OPTIMAL ORDER MULTIGRID METHODS FOR SOLVING EXTERIOR BOUNDARY VALUE PROBLEMS*

GEORGE C. HSIAO† AND SHANGYOU ZHANG†

Abstract. The coupling of boundary elements and finite elements combines the advantage of boundary elements for treating domains extended to infinity and that of finite elements in treating the nonhomogeneity of equations and the complexity of domains. In the case of the Laplacian, by taking a circle or a sphere as the artificial coupling boundary, it is shown that the corresponding boundary integral equation can be solved without any cost and the coupled system is reduced to a simple finite element system. Two multigrid methods are proposed to solve this finite element linear system. Both methods are of optimal order and can be used to solve such finite element equations as efficiently as to solve those arising from interior boundary value problems.

Numerical experiments are included to show the efficiency and advantages of the methods. An apparent significance of the methods is that the boundary elements appear neither in the discretization nor in the coding.

Key words. multigrid method, boundary element, Galerkin method, finite element

AMS subject classifications. 65N30, 65F10

1. Introduction. A good numerical method for boundary value problems should not only provide a discretization scheme, but also include an efficient algorithm for solving the resulting discrete linear systems. For exterior boundary value problems, especially those problems involving nonhomogeneity, nonlinearity, and/or irregularity of domains, the coupling method of boundary elements and finite elements provides a good approximation scheme. But the resulting discrete linear systems (see (2.5) below) are difficult to solve due to the complicated structure:

\[
\begin{pmatrix}
F & C_1 \\
C_2 & B
\end{pmatrix}
\begin{pmatrix}
X \\
Y
\end{pmatrix}
=
\begin{pmatrix}
f \\
0
\end{pmatrix},
\]

where \(F\) and \(B\) arise from the finite elements and the boundary elements discretizations, respectively, while \(C_1\) and \(C_2\) represent the coupling. Here the matrix \(B\) is fully populated. The purpose of this paper is to find some optimal order algorithms to invert the discrete problems (1.1), i.e., the number of arithmetic operations is proportional to the number of unknowns.

For exterior boundary value problems, we have a freedom in choosing the artificial outer boundary. In this paper, we will make the coupling at a circle in the two-dimensional case, or a sphere in three-dimensional. In such a case, the matrix \(B\) in (1.1) can be inverted exactly (in fact, not at discrete levels, but at the continuous level; cf. (2.5) and (3.7) below) and (1.1) is reduced to

\[
(F - C_1 B^{-1} C_2) X = f.
\]

Two optimal order methods will be proposed for solving (1.2). One is a multigrid-preconditioning iteration. We split the matrix in (1.2) and perform the Richardson

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† Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716 (hsiao@math.udel.edu and szhang@math.udel.edu). The work of the first author was supported in part by Office of Naval Research grant N00014-91-J-1700. The work of the second author was supported in part by the University of Delaware Research Foundation grant.
iteration with a relaxation parameter $\omega$:

$$
(1.3) \quad x_i = x_{i-1} + \omega \tilde{F}^{-1}(f - (F - C_1B^{-1}C_2)x_{i-1}), \quad i = 1, 2, \ldots,
$$

where the preconditioner $\tilde{F}^{-1}$ is a multigrid approximation of the inverse of the finite element stiffness matrix $F$. This iterative method is natural, like the Schwartz alternating method in the domain decomposition. Given a rough guess of the finite element solution, we can update the boundary element approximate solution by it. With the new boundary element solution, the initial guess of the finite element solution is improved. Nevertheless the boundary elements do not appear explicitly in the method (see (4.1)). However, unlike the Schwartz alternating method, this procedure may not converge if we do not use the relaxation parameter (i.e., $\omega = 1$ in (1.3)). A similar method has been used in [20], where a preconditioned conjugate gradient method is applied to the coupling system on the artificial boundary, where both interior and exterior problems are solved exactly.

The other method we will study for solving (1.2) is a direct multigrid method without splitting the matrix. We show that this multigrid iteration can solve (1.2) as efficiently as the multigrid method for solving standard finite element equations arising from interior boundary value problems. We remark again that unlike the work of multigrid methods for integral equations (cf. [11], [19], and [22]), our multigrid method is applied only to the finite element equations in the boundary-finite elements coupling system where the coupling integral equation is inverted exactly.

A great interest exists in the three-dimensional case. In this case, matrix $F$ in (1.1) is of size $O(h^{-3})$ with $O(h^{-3})$ nonzero entries, while $B$ is of size $O(h^{-2})$ but with $O(h^{-4})$ nonzeros. Here $h$ stands for the mesh size as usual. Since $FX$ in (1.2) can be evaluated in $O(h^{-3})$ operations, it is critical then to reduce the work from the possible $O(h^{-4})$ operations to $O(h^{-3})$ operations or less for evaluating the boundary term $(C_1B^{-1}C_2)X$ in (1.2). For the Galerkin boundary integral method, it needs, in general, $O(h^{-4})$ operations to generate the boundary integral matrix $B$ at least, and it needs $O(h^{-4})$ operations to compute the matrix-vector multiplication for threedimensional boundary elements. The method proposed in this note of inverting $B$ exactly has a practical significance. Due to the symmetry of the artificial coupling boundary, a sphere, the evaluation of $(F - C_1B^{-1}C_2)X$ needs only $O(h^{-3})$ operations if the fast Fourier transform is applied. Since $X$ has $O(h^{-3})$ entries, this is of optimal order. We note that $C_1B^{-1}C_2$ is a block circulant matrix (see (3.8)). We can apply the one-dimensional fast Fourier transform within each block. The evaluation of the coupling term $(C_1B^{-1}C_2)X$ might be done with only $O(h^{-2})$ operations if one can apply two-dimensional fast Fourier transform when the meshes on the sphere are well constructed.

2. Coupling of finite elements and boundary elements. We consider the following model exterior Dirichlet problem in two or three dimensions:

$$
(2.1) \quad \begin{aligned}
-\Delta u &= f \quad \text{in } \Omega^c := \mathbb{R}^d \setminus \Omega \cup \Gamma, \\
u &= 0 \quad \text{on } \Gamma := \partial \Omega, \\
u &= O(1) \quad \text{or } \quad O(|x|^{-1}) \quad \text{as } |x| \to \infty \text{ if } d = 2 \text{ or } 3,
\end{aligned}
$$

where $\Omega$ is a bounded domain, and $f$ is a given function with compact support in $\Omega_F := \Omega^c \cap \Omega_0$. Here $\Omega_0$ is a circular domain, $\Omega_0 := \{x : |x| < a\}$. We assume the
boundary of \( \Omega \) is sufficiently regular such that
\[
\|u_g\|_{H^{\alpha+1}(\Omega_F)} \leq c_1 \|g\|_{H^{\alpha-1}(\Omega_F)} \quad \forall g \in H^{\beta-1}(\Omega_F)
\]
for some constant \( \beta, 0 < \beta \leq 1 \), where \( u_g \) is the solution of the interior boundary value problem,
\[
-\Delta u_g = g \quad \text{in} \ \Omega_F,
\]
\[
u_0 = 0 \quad \text{on} \ \Gamma \cup \Gamma_0, \quad \text{where} \ \Gamma_0 := \partial \Omega_0.
\]
We note that Lipschitz bounded domains are not excluded in our study (cf. [10] and [9]). Throughout the paper, \( c_i \) will stand for positive constants independent of the functions in estimation and the multigrid level number, and \( H^\alpha(\Omega) \) stands for standard Sobolev spaces [1].

Problem (2.1) can be reformulated as a coupled problem of an interior and an exterior boundary value problem. The coupling is on the interface \( \Gamma_0 \) via the transmission conditions:
\[
\frac{\partial u^+}{\partial n} = \frac{\partial u^-}{\partial n} \quad \text{and} \quad u^- = u^+,
\]
where \( \pm \) denotes the limits from outside/inside of the interface \( \Gamma_0 \). The finite element method and the boundary element method are applied to subdomains \( \Omega_F \) and \( \Omega_0 := \mathbb{R}^d \setminus \Omega_0 \), respectively. The variational formulation of the coupled problem for (2.1) can be stated as follows (cf. [18] and [11]): Given \( f \in H^{-1}(\Omega_F) \), find \( (u_F, \sigma) \in H^1(\Omega_F) \times H^{-1/2}(\Gamma_0) \) such that
\[
\forall v \in H^1(\Omega_F),
\]
\[
\forall \chi \in H^{-1/2}(\Gamma_0).
\]
In this formulation, we adopt the following notation:
\[
H^1(\Omega_F) := \{ v \in H^1(\Omega_F) : v|_{\Gamma} = 0 \},
\]
\[
H^{-1/2}(\Gamma_0) := H^{-1/2}(\Gamma_0) \quad \text{if} \ d = 3,
\]
\[
H^{-1/2}(\Gamma_0) := \{ \chi \in H^{-1/2}(\Gamma_0) : \langle 1, \chi \rangle = 0 \} \quad \text{if} \ d = 2,
\]
\[
a(u, v) := \int_{\Omega_F} \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1(\Omega_F),
\]
\[
(f, v) := \int_{\Omega_F} f v \, dx \quad \forall f, v \in L^2(\Omega_F),
\]
\[
K_v(x) := \int_{\Gamma_0} \frac{\partial}{\partial n_y} \gamma(x, y) v(y) \, ds_y \quad \forall v \in H^{1/2}(\Gamma_0),
\]
\[
V\sigma(x) := \int_{\Gamma_0} \gamma(x, y) \sigma(y) \, ds_y \quad \forall \sigma \in H^{-1/2}(\Gamma_0),
\]
\[
\langle v, \chi \rangle := \int_{\Gamma_0} v \chi \, ds \quad \forall v \in H^{1/2}(\Gamma_0), \quad \forall \chi \in H^{-1/2}(\Gamma_0),
\]
where \( \gamma(x, y) \) denotes the fundamental solution of the Laplacian,
\[
\gamma(x, y) = \begin{cases} (-1/2\pi) \log |x - y| & \text{if} \quad d = 2, \\ 1/(4\pi|x - y|) & \text{if} \quad d = 3. \end{cases}
\]
The solution of (2.5) relates to that of (2.1) by \( u_F = u|_{\Omega_F} \) and \( \sigma = \partial u / \partial n|_{\Gamma_0} \).
3. Uncoupling of finite elements and boundary elements. In this section we show how to invert exactly the boundary integral equation in (2.5). Then we can have a weak variational formulation involving unknown \( u_F \) alone over a bounded domain \( \Omega_F \) for the exterior problem (2.1). Therefore, boundary elements will not be introduced in the discretized problem. This would cut the work of coding and computation by more than half. In fact, as we will see, after uncoupling \( \sigma \) and \( u_F \), the resulting finite element problems are almost the same as those arising from Neumann boundary value problems on a bounded domain.

We first consider the two-dimensional case. From Green’s formula, we represent the solution in the form (cf. [15])

\[
(\text{el. }[15])
\]

\[
(3.1) \quad u(x) = \int_{\Gamma_0} \frac{\partial}{\partial n_y} \gamma(x, y) u^+(y) \, ds_y - \int_{\Gamma_0} \gamma(x, y) \frac{\partial u^+}{\partial n_y} (y) \, ds_y + c \quad \forall x \in \Omega_0,
\]

where \( c \) is a constant. For our model problem (2.1), \( u^+=u_F \) and \( \partial u^+/\partial n = \partial u_F^+/\partial n \) on \( \Gamma_0 \). We denote the former by \( \mu \) and the latter by \( \sigma \). Letting \( x \) approach the artificial boundary \( \Gamma_0 \) in (3.1), and in the equation for the normal derivative obtained from (3.1), we get the following system of equations:

\[
\begin{align*}
(3.2) \quad & \mu = \left( \frac{1}{2} I + K \right) \mu - V \sigma + c, \\
(3.3) \quad & \sigma = -W \mu + \left( \frac{1}{2} I - K' \right) \sigma.
\end{align*}
\]

Here in addition to \( K \) and \( V \) defined by (2.6), we also introduce the following integral operators:

\[
K' \sigma = \int_{\Gamma_0} \frac{\partial \gamma}{\partial n_x} \sigma(y) \, ds_y, \quad x \in \Gamma_0,
\]

\[
W \mu := -\frac{\partial}{\partial n_x} \int_{\Gamma_0} \frac{\partial \gamma}{\partial n_y} \mu \, ds_y, \quad x \in \Gamma_0.
\]

The system (3.2)–(3.3) can be rewritten in the form:

\[
\begin{pmatrix} \mu \\ \sigma \end{pmatrix} - \begin{pmatrix} c \\ 0 \end{pmatrix} = C \begin{pmatrix} \mu \\ \sigma \end{pmatrix}, \quad \text{where } C := \begin{pmatrix} \frac{1}{2} I + K & -V \\ -W & \frac{1}{2} I - K' \end{pmatrix}
\]

is the Calderon projection (cf. [7]). We note that \( C(0,0) = (0,0) \), and that \( C^2 = C \). The latter implies that

\[
(3.4) \quad WV = \frac{1}{4} I - (K')^2.
\]

By (3.2) and (3.4), we see that

\[
(3.5) \quad \sigma = 4(K')^2 \sigma - 2W \mu + 4WK \mu.
\]

For \( \Gamma_0 \) being the circle of radius \( a \) centered at the origin,

\[
K' \sigma = -\frac{1}{2\pi} \int_{\Gamma_0} \frac{\partial}{\partial n_x} \log |x-y| \sigma(y) \, ds_y = -\frac{1}{2\pi} \int_{\Gamma_0} \frac{(x-y) \cdot n_x}{|x-y|^2} \sigma(y) \, ds_y
\]

\[
= -\frac{1}{2\pi} \int_{\Gamma_0} \frac{a^2 - x \cdot y}{2a(a^2 - x \cdot y)} \sigma(y) \, ds_y = -\frac{1}{4\pi a} \int_{\Gamma_0} \sigma(y) \, ds_y.
\]

Repeating the calculation gives

\[
(K')^2 \sigma = \left( -\frac{1}{4\pi a} \int_{\Gamma_0} \sigma(y) \, ds_y \right) \left( -\frac{1}{4\pi a} \int_{\Gamma_0} ds \right) = \frac{1}{8\pi a} \int_{\Gamma_0} \sigma(y) \, ds_y.
\]
Similar to the above computation, we have 
\[ K \mu = \left( -\frac{1}{4\pi a} \right) \int_{\Gamma_0} \mu(y) \, ds_y. \]
Then we can simplify (3.5) to
\[
\sigma = \frac{1}{2\pi a} \int_{\Gamma_0} \sigma(y) \, ds_y - 2W \mu - \frac{1}{\pi a} W \int_{\Gamma_0} \mu(y) \, ds_y
\]
\[ = \frac{1}{2\pi a} \int_{\Gamma_0} \sigma(y) \, ds_y - 2W \mu \]
(see also [16], p. 106). Noting that the first term in (3.6) is a constant, which vanishes for \( \sigma \in H_0^{-1/2}(\Gamma_0) \), we get
\[
\langle v^-, \sigma \rangle = -2\langle v^-, W \mu \rangle = -2 \left( \frac{d\mu}{ds}, \frac{dv^-}{ds} \right),
\]
where the second identity follows from a simple integration by parts (see, e.g., [13]), and \( d/ds \) denotes the differentiation in the tangential direction. Therefore, (2.5) is equivalent to the following variational problem: Find \( u \in H^1_\Gamma(\Omega_F) \), such that
\[
a(u, v) + 2\langle V \dot{u}, \dot{v} \rangle = \langle f, v \rangle \quad \forall v \in H^1_\Gamma(\Omega_F),
\]
where \( u \) and \( \dot{v} \) denote the tangential derivatives of \( u|_{\Gamma_0} \) and \( v|_{\Gamma_0} \) along \( \Gamma_0 \), respectively. For simplicity, in the sequel \( u \), as well as \( v \), will also be used to denote its restriction on the artificial boundary. We note that the kernel in the integral \( \langle V \dot{u}, \dot{v} \rangle \) is only weakly singular.

In the three-dimensional case, (3.1)-(3.4) remain valid except that the constant \( c \) is identically equal to zero. On the sphere, we have
\[
K' \sigma = \frac{1}{4\pi} \int_{\Gamma_0} \nabla_n \left( \frac{1}{|x-y|} \right) \sigma(y) \, ds_y = \frac{1}{4\pi} \int_{\Gamma_0} -\frac{1}{|x-y|^2} \frac{(x-y) \cdot n_x}{|x-y|} \sigma(y) \, ds_y
\]
\[ = \frac{1}{4\pi} \int_{\Gamma_0} \left( -\frac{1}{2a} \right) \frac{1}{|x-y|} \sigma(y) \, ds_y = -\frac{1}{2a} V \sigma.
\]
Similarly, \( K \sigma = \left( -\frac{1}{2a} \right) V \sigma \). Applying these several times, we can obtain the following identities from (3.2)-(3.4):
\[
\sigma = 4(K')^2 \sigma - 2W u + 4WK u = -\frac{2}{a} K'V \sigma - 2W u + \frac{2}{a} WV u
\]
\[ = \frac{1}{a} K' u + \frac{2}{a} K'K u - 2W u + \frac{1}{2a} u + \frac{2}{a} (K')^2 u
\]
\[ = -\frac{1}{2a} V u - 2W u + \frac{1}{2a} u.
\]
Therefore the uncoupled weak formulation in three dimensions is of the form:
\[
a(u, v) + \frac{1}{2a^2} \langle V u, v \rangle + 2\langle V n \times \nabla u, n \times \nabla v \rangle + \frac{1}{2a} \langle u, v \rangle = \langle f, v \rangle \quad \forall v \in H^1_\Gamma(\Omega_F).
\]
In both two-dimensional and three-dimensional cases, the final variational form contains neither the boundary integral operator \( K \) nor \( K' \).

4. A direct multigrid method. In this section the finite elements will be employed to discretize problem (3.7). We then apply a multigrid iteration to the finite element linear system. We will prove the constant-rate convergence (independent of the number of unknowns in the linear system) of the multigrid method. Consequently the method is shown to have the optimal computational order. To avoid repeating,
we will state and prove only the two-dimensional case. But the analysis holds for the three-dimensional case with only minor notation changes. We remark that we prove only the W-cycle multigrid method with the number of fine-level smoothings sufficiently large. But the multigrid methods of V-cycle and of one smoothing can be proved for our problem too with some further work and assumptions (cf. [3], [6], and [21]).

The existence and uniqueness of the solution for (3.7) are implied by the standard Lax–Milgram theorem as we note the following estimates.

**Theorem 4.1.** Let \( \Omega \) be a Lipschitz domain. It holds that

\[
c_1 \|u\|_{H^1(\Omega_F)}^2 \leq a(u, u) + 2 \langle V \dot{u}, u \rangle \leq c_2 \|u\|_{H^1(\Omega_F)}^2 \quad \forall u \in H^1_\Gamma(\Omega_F).
\]

**Proof.** By [17] and [14], we have

\[
c_3 \|\phi\|_{H_0^{-1/2}(\Gamma_0)}^2 \leq \langle V \phi, \phi \rangle \leq c_4 \|\phi\|_{H_0^{-1/2}(\Gamma_0)}^2 \quad \forall \phi \in H_0^{-1/2}(\Gamma_0).
\]

Noting \( u \in H_0^{-1/2}(\Gamma_0) \) when \( u \in H^1(\Omega_F) \), it follows that

\[
0 \leq \langle V \dot{u}, u \rangle \leq c_4 \|\dot{u}\|_{H^{-1/2}(\Gamma_0)}^2 \leq c_5 \|u\|_{H^1(\Gamma_0)}^2 \leq c_6 \|u\|_{H^1(\Omega_F)}.
\]

Therefore the theorem is proved by the Poincaré inequality. \( \square \)

**Fig. 1. Triangles with curved edges used in the paper.**

We then apply the finite element method to (3.7). For brevity, we will use piecewise linear elements on triangles or triangles with one curved edge as shown in Fig. 1. By connecting all midpoints of triangles, we can obtain a nested family of triangulations \( \{T_k\} \) and a nested sequence of finite element spaces: \( V_1 \subset V_2 \subset \cdots \subset H^1_\Gamma(\Omega_F) \).

We will consider applying the multigrid method to the discretized finite element problems: Find \( u_k \in V_k \), such that

\[
(4.1) \quad a(u_k, v) + 2 \langle \dot{u}_k, \dot{v} \rangle = (f, v) \quad \forall v \in V_k.
\]

An advantage of this method is apparently that one does not need to define the boundary element space. Generally speaking, a larger effort is needed in coding boundary elements than in coding finite elements. In this method, we just add a quadrature formula for evaluating \( \langle V \dot{u}, \dot{v} \rangle \) to a standard finite element multigrid code. We remark that these curved finite elements are practical too. But to be simpler, one can use straight edges to approximate the circular boundary (see Fig. 2). The theory provided in this paper can be easily extended to cover such a case. Treating the nonnestedness caused by meshes like Fig. 2 in the multigrid method has been studied in [23]. The multigrid method for more general nonnested meshes can be found in [4], [24], and the references therein.
Let $A_k : V_k \to V_k$ be defined by

$$\langle A_k u, v \rangle = a(u, v) + 2\langle Vu, v \rangle \quad \forall u, v \in V_k.$$  

By Theorem 4.1, $A_k$ is a symmetric, positive-definite operator in $(V_k, \langle \cdot, \cdot \rangle)$. Let $\rho(A_k)$ be the largest eigenvalues of $A_k$. We can define a family of norms on $V_k$ by

$$\|v\|_s^2 = (A_k^s v, v) \quad \forall v \in V_k, \ 0 \leq s \leq 2.$$  

Due to Theorem 4.1, $\| \cdot \|_s$ is equivalent to $\| \cdot \|_{H^s}$ for $0 \leq s \leq 1$ on $V_k$ with the equivalent constants independent of $k$ (a proof can be found in [2]).

**Lemma 4.2.** Let $I_k : C(\Omega_F) \cap H^1_F(\Omega_F) \to V_k$ be the standard nodal value interpolation operator. The following estimates hold for $0 < \alpha \leq 1$,

$$\|u - I_k u\|_1 \leq c_7\|u - I_k u\|_{H^1(\Omega_F)} \leq c_8 h_k^\alpha \|u\|_{H^{\alpha+1}(\Omega_F)} \quad \forall u \in H^{\alpha+1}_0(\Omega_F),$$

$$\|v\|_1 \leq c_9 h_k^{-\alpha}\|v\|_{H^{-\alpha+1}(\Omega_F)} \leq c_{10} h_k^{-\alpha}\|v\|_{H^{-\alpha+1}} \quad \forall v \in V_k.$$  

Let $\tilde{I}_k : C_0(\Omega_F) \cap H^1_F(\Omega_F) \to C_0(\Omega_F) \cap V_k$ be defined (cf. Fig. 3) such that $\tilde{I}_k u(x) = 0$ if $x$ is a vertex of some triangle touching the circular boundary $\Gamma_0$ and that $\tilde{I}_k u(x) = I_k u(x)$ at the rest nodes. It holds that for $0 < \alpha \leq \frac{1}{2}$,

$$\|u - \tilde{I}_k u\|_{H^1(\Omega_F)} \leq c_{11} h_k^\alpha \|u\|_{H^{\alpha+1}(\Omega_F)} \quad \forall u \in H^{\alpha+1}_0(\Omega_F).$$

**Fig. 3.** $\tilde{I}_k u$ vanishes at all circled nodes.
Proof. By Theorem 4.1 and a standard proof, for example, given in [8], we can show (4.3a) and (4.3b), though some triangular elements here have a circular edge. To prove (4.4), we let \( \omega_k \) be the two rings of the triangles near the artificial boundary circle:

\[
\omega'_k = \{ K \mid K \in \mathcal{T}_k, K \text{ has at least a vertex on } \Gamma_0 \},
\]

\[
\omega_k = \{ K \mid K \in \mathcal{T}_k, K \text{ has at least a vertex on } \overline{\omega'_k} \}.
\]

\( \tilde{I}_k u \) differs from \( I_k u \) only on \( \omega_k \) for \( u \in C_0(\Omega_F) \cap H^1(\Omega_F) \). As they are piecewise linear functions and vanish at the boundary, we have \( \|I_k u - \tilde{I}_k u\|_{H^1(\omega_k)} \leq \|I_k u - \tilde{I}_k u\|_{L^\infty(\omega_k) \sqrt{\omega_k}} \), where \( |\omega_k| \) is the area of \( \omega_k \). Therefore, by (4.3a) and the Sobolev embedding theorem, (4.4) is shown as follows.

\[
\|u - \tilde{I}_k u\|_{H^1(\Omega_F)} \leq \|u - I_k u\|_{H^1(\Omega_F)} + \|I_k u - \tilde{I}_k u\|_{H^1(\omega_k)} \\
\leq \|u - I_k u\|_{H^1(\Omega_F)} + c_{12} \|u\|_{L^\infty(\omega_k)} \sqrt{|\omega_k|} \\
\leq c_{12} h_k^{\alpha} \|u\|_{H^{1+\alpha}(\Omega_F)} + c_{12} h_k^{1/2} \|u\|_{H^{1+\alpha}(\Omega_F)}.
\]

**DEFINITION 4.3 (The multigrid scheme).**

1. If \( k = 1 \), (4.1) will be solved exactly by any method.
2. If \( k > 1 \), a new approximation \( w_{m+1} \) will be generated by an initial approximation \( w_0 \) of \( u_k \) as follows.
   
   **2a.** \( m \) smoothings are performed: For \( 1 \leq l \leq m \),

   \[
   (w_l - w_{l-1}, v) = \rho(A_k)^{-1} [(f, v) - (A_k w_{l-1}, v)] \quad \forall v \in V_k.
   \]

   **2b.** A coarse level correction is performed:

   \[
   w_{m+1} = w_m + q,
   \]

   where \( q \) is obtained by doing \( p \) multigrid iteration(s) with initial guess zero for solving (4.1) on the \((k-1)\)st level with the right-hand side being replaced by the residual:

   \[
   (f, v) - (A_k w_m, v).
   \]

Readers may consult [2] and [4] for more information about multigrid methods. The multigrid method is a recursive iteration. A proof is based on the principle of mathematical induction. The convergence of \( W \)-cycle multigrid method follows the convergence of the two-level method by a standard approach (see [2]). So we will show only the two-level convergence.

**THEOREM 4.4 (two-level multigrid method).** Let \( k = 2 \) in Definition 4.3. Under the elliptic regularity assumption (2.2), for any \( 0 < \gamma < 1 \), there is an \( m \), independent of the number of unknowns in (4.1), such that

\[
\|u_k - w_{m+1}\|_1 \leq \gamma \|u_k - w_0\|_1.
\]

**Proof.** The estimate (4.7) states that the error reduction factor is \( \gamma \) which does not depend on the number of unknowns. The proof is standard as given, for example, in [2]. The key point is to construct an auxiliary boundary value problem (2.3). We will sketch the proof. Let \( e_1 = u_k - w_1 \) be the errors. By (4.2) and (2b) in Definition 4.3, it follows that \( (A_k (e_m - q), v) = 0 \) for all \( v \in V_{k-1} \) and that

\[
\|e_{m+1}\|_1^2 = (A_k (e_m - q), e_m) \leq \|e_{m+1}\|_{-\alpha+1} \|e_m\|_{\alpha+1}.
\]
By ((4.5)) and (4.3b), we have the following smoothing property (cf. [2])

\[(4.9)\]
\[\|e_m\|_{\alpha+1} \leq c_{11}m^{-\alpha/2}\rho(A_k)^{\alpha/2}\|e_0\|_1 \leq c_{12}m^{-\alpha/2}h^{-\alpha}_k\|e_0\|_1.\]

Next, a standard duality argument will be used (cf. [2]). Since a special interior boundary value problem (2.3) needs to be introduced, part of the process of [2] will be repeated here. We let \(\alpha = \min\left\{\frac{1}{2}, \beta\right\}\), where \(\beta\) is defined in (2.2). Let \(u_g\) be the solution of (2.3) for \(g \in H^{-1+\alpha}(\Omega_F)\) with \(\|g\|_{H^{-1+\alpha}(\Omega_F)} = 1\).

\[(4.10)\]
\[\|e_{m+1}\|_{-\alpha+1} \leq c_{13}\|e_{m+1}\|_{H^{-\alpha+1}(\Omega_F)} = c_{13}\sup_g\|e_{m+1}\|_{H^{-\alpha+1}(\Omega_F)} = c_{13}\sup_g a(u_g, e_{m+1}) = c_{13}\sup_g a(u_g, e_m - q)\]
\[\leq c_{14}\sup_g \|u_g - \tilde{I}_k u_g\|_{H^1(\Omega_F)} \|e_m - q\|_1\]
\[\leq c_{15}h^{-\alpha}_k\|e_m - q\|_1 \sup_g \|u_g\|_{H^{1+\alpha}(\Omega_F)}\]
\[\leq c_{16}h^{-\alpha}_k\|e_{m+1}\|_1 \sup_g \|g\|_{H^{-1+\alpha}(\Omega_F)} = c_{16}h^{-\alpha}_k\|e_{m+1}\|_1.\]

Here we made use of the elliptic regularity (2.2) and the approximability (4.4). We need to make two remarks. The boundary value problem (2.3) is constructed in such a way that the boundary integral in (4.10) vanishes for all \(u_g\). Due to the curved boundary elements, to make \(\tilde{I}_k u_g\) vanish at the boundary \(\Gamma_0\) will force it vanish on all triangles touching the curved boundary (see \(\omega_k\) in the Proof of Lemma 4.2 and Fig. 3). As a matter of fact, \(\tilde{I}_k u_g\) can be considered defined in the part of \(\Omega_F\) excluding the outer ring of triangles. The multigrid method on such varying multilevel domains has been considered by Bramble in [5]. Finally, (4.7) follows (4.8)–(4.10) by choosing \(m\) large enough such that \(m \geq (c_{12}c_{16}/\gamma)^{2/\alpha}\).

We now consider the operation counts for the multigrid method. Let \(\{v_i, i = 1, \ldots, N_k\}\) be the standard nodal basis for \(V_k\). Given any \(u \in V_k\), \(\{a(u, v_i), i = 1, \ldots, N_k\}\) can be evaluated with \(O(\dim V_k)\) operations. This is standard in the finite element method. The support for the discrete bilinear product \(\langle V\hat{u}, v_i \rangle\) is not local, unlike that for \(a(u, v_i)\). However, asymptotically there are only \(\sqrt{\dim V_k}\) boundary edges for each boundary integration. Therefore the operation counts remain the same as that for finite elements, \(O(\dim V_k)\). But one could reduce this computational costs to \(O(\sqrt{\dim V_k})\) if one uses the symmetry of the circle and the fast Fourier transform. This reduction does not affect the order of total work for the multigrid method. Nevertheless it is of great importance when we apply this method to three-dimensional problems, where the evaluations of \(\{a(u, v_i)\}\) needs \(O(\dim V_k)\) operations, but \(O((\dim V_k)^{4/3})\) operations are needed for \(\{\langle V\hat{u}, v_i \rangle\}\) if no symmetry is used in evaluating the boundary integrals. Since the work per iteration is proportional to the number of unknowns, it is standard to get the following theorem on the optimal order of the multigrid method (see [2] for the mathematical statement and the full multigrid method, i.e., the initial guesses need to be the multigrid solutions on previous coarser levels).

**Theorem 4.5.** Up to the order of accuracy of the discretization, the linear system (4.1) can be solved by the multigrid method defined in Definition 4.3 such that the number of arithmetic operations needed is proportional to the number of unknowns in it. \(\square\)
5. A double iterative multigrid method. In addition to the multigrid method defined in Definition 4.3, we will consider another iterative method where the multigrid method is used as an inner iteration, applied to solve a Poisson equation inside each outer iteration. Let \( \tilde{A}_k : V_k \rightarrow V_k \) defined by \( (\tilde{A}_ku, v) := a(u, v) \) for all \( u, v \in V_k \) and

\[
\|u\|^*_s := \sqrt{(A_k^s u, u)} \quad \forall u \in V_k, 0 \leq s \leq 2.
\]

**Definition 5.1** (a multigrid preconditioned Richardson iteration). One such iteration will produce \( w^{i+1} \) from \( w^i \) by

\[
w^{i+1} = w^i + \omega e_i,
\]

where \( \omega \) is a relaxation parameter and \( e_i \) is the multigrid solution defined by Definition 4.3 for the following Poisson equation:

\[
a(e, v) = (f, v) - a(w^i, v) - 2(Vw^i, v) \quad \forall v \in V_k.
\]

To be precise, \( e_i \) is defined by the following two steps:

1. If \( k = 1 \), \( e_i = e \).
2. If \( k > 1 \), \( e_i = w_{m+1} \), which is generated as follows.
   2a. Smoothenings: \( w_0 = 0 \). For \( 1 \leq l \leq m \),

\[
(w_l - w_{l-1}, v) = \rho(\tilde{A}_k)^{-1} [(f_l, v) - a(w_{l-1}, v)] \quad \forall v \in V_k.
\]

2b. A coarse level correction: \( w_{m+1} = w_m + q \), where \( q \) is obtained by performing \( p \) multigrid iteration(s) with initial guess zero for solving (5.2) on the \((k-1)\)st level with the right-hand side being replaced by the residual:

\[
(f_l, v) - a(w_m, v).
\]

Let \( M_k \) be the multigrid approximate inverse operator for \( \tilde{A}_k \), that is, after one \( k \)th level multigrid iteration, the iterative error \( e \) will be reduced to \( (I - M_k \tilde{A}_k)e \). By the standard multigrid method theory (for example, [2]) or by replacing \( A_k \) in Theorem 4.4 by \( \tilde{A}_k \), the following lemma holds.

**Lemma 5.2.** Under the elliptic regularity assumption (2.2), there is a constant \( \gamma_1 < 1 \) depending only on \( m \) and \( p \) in Definition 5.1 such that

\[
\|I - M_k \tilde{A}_k u\| \leq \gamma_1 \|u\| \quad \forall u \in V_k.
\]

**Theorem 5.3.** For \( w \in H^1(\Omega_F) \), let \( u \in H^1(\Omega_F) \) be the solution of the following Poisson equation in variational form:

\[
a(u, v) = -2(Vw, v) \quad \forall v \in H^1(\Omega_F).
\]

There exists a positive constant \( c_1 \) depending only on \( \Omega_F \) such that

\[
a(u, u) \leq c_1 a(w, w).
\]

**Proof.** This theorem is a simple corollary of Theorem 4.1. The solution of the above Poisson equation defines an operator: \( H^1(\Omega_F) \rightarrow H^1(\Omega_F) \) composed of the following sequence of mappings:

\[
H^1(\Omega_F) \xrightarrow{\gamma_0} H^{1/2}(\Gamma_0) \xrightarrow{d/ds} H^{-1/2}(\Gamma_0) \xrightarrow{V} H^{1/2}(\Gamma_0) \xrightarrow{-d/ds} H^{-1/2}(\Gamma_0) \xrightarrow{\gamma_0'} H^{-1}(\Omega_F) \xrightarrow{-\Delta} H^1(\Omega_F),
\]
where $\gamma_0$ and $\gamma'_0$ denote the trace and the transpose of the trace operator, respectively. We note that each mapping in (5.5) is continuous, and this completes the proof of the theorem.

**Theorem 5.4.** Assume that (2.2) holds. Then there exist constants $m$, $p$, and $\omega$ in Definition 5.1, independent of the level number $k$ such that the error reduction factor for the iteration (5.1) remains constant on all levels:

\[
\|u_k - w^{i+1}\|_1^{\sim} \leq \frac{c_3 + 2\gamma_1(c_3 + 1)}{c_3 + 2} \|u_k - w^i\|_1^{\sim},
\]

where

\[
\gamma_1 < \frac{1}{c_3 + 1}
\]

is defined in (5.3) and $c_3$ is a positive constant defined in (5.6) below.

Proof. Let $B_k : V_k \rightarrow V_k$ be defined by $(B_k u, v) := 2(V \dot{u}, \dot{v})$ for all $u, v \in V_k$. Then, for all $u, v \in V_k$, we see that

\[
a(-lB_k u, v) = (-lB_k u, v) \leq (V, )
\]

Hence, the operator $-lB$ is selfadjoint in the Hilbert space $(V_k, a(\cdot, \cdot))$. Furthermore, due to (5.4), we have that

\[
a(-lB_k u, v) = (B_k u, v) \leq c_2\|Bu\|_{H^{-1}(\Omega)}\|v\|_{H^1(\Omega)}
\]

(5.6)

where $B : H^1(\Omega) \rightarrow H^{-1}(\Omega)$ is defined by $(Bu, v) = 2(V \dot{u}, \dot{v})$. Therefore, $-lB_k$ is uniformly bounded (independent of $k$), symmetric and positive definite in $(V_k, a(\cdot, \cdot))$.

By the Definition 5.1, the error reduction operator is

\[
I - \omega M_k(\tilde{A}_k + B_k) = [I - \omega(I + \tilde{A}_k^{-1}B_k)] + [\omega(I - M_k\tilde{A}_k)(I + \tilde{A}_k^{-1}B_k)].
\]

The first operator on the right-hand side is selfadjoint. In fact, it follows from (5.6) that for $\omega = 2/(c_3 + 2)$,

\[
\|\omega(I - M_k\tilde{A}_k)(I + \tilde{A}_k^{-1}B_k)\|_1^{\sim} \leq \frac{c_3}{c_3 + 2} \|u\|_1^{\sim} \quad \forall u \in V_k.
\]

Also the second operator on the right-hand side of (5.7) can be bounded by (5.3) and the nonnegativeness of $\tilde{A}_k^{-1}B_k$. More precisely, we see that

\[
\|\omega(I - M_k\tilde{A}_k)(I + \tilde{A}_k^{-1}B_k)\|_1^{\sim} \leq \omega\gamma_1(c_3 + 1)\|u\|_1^{\sim}.
\]

Summing up these two estimates, (5.5) is proven if we choose $m$ and $p$ large enough in Lemma 5.2.

We remark that the number of smoothings $m$ in Definition 5.1 needs to be sufficient large (such that $\gamma_1 < (c_3 + 1)^{-1}$) in order to guarantee the convergence of (5.1). However, this is not observed in our numerical tests. Nevertheless, the numerical results depend critically on the relaxation parameter $\omega$ in (5.1) as revealed in our computation (see §6). It looks as if the method proposed in this section is more complicated than the method defined in §4. However, these two methods are about the same in terms of the complexity in their implementation. The second method requires an extra correction (5.1) on the top level, but it has simpler bilinear forms for
all lower level residual problems (cf. (2b) in Definition 5.1 and Definition 4.3). Again, if we take the iterative solution on previous level as the initial guess, then the method defined in Definition 5.1 is of optimal order (see Theorem 4.5). In Definition 5.1, we take the simple Richardson iteration as our outer iteration for simplicity only. As the linear system is well-preconditioned by the multigrid method, the theory carries over to other types of outer iterations. In practice, one would use the conjugate gradient method or other minimal residual methods as the outer iteration.

6. Numerical experiments. First, we consider a boundary value problem which has a smooth solution. Let \( \tilde{u} = x_1/(x_1^2 + x_2^2) \). Then \( \tilde{u} \) is the solution of \( \Delta u = 0 \) on the exterior of the square \([-1, 1] \times [-1, 1] =: \Omega \) with boundary value \( u = \tilde{u} \). We make the coupling at the circle \( \Gamma_0 = \{|x| = 3\} \). The meshes are displayed in Fig. 4(a)–(d).

To compare the coupling method for exterior problems and the finite element method for interior problems, we apply the finite element method to the Dirichlet problem \( -\Delta u = 0 \) with boundary condition \( u = \tilde{u} \) on both the square and the circle. The solution \( u \) is plotted in Fig. 4(e). In Fig. 4(f), we plot the error of the finite element solution of the coupling system (4.1), where the scaling is 100 times of that for Fig. 4(e). The error for the finite solution of the above Dirichlet problem is shown in Fig. 4(g). We can see the difference of the two finite element solutions in Fig. 4(h). The coupling method does provide an accurate solution. In fact, the error for the coupling method is a little smaller than that from the simple finite element problem.

Next, we compare the number of iterations needed for the two methods defined in Definitions 4.3 and 5.1. The criteria for terminating the iterations is that the pointwise iterative errors are smaller than \( 0.1 \times 2^{-2k} \) for the \( k \)th level finite element problems. The number of \( V \)-cycle (with four pre- and four post-smoothings of conjugate gradient iterations) iterations are listed in Table 1, where we also listed the number of iterations for the standard multigrid method in solving the Laplace equation with the Dirichlet boundary condition on the outer circle. The efficiency of our methods for solving exterior problems can be seen in Table 1. We remark that it would be better to use Jacobi-like fine-level smoothings in multigrid method in practice, also from theoretic point of view. But the conjugate gradient method cost very little more work in computation, compared with the Jacobi iteration. The convergence rates are about the same in the multigrid method.

<table>
<thead>
<tr>
<th>level</th>
<th># elements</th>
<th>Method 1</th>
<th>Method 2</th>
<th>Interior (-\Delta u = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3072</td>
<td>6</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>12288</td>
<td>7</td>
<td>7</td>
<td>5</td>
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<tr>
<td>7</td>
<td>49152</td>
<td>7</td>
<td>7</td>
<td>5</td>
</tr>
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</table>

In the second method, a relaxation parameter \( \omega \) is introduced in (5.1). As we pointed out earlier that omitting of \( \omega \), i.e. \( \omega = 1 \), may lead to a divergent iteration. We tested numerically the dependence of the convergence rate on the relaxation factor on the mesh level 4. The convergence rates are listed in Table 2, where the multigrid iteration uses the symmetric \( V \)-cycle with 4 pre- and post-smoothings.

We have assumed certain elliptic regularity for the Poisson equation (2.3) on the nonconvex domain \( \Omega_F \). In the following we numerically test the case for \( \Omega^c = \{[-1, 1] \times [-1, 1]\}^c \). Again the coupling is made at the circle \( \Gamma_0 = \{|x| = 3\} \). We solve the following exterior problem by the coupling method of finite elements and
**FIG. 4.** (a–d). Level 1 to 4 meshes of the finite element method. (e): Exact solution $u$ on level 4 mesh. (f): (100×) The error for the coupling method. (g): (100×) The error for finite elements. (h): (100×) Difference between the two solutions.

**TABLE 2**

<table>
<thead>
<tr>
<th>$\omega$ in (5.1)</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
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<tr>
<td>rate</td>
<td>.79</td>
<td>.69</td>
<td>.59</td>
<td>.49</td>
<td>.39</td>
<td>.29</td>
<td>.27</td>
<td>.52</td>
<td>.68</td>
<td>1.01</td>
</tr>
</tbody>
</table>

The dependence of the convergence rate on the parameter $\omega$.

boundary elements:

$$-\Delta u = \begin{cases} 
1 & \text{in } \Omega^c \cap \{|x| \leq 3\} \\
0 & \text{in } \{|x| > 3\}
\end{cases}$$

(6.1)

$$u = 0 \quad \text{on } \partial\Omega, \quad u = O(1) \quad \text{as } |x| \to \infty.$$  

The exact solution for (6.1) is not known. In Fig. 5(a), the numerical solution of the coupling method on mesh level 4 is plotted. The differences of the two numerical solutions on level 4 and level 3 can be found in Fig. 5(b). As can be seen, the larger
errors occur at the four corners of the inner square. Table 3 shows the numerically computed, convergence orders of the scheme for (6.1). Also in Table 3, one can find the order of convergence for the scheme for the previous test problem, where we have a smooth solution. Clearly the singularity near the four corners of the square affects the convergence of the finite element method.

![Figure 5](image)

**Fig. 5.** (a): The numerical solution of (6.1) at level 4 mesh. (b): (10x) The differences of level 4 and 3 solutions for (6.1).

<table>
<thead>
<tr>
<th>level</th>
<th>smooth solution</th>
<th>(6.1) solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>energy norm</td>
<td>conv-order</td>
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<td>7</td>
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<td>1.0203</td>
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**REFERENCES**


