The $p$ and $h$–$p$ versions of some finite element methods for Stokes’ problem

Søren Jensen *,1
Department of Mathematics, University of Maryland Baltimore County, Baltimore, MD 21228, USA

Shangyou Zhang 2
Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

Abstract

We investigate discretizations of Stokes’ problem (as a sample saddle point problem) from the point of view of stability of increasing order mixed finite element methods. For the cases of velocity–pressure space pairs where the stability can be characterized in terms of the growth of a right-inverse of the divergence operator, we exhibit algebraic instabilities of varying size depending on boundary conditions and the kind of element employed.

1. Introduction

Some interesting questions arise in problems from continuum mechanics in which an incompressibility constraint is enforced—or nearly so—such as viscous, incompressible fluid flows or nearly incompressible, plane elastic deformations. We shall study certain high-order discretizations of steady Stokes flows in two and three space dimensions.

One way to achieve high accuracy is to use a mixed finite element technique with high-order piecewise-polynomials on a subdivision of the domain. This note will concentrate on the cause and effects of lack of stability (inf-sup constant going to zero as the dimension parameter tends to infinity). We consider Stokes’ problem on a bounded, open, simply connected, polygonal or polyhedral, domain $\Omega$ in $\mathbb{R}^D$, where the dimension $D = 2$ or 3. The velocity ($U$)–pressure ($P$) formulation with kinematic viscosity $\nu$ is

\begin{align}
-\nu \Delta U + \nabla P &= F \quad \text{in } \Omega \subseteq \mathbb{R}^D, \\
\nabla \cdot U &= 0 \quad \text{in } \Omega,
\end{align}

(1)

along with some appropriate boundary conditions (no-slip or stress-free, e.g.) on $\partial \Omega$. These influence the choice of $\mathcal{V} \subseteq [H^1(\Omega)]^D$ and $\mathcal{W} \subseteq L^2(\Omega)$ in the weak formulation, e.g. [1]. Using a conforming mixed method discretization, one chooses $\mathcal{V}_N \subseteq \mathcal{V}$, $\mathcal{W}_N \subseteq \mathcal{W}$ and

* Corresponding author.

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find $U_N \in \mathcal{V}_N$ and $P_N \in \mathcal{W}_N$ so that

$$a(U_N, \nu) + b(\nu, P_N) = (F, \nu) \quad \forall \nu \in \mathcal{V}_N,$$

$$b(U_N, q) = 0 \quad \forall q \in \mathcal{W}_N.$$  \hfill (2)

The bilinear forms $a$ and $b$ are given by

$$a(U, \nu) = 2\nu \int \sum_{i,j} e_{ij}(U) e_{ij}(\nu) \, dx,$$

where $(e_{ij})$ is the symmetric part of the velocity (time-rate of change of the deformation) gradient, and

$$b(\nu, P) = -\int \nabla \cdot \nu P \, dx.$$

It now becomes natural to wonder about the questions of stability and approximability. One may ask for stability in the following form,

$$\|U - U_N\|_{H^1} + \|P - P_N\|_{L^2} \leq C \left\{ \min_{\nu \in \mathcal{V}_N} \|U - \nu\|_{H^1} + \min_{q \in \mathcal{W}_N} \|P - q\|_{L^2} \right\},$$

with $C$ independent of $N$, to be guaranteed by

$$\min_{q \in \mathcal{W}_N \setminus \{0\}} \max_{\nu \in \mathcal{V}_N \setminus \{0\}} \frac{\int \nabla \cdot \nu q \, dx}{\|\nu\|_{H^1} \|q\|_{L^2}} \geq \mu > 0,$$  \hfill (3)

$\mu$ independent of $N$ [2,3]. Take $\mathcal{V}_N = \nabla \cdot \mathcal{V}_N.$ Then the Babuška-Brezzi condition (3) is equivalent to $\nabla \cdot : \mathcal{V}_N \to \mathcal{W}_N$ having right inverses (for any given $q \in \mathcal{W}_N$ choose any $\nu \in \mathcal{V}_N$ with minimal $H^1(\Omega)$ norm and $\nabla \cdot \nu = q$)

$$(\nabla \cdot)^{-1} : \mathcal{W}_N \to \mathcal{V}_N \quad \text{with} \quad \|((\nabla \cdot)^{-1})\|_{B(L^2; H^1)} \leq C,$$ \hfill (4)

uniformly bounded in $B(L^2; H^1)$. The equivalence of (3) with (4) was shown in [1]. There one finds a lucid discussion of some of the troubles encountered when one tries to use low-order polynomials and decreasing mesh size. They show that if one takes an $h$ version with the right combination of subspaces of polynomials of sufficiently high (but fixed) degree, then the inf-sup condition is satisfied. See the references in [1]. We are here interested in a situation with sequences of subspaces of increasing polynomial degree and fixed mesh size. They show that if one takes an $h$ version with the right combination of subspaces of polynomial degree and a fixed mesh (sometimes called the $p$ version) and to some extent also the case when one enriches the subspaces by choosing smaller element sizes (at times called the $h$ version) or combinations of these two enrichment strategies (then termed an $h-p$ version).

The plan of the paper is: Investigate stability of $h$ (of high order), $p$, and $h-p$ versions of finite element methods for Stokes’ problem in terms of estimates for the growth (with respect to $p$) of operator norms of right inverses of the divergence operators restricted to polynomial subspaces. In the next two sections we will describe some results for Stokes’ problem, in particular we will establish that some $h$-‘proven’ choices will actually be $p$-unstable and we will see, in Section 4, what (if any) can be done about that. The stability discussion is divided in two parts: a theoretical component in Section 2 and a computational one in Section 3. A new mixed method for Poisson’s problem is presented in Section 4 as a mean for ‘recovering’ the pressure. It turns out that some ‘proven’ choices (Brezzi-Douglas-Marini and Raviart-Thomas) in fact are $p$-stable. See [4,5] for estimates of right inverses of first order operators in general (at times supplemented with estimates of growths of projection operators composed with the adjoint operators).

2. Stability results for the $p$ version of Stokes’ problem; norm estimates for $(\nabla \cdot)^{-1} \in B(L^2, H^1)$

Allow us first to look at the plane. Let $R = (-1,1)^2$ and $T = \{(x,y) : |x| < 1, -1 < y < x\}$ denote a reference square and triangle, respectively. Let $F_i$ be an affine, orientation preserving (i.e. the Jacobian $\det(DF_i) > 0$) mapping which maps $\Omega_i$ onto $R$ if $\Omega_i$ is a parallelogram and onto $T$ if $\Omega_i$ is a triangle. Let
be polynomial spaces of \textit{total} and \textit{separate} degree, respectively.

Then we define the space of piecewise polynomials

\[
S_p = \left\{ u \in L^2(\Omega) : u|_{\Omega_i} \circ \text{(F}_i)^{-1} \in \begin{cases} Q_p(R) & \text{if } F_i(\Omega_i) = R \\ P_p(T) & \text{if } F_i(\Omega_i) = T \end{cases}, 1 \leq i \leq M \right\}
\]

and we choose (for no-slip boundary conditions)

\[
V_N = [H^1_0(\Omega) \cap S^p]^2, \quad \mathcal{V}_N = \nabla \cdot \mathcal{V}_N.
\]

Thus \( V_N \subseteq [C_0(\overline{\Omega})]^2 \) [6]. The pressures are allowed to be discontinuous.

\( ([P^p]^2, P^{p-1}) \), or any other pair with a similar degree selection and (6), is a natural choice in the sense that it reflects the approximability of the velocity–pressure pair given by their regularity and that it was proven \( h \)-stable for \( p \geq 4 \) when excluding exceptional meshes [1]. This was proven for a quasi-uniform family of triangulations in [7] for the \( h \) version of the mixed finite element method. We will allow \( h \), the diameter of the elements, to tend to zero also but will add the consideration of increasing \( p \). In Remark 2.2 in [8] it is shown that one may allow curvilinear triangles that are curved on one side.

We will confine ourselves to the case, where there are no nonsingular vertices which (as \( h \to 0 \)) degenerate to a singular vertex either in the interior or at the boundary [7,8]. It is not realistic to entirely avoid singular vertices, for example at the boundary. We note in passing that a constructive characterization of \( \mathcal{V}_N = \nabla \cdot \mathcal{V}_N \) is particularly simple when one excludes “singular vertices” [8]. When “singular vertices” are present in a pure triangulation, an alternate description of the range of the divergence operator is given in [7,8]. (Note that for polynomials of separate degree: \( x^j \cdot (Q_p)^2 = \{ q = \sum a_{lm}x^l y^m ; 0 \leq l, m \leq p, l + m < 2p \} \subset Q_p \).)

**LEMMA 1.** Let us consider solving (2) with the choice of spaces given in (5) and (6) with a quasi-uniform mesh and no nonsingular vertices degenerating to singular. Then the discrete problem (2) is well-posed.

Furthermore, the discrete inf-sup constant satisfies a bound from below which decays at most algebraically in the degree \( p \) but is independent of the mesh-size \( h \):

\[
\min_{q \in \mathcal{V}_N \setminus \{0\}} \max_{v \in \mathcal{V}_N} \frac{\int_B \nabla \cdot v q \, dx}{\|v\|_{H^1(B)} \|q\|_{L^2}} = \mu(N) \geq c p^{-K};
\]

for some positive \( K \) and \( c \) which do not depend on \( h \) or \( p \). (For \( K \) we can take any \( K > 9 \) in general and \( K = 4 \) if we only employ parallelograms.)

**PROOF.** This is a consequence of some norm estimates of \((\nabla \cdot)^{-1}\). Problem (2) has a unique solution \((U_N, P_N)\) since the inf-sup constants in (3) are bounded below by (the pessimistic) \( cp^{-K} > 0 \) with \( K \) and \( c \) independent of \( h \) and \( p \). This is seen by the following argument. For a given \( q \) vanishing at all vertices and having zero mean over each \( \Omega_i \), choose \( v \) in (3) such that \( v|_{\Omega_i} \) satisfies \( \nabla \cdot v = q \) in \( \Omega_i \), \( v = 0 \) on \( \partial \Omega_i \), with \( \|v\|_{1, \partial \Omega_i} \leq Cp^K \|q\|_{0, \partial \Omega_i} \), \( 1 \leq i \leq M \). Such \( v \) exist by Lemma 2.5 in [8] for \( \Omega_i \) a triangle and Lemma V.5, cf. (V.40) in [9] for \( \Omega_i \) a parallelogram. The norm estimates quoted are based on direct construction of an inverse and inverse inequalities in \( L^2 \)-based norms. It is possible to reduce to these \( q \) by a special construction at \( p = 4 \). \( \square \)

Similar polynomial spaces can be introduced in 3-D using combinations of \( Q^p \) on reference cubes and \( P^p \) on reference tetrahedra (as well as others). In 3-D, denote the unit ball in max-norm by \( B = (-1, 1)^3 \) and let velocities and pressures belong to \((Q^p)^3\) and \( \nabla \cdot (Q^p)^3 \), respectively, where \( Q^p \) was defined above.

**PROPOSITION 2.** The operator \((\nabla \cdot)^{-1} : \nabla \cdot (Q^p)^3 \to (Q^p)^3, \, p \geq 1 \), considered as an operator from a subspace of \( L^2(B) \) to a subspace of \((H^1(B))^3 \) satisfies

\[
cp \leq \|((\nabla \cdot)^{-1})\|_{B(L^2, H^1)} \leq C_p,
\]
with positive constants \(c\) and \(C\) independent of \(p\).

**Proof.** This will follow from some norm estimates of \((\nabla \cdot \cdot)^{-1}\) over \(B\) where some observations and notation relating to Legendre polynomials are useful. Let \(\ell_n(x)\) denote the Legendre polynomial of degree \(n\) with squared \(L^2\) norm:

\[
\int_{-1}^{1} \ell_n^2(x) \, dx = \frac{2}{2n+1}.
\]

Fix a positive integer \(p\). Let \(q_n, 0 \leq n \leq p\), denote the polynomials

\[
q_n(x) = \ell_n(x) \quad 0 \leq n < p.
\]

\[
q_p(x) = \int_{-1}^{x} \ell_{p-1} \, dx.
\]

We first prove the upper bound in the proposition: Let \(q \in \nabla \cdot (Q^p)^3\), so for some \(\{a_{lmn}\}\),

\[
q = \sum_{l+m+n < 3p} a_{lmn} q_l(x_1) q_m(x_2) q_n(x_3).
\]

Note that the following equivalence holds with positive constants, independent of \(p\):

\[
\|q\|^2_{L^2(B)} \approx \sum_{l,m,n < p} (l+1)^{-1}(m+1)^{-1}(n+1)^{-1} \alpha_{lmn}^2
\]

\[
+ \sum_{m,n < p} (p+1)^{-3}(m+1)^{-1}(n+1)^{-1} \left[ \alpha_{lpm}^2 + \alpha_{mnp}^2 + \alpha_{lnp}^2 \right]
\]

\[
+ \sum_{n < p} (p+1)^{-6}(n+1)^{-1} \left[ \alpha_{lpm}^2 + \alpha_{mpn}^2 + \alpha_{lnp}^2 \right].
\]

We set

\[
u = \begin{bmatrix}
\sum_{l} \alpha_{lmn} \left( \int_{-1}^{x_1} q_l(x_2) q_n(x_3) \right)
\sum_{ll} \alpha_{lmn} q_l(x_1) \left( \int_{-1}^{x_2} q_m(x_2) q_n(x_3) \right)
\sum_{lll} \alpha_{lmn} q_l(x_1) q_m(x_2) \left( \int_{-1}^{x_3} q_n \right)
\end{bmatrix},
\]

where \((0, p)^3\) is partitioned into three sets \(I \cup II \cup III\) in the following way:

\[I = \{(l, m, n) \in [0, p]^3 : m \leq l < p \wedge n \leq l < p\} \cup \{(l, m, n) \in [0, p]^3 : m = p \wedge n \leq l < p\}
\]

\[\cup \{(l, m, n) \in [0, p]^3 : m = p \wedge m \leq l < p\} \cup \{(l, m, n) \in [0, p]^3 : m = p \wedge m \leq l < p\}.
\]

\[II = \{(l, m, n) \in [0, p]^3 : l < m < p \wedge n \leq m < p\} \cup \{(l, m, n) \in [0, p]^3 : l = p \wedge n \leq m < p\}
\]

\[\cup \{(l, m, n) \in [0, p]^3 : n = p \wedge l < m < p\} \cup \{(l, m, n) \in [0, p]^3 : n = p \wedge m < p\}.
\]
III = \{(l, m, n) \in [0, p)^3 : l < n < p \land m < n < p\} \cup \{(l, m, n) \in [0, p)^3 : l = p \land m < n < p\} \\
\cup \{(l, m, n) \in [0, p)^3 : m = p \land l < n < p\} \cup \{(l, m, n) \in [0, p)^3 : l = m = p \land n < p\}.

One may inspect that \([0, p)^3 = I \cup II \cup III\). (Perhaps the easiest way to see this is graphically by splitting each of the three sets' first component into two along the 'A's.)

Then \(\|\partial u_1 / \partial x_1\|_{L^2(B)} \leq C \|q\|_{L^2(B)}\) and, using that \(m \leq l\), along with estimates (11), (14), and (18) established in [10],

\[
\left\| \frac{\partial u_1}{\partial x_2} \right\|^2_{L^2(B)} = \left\| \sum_{i} \alpha_{l,m,n} \left( \int_{-1}^{1} q_l(x_1) q_m'(x_2) q_n(x_3) \right) \right\|^2 \\
\leq C \sum_{i} (l + 1)^{-3} \left\| \sum_{m,n} \alpha_{l,m,n} q_m'(x_2) q_n(x_3) \right\|^2 \\
\leq C \sum_{i} (l + 1)^{-3} (n + 1)^{-1} \left( \sum_{m} \alpha_{l,m,n}^2 \right) \left( \sum_{n} (m + 1)^2 \right) \\
\leq C \sum_{i} \alpha_{l,m,n}^2 (n + 1)^{-1} \\
\leq C p^2 \sum_{i} (l + 1)^{-1} (m + 1)^{-1} (n + 1)^{-1} \alpha_{l,m,n}^2 \\
= C p^2 \|q\|^2_{L^2(B)}.
\]

Using that \(n \leq l\) we may likewise show that \(\|\partial u_1 / \partial x_3\|_{L^2(B)} \leq C p \|q\|_{L^2(B)}\). The other components are bounded similarly. Finally,

\[
\int_{B} u \ dx = \begin{bmatrix}
8\alpha_{000} - \frac{8}{3}\alpha_{001} \\
-\frac{8}{3}\alpha_{010} \\
-\frac{8}{3}\alpha_{001}
\end{bmatrix},
\]

for \(p > 2\), each of which is bounded, in absolute value, by \(C \|q\|_{L^2(B)}\). We have proved so far that

\[\|u\|_{H^1(B)} \leq C p \|q\|_{L^2(B)}\]

so that

\[\|\nabla^{-1}\|_{L^2(B; H^1)} \leq C p.\]

Now, to prove the lower bound, let

\[q^*(x) = r(x_1) \ell_p(x_2) \ell_p(x_3),\]

and

\[u = \sum_{l,m,n} \begin{bmatrix}
\alpha_{l,m,n} \\
\beta_{l,m,n} \\
\gamma_{l,m,n}
\end{bmatrix} q_l(x_1) q_m(x_2) \ell_n(x_3).\]

We may rewrite \(\int_{-1}^{1} \int_{-1}^{1} \nabla \cdot u \ell_p(x_2) \ell_p(x_3) \ dx_2 \ dx_3 \) as

\[
\sum_{l=0}^{p} \alpha_{l,p} q_l^2(x_1) \left( \int_{-1}^{1} q_p(x_2) \ell_p(x_3) \right) \left( \int_{-1}^{1} \ell_p^2(x_3) \right) = \int_{-1}^{1} \int_{-1}^{1} q^* \ell_p(x_2) \ell_p(x_3) \ dx_2 \ dx_3.
\]
and as
\[
\frac{d}{dx} \left( \sum_{l=0}^{p} \alpha_{lpp} q_l \right) = (2p - 1) r(x).
\]
Then
\[
u_1 = \sum_{l,m,n} \alpha_{lmm} q_l(x_1) q_m(x_2) \ell_n(x_3) + q_p(x_2) \ell_p(x_3)(2p - 1) \left( \int_{-1}^{x_1} r + c \right),
\]
so that
\[
\frac{\partial \nu_1}{\partial x_1} = \sum_{l,m+n<2p} \alpha_{lmm} q_l(x_1) q_m(x_2) \ell_n(x_3) + \ell_{p-1}(x_2) \ell_p(x_3)(2p - 1) \left( \int_{-1}^{x_1} r + c \right)
\]
and, by orthogonality,
\[
\left\| \frac{\partial \nu_1}{\partial x_1} \right\|_{L^2(B)}^2 \geq (2p - 1)^2 \frac{2}{2p - 1} \left\| \int_{-1}^{x_1} r + \frac{1}{2} \int_{-1}^{x_1} r \, dx \right\|_{L^2((-1,1))}^2
\]
\[
\left\| q^* \right\|_{L^2(B)}^2 = \left( \frac{2p}{2p + 1} \right)^2 \left\| r \right\|_{L^2((-1,1))}^2,
\]
so that
\[
\| \nu \|_{H^1(B)} \geq c \| q^* \|_{L^2(B)}
\]
for a fixed \( r \) (one may pick \( r(x_1) = x_1 \), e.g.).

The same algebraic instability (\( \sim 1/p \)) holds in 2-D [10]. Depending on the particular choice of spaces and boundary conditions, [10] showed that there is an algebraic decay of the stability constant \( \mu_p \) of varying degree. Such results on single elements are related to ‘inverting’ the gradient operator on polynomial subspaces [11]. When no boundary conditions are enforced we will have to use the adjoint operator to the divergence operator rather than the actual gradient operator. The same lower bound for \( \| (\nabla \cdot)^{-1} \| \) holds under no-slip boundary conditions [4]. Now the adjoint operator simply becomes the gradient.

We do not know whether Proposition 2 also holds for \( \mathcal{P}^p \), the polynomials of total degree \( p \). The upper bound in (8) can be proven to hold also posed over \( B \). Some estimates are given in [10] for 2-D. Also, we emphasize that the \( p \)-dependence is strongly dependent on boundary conditions.

There do exist examples of \( p \)-stable elements, see Example 5 in [4].

When the geometry does not allow a tensor product factorization it seems very hard to give sharp estimates. We will instead rely on some eigenvalue approximations.

3. The degree of instability determined numerically

One may compute the inf-sup constant via a generalized eigenvalue computation involving the matrices induced by the divergence operator restricted to the discrete velocity space and the matrices associated with the quadratic forms defined by the square of the norms entering \( (H^1 \) and \( L^2 \). Relative to choices of bases for \( \mathcal{V}_N \) and \( \mathcal{W}_N \), let the discrete representatives of these operators be denoted by the matrices \( A, B \) and \( C \), respectively (see e.g. [10]). Then \( \mu(p) \) is the smallest singular value of the matrix \( B^{-1/2} A^T C^{1/2} \) or the square root of the smallest eigenvalue of the symmetric, positive definite \( D = C^{1/2} A B^{-1} A^T C^{1/2} \). In [10] we computed the inf-sup constants for a range of \( p \)-values and observed that, if no boundary conditions are enforced on the velocity spaces, the inf-sup constant seemed to decay at a much lesser rate, inf-sup\(_p \) (no-slip) \( \sim 1/p \) whereas inf-sup\(_p \)
The inf-sup constant vs. polynomial degree $p$. The case of one or two triangles.

(no boundary conditions) $\sim (1/p)^{0.2}$ for polynomial spaces $\mathcal{P}^p$. In addition, it is clearly possible to define other pairings of polynomial spaces (such as the ones used by the software PROBE) and also take triangles (tetrahedra) instead.

In Figs. 1 and 2 is given the computed inf-sup constants for the cases where we have taken $\mathcal{P}_p$ over a triangle or two triangles (Fig. 1) and a tetrahedron (Fig. 2), respectively with no or zero boundary conditions imposed on the discrete velocities. We employed NMGTM [12] for this.

It will seem that one again observes an algebraic instability with respect to $p$.

4. A priori estimate for asymptotic convergence of approximations in 2-D

One still has quasi-optimality of the velocity approximation for stress-free or no-slip boundary conditions by observing that the discrete velocity is an elliptic projection onto solenoidal polynomial vector fields. For planar problems, one may characterize these for triangulations, lattices and combinations of these and show that they possess quasi-optimal approximation properties by using [13] and the interpolation result [14] (for no-slip boundary conditions). We refer to [4,10,14,15]. One may include in these estimates the dependence upon $h$ for pure triangulations since we avoided singular vertices arising in the limit $h \to 0$ (without too much work one could include parallelograms).

**Proposition 3.** The mixed method finite element solution $(U_N, P_N)$ given by (2) and the hypotheses in Lemma 1 satisfies the following error estimates:

$$
\|U - U_N\|_{H^1} + \|P - P_N\|_{L^2} \leq Ch^{\min(p,s)} p^{K-s} \left( \|U\|_{H^{s+1}} + \|P\|_{H^s} \right),
$$

$$
\|U - U_N\|_{H^1} \leq Ch^{\min(p,s)} p^{-s} \|U\|_{H^{s+1}},
$$

with $C$ and $K$ independent of $h$ and $p$. 

Fig. 1. The inf-sup constant vs. polynomial degree $p$. The case of one or two triangles.
If furthermore the original datum $F$ is piecewise analytic and one uses especially designed mesh- and degree progressions, one may use the arguments in [16] to prove exponential convergence.

One can think of computing the velocity independently first (using the solenoidal spaces and solving bi-Poisson problems) and then the pressure, or alternatively think of recovering the pressure (at a larger convergence rate) within the mixed method. See [15] where this is done by (1) local $H^1$ projections or (2) a single layer potential recovery.

$P$ satisfies the following Poisson problem

$$
\Delta P = \nabla \cdot F \quad \text{in } \Omega,
$$

$$\frac{\partial P}{\partial n} = (F + \nu \Delta U) \cdot n \quad \text{on } \partial \Omega,
$$

$$\int_\Omega P \, dx = 0,
$$

where $n$ denotes the unit outward normal. This boundary value problem is solvable since $U$ is solenoidal and due to the Gauss divergence theorem.

We can interpret (12) as local averaging, bringing this method into the same general frame of ideas as that of Bramble and Schatz [17] or Johnson and Pitkäranta [18]. Problem (12) can also be posed on $\Omega_m$, an element $\subseteq \Omega$ or can be employed locally in the interior of $\Omega$.

We could also consider (3) solving (12) with $U_N$ replacing $U$ using a mixed method using BDM or RT elements for example. In [4] it is shown that these are $p$ (in addition to $h$) stable via a norm estimate on the right inverse of the divergence operator but now in $B(L^2, H(\text{div}))$. One may now prove error estimates in quite a standard fashion using a trace boundedness result from [15]. One may have to employ Babuška's interpolation trick once more. Again, one may accommodate for the $h-p$ version with estimates as they depend on $h$ also.
References