \right). \]

Note that the above definition uses the \( r \)-point Gauss quadrature rule \( (A_i^{(r)} \) are weights) in the \( x \)-direction and exact integration in the \( y \)-direction to treat the term \( (\partial w/\partial x)^2 \), and uses the \( r \)-point Gauss quadrature rule in the \( y \)-direction and exact integration in the \( x \)-direction to treat the term \( (\partial w/\partial y)^2 \). Obviously

\[ |w|^2 = |w|^2, \quad \forall \ w \in Q^h. \]

and hence \( \| \cdot \|_h \) is a norm on \( \tilde{Q}^h = \{ w \in Q^h, \langle w, 1 \rangle = 0 \} \).

Also define for \( w \in H^{1}_h(\Omega) + Q^h \),

\[ |w|^2 = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{1}{4} A_i A_j \| \hat{w}_i \|_{L^2} \left( \int_{-1}^{1} \left[ \frac{\partial \hat{w}_i}{\partial \xi} (g_i^{(r)}, \eta) \right]^2 \, d\eta \right) \left( \int_{-1}^{1} \left[ \frac{\partial \hat{w}_j}{\partial \eta} (g_j^{(r)}, \xi) \right]^2 \, d\xi \right). \]

Clearly, \( \| \cdot \|_h \) uses the \( r \)-point Gauss quadrature rule for both \( x \) and \( y \). Thus, \( \| \cdot \|_h \) is a semi-norm on \( \tilde{Q}^h \) and

\[ |w|^2 \leq C |w|, \quad \forall \ w \in Q^h, \]

where \( C \) is a constant independent of \( h \) and \( w \).
REMARK. For \( r = 1 \) and \( MN \) being even, \( | \cdot |_* \) is not a norm on \( Q^h_{w} \). In fact, for \( w \in Q^h_{w} \) satisfying \( \hat{w} = (-1)^{i} \xi \eta, 1 \leq i \leq M, 1 \leq j \leq N \), one gets \( |w|_* = 0 \). But the fact that it is a semi-norm is sufficient for the error estimates in this paper.

To verify (4.15), we use (3.1), the fact \( A^l_i > 0 \), \( l = 1, \ldots, r \) and the inverse inequality,

\[
|w|_* \leq \left( \sum_{i=1}^{M} \sum_{j=1}^{N} |Vw|_{L^2(R_{i,j})}^2 |R_{i,j}| \right)^{1/2} \leq C \left( \sum_{i=1}^{M} \sum_{j=1}^{N} |Vw|_{L^2(R_{i,j})}^2 \right)^{1/2} = C|w|_1.
\]

Further, define for \( \phi \in (H^3_{\eta}(\Omega) + Q^h_{\eta})^2 \),

\[
|\phi|_h^2 = |\phi_1|^2 + |\phi_2|^2, \quad |\phi|_*^2 = |\phi_1|^2 + |\phi_2|^2,
\]

and define for \( u = (\phi, w) \in H^3_{\eta}(\Omega) + (Q^h_{\eta})^3 \),

\[
|u|_h^2 = |\phi|^2 + |w|^2, \quad |u|_*^2 = |\phi|^2 + |w|^2.
\]

Obviously, \( |u|_* = |u|_{1,1} \) if \( u \in (Q^h_{\eta})^3 \), and

\[
|u|_* \leq C|u|_{1,1}, \quad \forall \, u \in (Q^h_{\eta})^3. \tag{4.16}
\]

**Lemma 4.3.** For \( u = (\phi, w) \in H^{r+2, r+2}_{\eta}(\Omega), \ r \geq 1, \) let \( u_i = (\phi_i, w_i) \in (Q^h_{\eta})^2 \times Q^h_{\eta} \) be its Lobatto interpolant. Then

\[
|u - u_i| \leq Ch^{r+1}|u|_{r+2, r+2}, \quad |u - u_i|_* \leq Ch^{r+1}|u|_{r+2, r+2}.
\]

**Proof.** Only an estimate \( |w - w_i|_* \leq Ch^{r+1}|w|_{r+2} \) will be proved. The estimates for terms \( |w - w_i|_h, |\phi - \phi_i|_h, \) and \( |\phi - \phi_i|_* \) are similar. From definition (4.14), we need to estimate the functional

\[
E(\hat{w}) = \frac{\partial}{\partial \xi}(\hat{w} - \hat{w}_i)(g_{m}^{(r)}, g_{n}^{(r)}).
\]

It is bounded on \( H^{r+2, R}(\hat{R}) \) for \( r \geq 1 \), and it vanishes for \( \hat{w} \in Q(r, r) \) and for \( \hat{w} = \eta^{r+1} \). By (4.8), it is also vanishes for \( \hat{w} = \xi^{r+1} \) because the coordinates \( g_{m}^{(r)}, l = 1, \ldots, r \) are zeros of \( L_\eta \). Hence \( E(\hat{w}) \) vanishes for \( P(r+1) \) and, according to the Bramble–Hilbert Lemma,

\[
\left| \frac{\partial}{\partial \xi}(\hat{w} - \hat{w}_i)(g_{m}^{(r)}, g_{n}^{(r)}) \right| \leq C|\hat{w}|_{r+2, R} \leq Ch^{r+1}|w|_{r+2, R_{\eta}}.
\]

In the same way, we get

\[
\left| \frac{\partial}{\partial \eta}(\hat{w} - \hat{w}_i)(g_{m}^{(r)}, g_{n}^{(r)}) \right| \leq Ch^{r+1}|w|_{r+2, R_{\eta}}.
\]

From (4.14) and (3.1), we obtain

\[
|w - w_i|_* \leq C \left( \sum_{i=1}^{M} \sum_{j=1}^{N} h^{2(r+1)}|w|_{r+2, R_{i,j}}^2 \right)^{1/2} \leq Ch^{r+1}|w|_{r+2}.
\]

The estimate for the term \( |\phi - \phi_i|_* \) can be proved similarly. For the case \( | \cdot |_h \), we need to estimate

\[
E_1(\hat{w}) = \int_{-1}^{1} \frac{\partial}{\partial \xi}(\hat{w} - \hat{w}_i)(g_{m}^{(r)}, \eta) \, d\eta, \quad E_2(\hat{w}) = \int_{-1}^{1} \frac{\partial}{\partial \eta}(\hat{w} - \hat{w}_i)(\xi, g_{n}^{(r)}) \, d\xi.
\]

but the argument is similar. \( \square \)

Now we are ready to prove the main theorems of this paper.
THEOREM 4.4. Let \( u^h = (\phi^h, w^h) \) be the solution of \((S^h)\), let \( u_i = (\phi_i, w_i) \) be the Lobatto interpolant of the solution of \((P)\). Then there exists a constant \( C \) independent of \( h \) and \( g \) such that
\[
\| u^h - u_i \|_{E,1} \leq C h^{r+1}(\|g\| + t^{-1} \|g\|_{r-1} + t^{-2} \|g\|_{r-3}).
\]

PROOF. From the assumption, we have, for all \( v = (\psi, z) \in (Q^n, X) \times Q^n x Q^n, \)
\[
a(\phi^h, \psi) + \frac{\lambda}{t^2} \langle \nabla w^h - \phi^h, \nabla z - \psi \rangle = \langle g, z \rangle
\]
\[
= a(\phi_i, \psi) + \frac{\lambda}{t^2} \langle \nabla w_i - \phi_i, \nabla z - \psi \rangle . \tag{4.17}
\]
Subtracting
\[
a(\phi^h, \psi) + \frac{\lambda}{t^2} \langle \nabla w^h - \phi^h, \nabla z - \psi \rangle
\]
from both sides of (4.17), we have
\[
a(\phi^h - \phi_i, \psi) + \frac{\lambda}{t^2} \langle \nabla (w^h - w_i) - (\phi^h - \phi_i), \nabla z - \psi \rangle
\]
\[
= a(\phi_i - \phi_i, \psi) + \frac{\lambda}{t^2} \langle \nabla (w_i - w_i) - (\phi_i - \phi_i), \nabla z - \psi \rangle
\]
\[
= a(\phi^h - \phi_i, \psi) + \frac{\lambda}{t^2} \langle \nabla (w_i - w_i) - (\phi_i - \phi_i), \nabla z - \pi^n \psi \rangle
\]
\[
+ \frac{\lambda}{t^2} \langle \nabla (w_i - w_i) - (\phi_i - \phi_i), \nabla \pi^n \psi - \psi \rangle , \tag{4.18}
\]
where \( \pi^n \) is the \( L^2 \) orthogonal projection operator from \( Q^n \times Q^n \) to \( I^{n-1} \). Setting \( \psi = \phi^h - \phi_i \) and \( z = w_i - w_i \) in (4.18), we have
\[
\| u^h - u_i \|_{E,1}^2 = a(\psi, \psi) + \frac{\lambda}{t^2} \| \nabla \psi \|_0^2
\]
\[
= a(\phi^h - \phi_i, \psi) + \frac{\lambda}{t^2} \langle \nabla (w_i - w_i) - (\phi_i - \phi_i), \nabla z - \pi^n \psi \rangle
\]
\[
+ \frac{\lambda}{t^2} \langle \nabla (w_i - w_i) - (\phi_i - \phi_i), \nabla \pi^n \psi - \psi \rangle . \tag{4.19}
\]
For the first term on the right-hand side of (4.19), we have, from Lemma 4.2 and the regularity result,
\[
|a(\phi_i - \phi_i, \psi)| \leq C h^{r+1} \| \phi^h_i \|_{r+2} \| \psi \|_1 \leq C h^{r+1} \| g \|_{r-1} \| \psi \|_1 . \tag{4.20}
\]
For the second term on the right-hand side of (4.19), we have, from Lemma 4.1 with \( \chi = \nabla z - \pi^n \psi \), and the regularity result,
\[
|\langle \nabla (w_i - w_i) - (\phi_i - \phi_i), \nabla \pi^n \psi \rangle|
\]
\[
\leq C h^{r+1} \| u^h_i \|_{r+1, r+2} \| \nabla \pi^n \psi \|_0
\]
\[
\leq C h^{r+1} (\| g \|_{r-2} + t \| g \|_r) \| \nabla \pi^n \psi \|_0 . \tag{4.21}
\]
Applying the standard interpolation theory to the last term in (4.19) yields,
\[
|\langle \nabla (w_i - w_i) - (\phi_i - \phi_i), \pi^n \psi - \psi \rangle|
\]
\[
\leq C h^{r+1} (\| w^h_i \|_{r+1} + \| \phi_i \|_r) \| \psi \|_1 \leq C h^{r+1} (t \| g \|_{r-1} + \| g \|_{r-3}) \| \psi \|_1 . \tag{4.22}
\]
Then we have from (4.20)–(4.22),
THEOREM 4.5. Let \( u_i, u^h_i \) be solutions of \((P_i), (S^h_i)\), respectively. Then there exists a constant \( C \) independent of \( h, t \) and \( g \) such that

\[
\|u_i - u^h_i\|_{1,1} \leq C h^{-1}(\|g\|_r + t^{-1}\|g\|_{r-1} + t^{-2}\|g\|_{r-3}) ;
\]

(4.23)

\[
\|u_i - u^h_i\|_a \leq C h^{-1}(\|g\|_r + t^{-1}\|g\|_{r-1} + t^{-2}\|g\|_{r-3}) ;
\]

(4.24)

\[
\|u_i - u^h_i\|_{0,0} \leq C h^{-1}(\|g\|_r + t^{-1}\|g\|_{r-1} + t^{-2}\|g\|_{r-3}) .
\]

(4.25)

PROOF. First from (2.8) and Theorem 4.4,

\[
\|u^h_i - u_i\|_{E,i} \approx \|u^h_i - u_i\|_{E,i} \leq C h^{-1}(\|g\|_r + t^{-1}\|g\|_{r-1} + t^{-2}\|g\|_{r-3}) .
\]

(4.26)

By the triangle inequality, Lemma 4.3, and (4.26),

\[
\|\phi_i - \phi^h_i\|_n \leq \|\phi_i - \phi_h\|_n + \|\phi_h - \phi^h_i\|_n
\]

\[
\leq C h^{-1}(\|\phi_i\|_{r+2} + \|\phi_h - \phi^h_i\|_1)
\]

\[
\leq C h^{-1}(\|g\|_r + t^{-1}\|g\|_{r-1} + t^{-2}\|g\|_{r-3}) .
\]

In the same way, we can prove that

\[
\|w_i - w^h_i\|_n \leq C h^{-1}(\|g\|_r + t^{-1}\|g\|_{r-1} + t^{-2}\|g\|_{r-3}) .
\]

Hence we have proved (4.23). The argument for (4.24) is the same. For (4.25), we apply the triangle inequality, standard interpolation theory, and (4.26) to get

\[
\|\phi_i - \phi^h_i\|_0 \leq \|\phi_i - \phi_l\|_0 + \|\phi_l - \phi^h_i\|_0
\]

\[
\leq C h^{-1}(\|\phi_i\|_{r+1} + \|\phi_l - \phi^h_i\|_0)
\]

\[
\leq C h^{-1}(\|g\|_r + t^{-1}\|g\|_{r-1} + t^{-2}\|g\|_{r-3}) .
\]

The estimate for \( \|w_i - w^h_i\|_0 \) is the same. \( \square \)

REMARK. Usually, the \( L^2 \) estimate (4.25) is achieved by the duality argument. Here it comes out as a byproduct of the superconvergence analysis.

The above theorem tells us that the Gaussian points are derivative superconvergence points for the Reissner-Mindlin plate just as they are for the linear elastic equation and the Poisson equation. The result here makes it possible to apply a superconvergence recovery technique developed recently by O.C. Zienkiewicz and J.Z. Zhu [17] to the Reissner-Mindlin plate. This leads further to an a posteriori error estimate of the Reissner-Mindlin plate. We would like to point out that since the error estimate in Theorem 4.5 is \( t \) dependent, it only works for plates of moderate thickness, i.e. for a family of plates with thickness bounded below by a fixed positive constant \( t_0 \).
5. Numerical experiments

Instead of performing a test on the periodic setting, we choose a more practical and popular model, the clamped unit square plate with a uniformly distributed load. Also, we shall investigate the pointwise superconvergence property of the model. In this sense, the numerical experiments here can be seen as a further extension of the analysis.

In all our calculations, we choose the Young's modulus $E = 1$, the Poisson ratio $\nu = 0.3$, and the shear correction factor $k = 5/6$. We only consider the simplest case, the bilinear element where the Gauss point is the element center. As a comparison, we will also perform the same computations for the Bathe–Dvorkin (BD) element (it is equivalent to the Hughes–Tezduyar element [9] for the rectangular mesh). Because of the boundary layer behavior (cf. [18]), we can only observe the superconvergence at an interior point which we choose as the quarter point (0.25, 0.25). In order to make this point an element center, we subdivide the unit square into $n \times n$ uniform squares with $n = 6, 10, 14, 18, 22, 26$. Then we calculate the rotation $\phi^h_{11}$, its derivative $\partial \phi^h_{11} / \partial x$, the derivative of the transverse displacement $\partial w^h_{11} / \partial x$, and the shear stress $\gamma^h_{11}$ for $t = 0.2, 0.1, 0.05, 0.01$. We list some computed data for $t = 0.1$ and $t = 0.01$ in Tables 1 and 2. Note that all values are multiplied by 100.

### Table 1
Computed data [(1/2, 1/2) as an element vertex]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\partial w^h_{11} / \partial x(1/4, 1/4)$</th>
<th>$(\phi^h_{11} - \partial w^h_{11} / \partial x)(1/2, 1/2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bilinear</td>
<td>BD</td>
</tr>
<tr>
<td>-----</td>
<td>----------</td>
<td>--------</td>
</tr>
<tr>
<td>$t = 0.1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.262245</td>
<td>-0.467991</td>
</tr>
<tr>
<td>10</td>
<td>-0.384584</td>
<td>-0.491996</td>
</tr>
<tr>
<td>14</td>
<td>-0.436259</td>
<td>-0.497945</td>
</tr>
<tr>
<td>18</td>
<td>-0.461020</td>
<td>-0.500336</td>
</tr>
<tr>
<td>22</td>
<td>-0.474488</td>
<td>-0.501535</td>
</tr>
<tr>
<td>26</td>
<td>-0.482544</td>
<td>-0.502220</td>
</tr>
<tr>
<td>$t = 0.01$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.005888</td>
<td>-0.446601</td>
</tr>
<tr>
<td>10</td>
<td>-0.017378</td>
<td>-0.491832</td>
</tr>
<tr>
<td>14</td>
<td>-0.033553</td>
<td>-0.497359</td>
</tr>
<tr>
<td>18</td>
<td>-0.053449</td>
<td>-0.499369</td>
</tr>
<tr>
<td>22</td>
<td>-0.076014</td>
<td>-0.500336</td>
</tr>
<tr>
<td>26</td>
<td>-0.100224</td>
<td>-0.500879</td>
</tr>
</tbody>
</table>

### Table 2
Computed data at (1/4, 1/4) (an element center)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\partial w^h_{11} / \partial x$</th>
<th>$\phi^h_{11}$</th>
<th>$\gamma^h_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bilinear</td>
<td>BD</td>
<td>Bilinear</td>
</tr>
<tr>
<td>-----</td>
<td>----------</td>
<td>--------</td>
<td>----------</td>
</tr>
<tr>
<td>$t = 0.1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.155847</td>
<td>0.238653</td>
<td>0.134266</td>
</tr>
<tr>
<td>10</td>
<td>0.207711</td>
<td>0.248954</td>
<td>0.184892</td>
</tr>
<tr>
<td>14</td>
<td>0.228342</td>
<td>0.251694</td>
<td>0.205571</td>
</tr>
<tr>
<td>18</td>
<td>0.238040</td>
<td>0.252824</td>
<td>0.212449</td>
</tr>
<tr>
<td>22</td>
<td>0.243263</td>
<td>0.253398</td>
<td>0.220453</td>
</tr>
<tr>
<td>26</td>
<td>0.246371</td>
<td>0.253728</td>
<td>0.223551</td>
</tr>
<tr>
<td>$t = 0.01$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.003384</td>
<td>0.203393</td>
<td>0.003158</td>
</tr>
<tr>
<td>10</td>
<td>0.009194</td>
<td>0.213297</td>
<td>0.008971</td>
</tr>
<tr>
<td>14</td>
<td>0.017339</td>
<td>0.215848</td>
<td>0.017118</td>
</tr>
<tr>
<td>18</td>
<td>0.027260</td>
<td>0.216885</td>
<td>0.027040</td>
</tr>
<tr>
<td>22</td>
<td>0.038865</td>
<td>0.217411</td>
<td>0.038145</td>
</tr>
<tr>
<td>26</td>
<td>0.050098</td>
<td>0.217713</td>
<td>0.049878</td>
</tr>
</tbody>
</table>
In Fig. 1, we plot the convergence rate for the derivative of the rotation $\partial \phi^h_{i1}/\partial x$ at the quarter point (as an element center) for $t = 0.1$ and $t = 0.01$. We see that the bilinear element exhibits a quadratic convergence rate for $t = 0.1$ but 'locks' in the case of $t = 0.01$, and the BD element shows quadratic convergence for both cases. It seems that the derivative superconvergence of the BD element is uniform with respect to $t$. We shall see the same phenomenon in Fig. 3 for $\partial w^h_i/\partial x$. Next, we graph, in Fig. 2 (for $t = 0.1$) and Fig. 3 (for $t = 0.01$), the convergence rate for the rotation $\phi^h_{i1}$ ($U$ denotes its nodal error) and the derivative of the transverse displacement $\partial w^h_i/\partial x$ ($dW/dx$ denotes its nodal error) at $(0.25, 0.25)$ as an element center. Here all values of $U$ are multiplied by 0.7 since the curves for $U$ and $dW/dx$ actually overlap. Furthermore, in order to show the convergence rate clearly, we multiply all the data in Figs. 1–3 by $10^4$. Again, superconvergence is observed for both the bilinear element and the BD element when $t = 0.1$ and the bilinear element shows locking in the case $t = 0.01$. We also plot the nodal error of $\phi^h_{i1} - \partial w^h_i/\partial x$ ($U - dW/dx$) at $(0.5, 0.5)$ as an element vertex. The convergence rate is linear as expected and the bilinear element has locking for $t = 0.01$.

We investigate the convergence for the shear stress $\gamma^h_{i1}$ in Figs. 4–7. We plot the computed stress for...
different values of $t$ at $(0.25, 0.25)$ as an element center and at $(0.5, 0.5)$ as an element vertex. Here all values are multiplied by $10^3$. As expected, the BD element converges quadratically at the element center and linearly at the element vertex uniformly with respect to $t$. A surprising fact is that the bilinear element does not have locking for the shear stress at the element center (it does have locking at the element vertex), the calculated values for the shear stress are even a little better than that of from the BD element. But in order to observe the quadratic convergence rate for the bilinear element, $h$ must sufficiently small when $t$ is small.

In all our graphs, the 'exact' solutions are based on the computed data and the extrapolation technique.

Some observations

(1) The derivative superconvergence for the linear element can only be observed when $h$ is sufficiently small for small $t$. 

![Fig. 3. Convergence rate for the rotation and $dw/dx$ ($t = 0.01$).](image)

![Fig. 4. Convergence rate for the shear stress ($t = 0.2$).](image)
(2) The derivative superconvergence for the Bathe–Dvorkin element is independent of $t$ although we do not have the theoretical justification for this.

(3) Away from the boundary-layer, bilinear element is able to predict quite accurate shear stress at the center of the interior element even for very small $t$, in spite of the locking for both rotations and the displacement.

As a final remark, we refer the reader to [20] for a superconvergence analysis on the arch beam models which includes the Timoshenko beam, the one-dimensional version of the Reissner–Mindlin plate model, as a special case. Also see [19] for a computer-aided approach to find all superconvergence points of gradients and stresses for plane elasticity.
Acknowledgments

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References