Solutions to Homework MATH 350

Chapter 4:

4.2-1 $S=\{(i,j) : i,j \in \{1,\dots,6\}\}$

$j \backslash i$	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

$X : S \to \mathbb{N}, \quad (i,j) \xmapsto{X} i+j$

From the table we see that X takes values from $\{0,\dots,5\}$

$X : S \to \{0,\dots,5\}$

Also

$P(X=0) = \frac{6}{36} = \frac{1}{6}$

$P(X=1) = \frac{10}{36} = \frac{5}{18}$

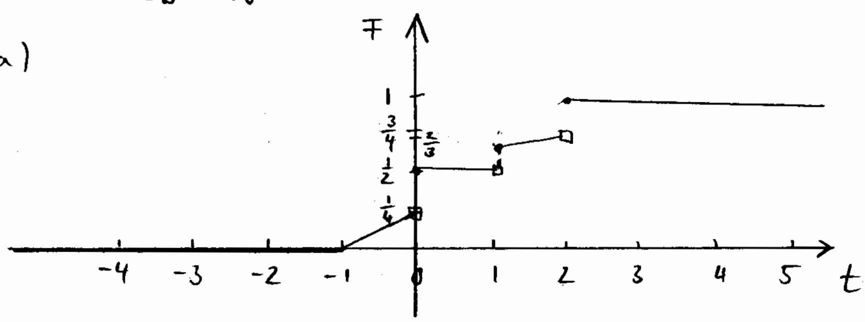
$P(X=2) = \frac{8}{36} = \frac{2}{9}$

$P(X=3) = \frac{6}{36} = \frac{1}{6}$

$P(X=4) = \frac{4}{36} = \frac{1}{9}$

$P(X=5) = \frac{2}{36} = \frac{1}{18}$

4.2-5 (a)



(b) $P(X<1) = F(1-) = \frac{1}{2}$, $P(X=1) = F(1) - F(1-) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

$P(1 \le X < 2) = F(2-) - F(1-) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$, $P(X > \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(X = \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{3}{2}-) = 0$, $P(1 < X \le 6) = F(6) - F(1) = 1 - \frac{2}{3} = \frac{1}{3}$.

4.2-11

$$F(t) = \begin{cases} \frac{t}{1+t} & t \ge 0 \\ 0 & t < 0. \end{cases}$$

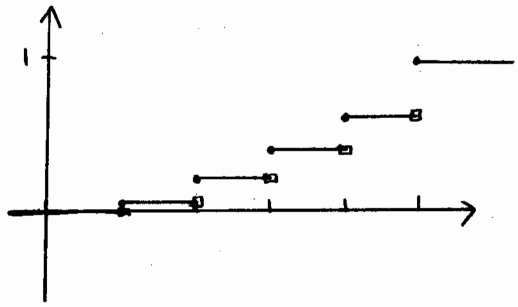
- $\lim_{t\to-\infty} F(t) = 0$ ✓
- $\lim_{t\to+\infty} F(t) = 1$ ✓
- F is right-continuous ✓
- For $t \ge 0$ $F'(t) = \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0$. (increasing on $[0,\infty)$)

  $\Rightarrow F$ is non-decreasing on $(-\infty,\infty)$ ✓

$\Rightarrow F$ is a distribution fct.

4.3-1 $p(x) = \frac{x}{15}$, $x = 1,2,3,4,5$

$$F(x) = \sum_{k=1}^{5} p(k)\, 1\{k \geq x\} = \begin{cases} 0 & x < 1 \\ \frac{1}{15} & 1 \leq x < 2 \\ \frac{3}{15} = \frac{1}{5} & 2 \leq x < 3 \\ \frac{6}{15} = \frac{2}{5} & 3 \leq x < 4 \\ \frac{10}{15} = \frac{2}{3} & 4 \leq x < 5 \\ 1 & 5 \leq x \end{cases}$$



4.3-3 Referring back to 4.2-1:
$X : S \to \{2, \dots, 12\}$ so $p(x) = P(X = x)$ for $x \in \{2, \dots, 12\}$
is the probability fct.

4.3-7 (a) $\sum_{x=1}^{5} kx = k \cdot (1+2+3+4+5) = 15k \overset{!}{=} 1 \Rightarrow \underline{\underline{k = \frac{1}{15}}}$

(b) $\sum_{x \in \{-2,0,1,2\}} k(1+x)^2 = k\left((-1)^2 + 1^2 + 2^2 + 3^2\right) = 15k \Rightarrow \underline{\underline{k = \frac{1}{15}}}$

(c) $\sum_{x=1}^{\infty} k\left(\frac{1}{9}\right)^x = k \frac{1}{9} \cdot \underbrace{\sum_{x=0}^{\infty} \left(\frac{1}{9}\right)^x}_{\text{geom. series}} = k \frac{1}{9} \frac{1}{1 - \frac{1}{9}} = \frac{1}{8} k \Rightarrow \underline{\underline{k = 8}}$

(d) $\sum_{x=1}^{n} kx = k \frac{(n+1)n}{2} \Rightarrow \underline{\underline{k = \frac{2}{n(n+1)}}}$

(e) $\sum_{x=1}^{n} kx^2 = k \frac{n(n+1)(2n+1)}{6} \Rightarrow \underline{\underline{k = \frac{6}{n(n+1)(2n+1)}}}$

4.3-13 The expression has 6 vowels and 9 consonants.
$p(k) = \binom{6}{k}\binom{9}{5-k} \cdot \binom{15}{5}^{-1}$ $k = 0, \dots, 5$ (Hypergeometric)

k	0	1	2	3	4	5
p(k)	$\frac{42}{1001}$	$\frac{252}{1001}$	$\frac{420}{1001}$	$\frac{240}{1001}$	$\frac{45}{1001}$	$\frac{2}{1001}$

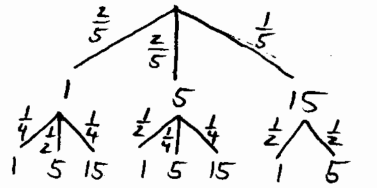
4.4-3 $S = \{1, \dots, 2{,}000\,000\}$
$X : S \to$ prize money.

$$E(X) = 30\, P(X=30) + 800 \cdot P(X=800) + 1{,}200{,}000 \cdot P(1{,}200{,}000)$$
$$= 30 \cdot \frac{4000}{2000\,000} + 800 \cdot \frac{500}{2000\,000} + 1{,}200{,}000 \cdot \frac{1}{2{,}000{,}000}$$
$$= \frac{12}{200} + \frac{40}{200} + \frac{12}{20} = \underline{\underline{\frac{43}{60}}} = 0.86$$

Therefore, the prize money paid is on average lower than the cost of a ticket.

4.4−5 $S=\{(i,j) : \{i,j\}\subset\{1,1,5,5,15\}\}$

$X : S \to \{2,6,10,16,20\}$



X	2	6	16	6	10	20	16	20
P	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

From the tree diagram:

$P(X=2)=\frac{1}{10}$

$P(X=6)=\frac{2}{5}$

$P(X=10)=\frac{1}{10} \Rightarrow E(X)=2\cdot\frac{1}{10}+6\cdot\frac{2}{5}+10\cdot\frac{1}{10}+16\cdot\frac{1}{5}+20\cdot\frac{1}{5}$

$P(X=16)=\frac{1}{5}$

$P(X=20)=\frac{1}{5}$ $\qquad = \frac{54}{5}=10.8>10$

Hence, this is not a fair game.
On average more is paid out than paid in.

4.4.−7 Note that we cannot sell more magazines that there are in stock.
Let X^n be the profit when n magazines are ordered.

demand k	4	5	6	7	$E(X^n)$
$p(k)$	$\frac{6}{18}$	$\frac{5}{18}$	$\frac{4}{18}$	$\frac{3}{18}$	
X^4	$4a-4\frac{2}{3}a$	$4a-4\frac{2}{3}a$	$4a-4\frac{2}{3}a$	$4a-4\frac{2}{3}a$	$\frac{4}{3}a$
X^5	$4a-5\frac{2}{3}a$	$5a-5\frac{2}{3}a$	$5a-5\frac{2}{3}a$	$5a-5\frac{2}{3}a$	$\frac{4}{3}a$
X^6	$4a-6\frac{2}{3}a$	$5a-6\frac{2}{3}a$	$6a-6\frac{2}{3}a$	$6a-6\frac{2}{3}a$	$\frac{19}{18}a$
X^7	$4a-7\frac{2}{3}a$	$5a-7\frac{2}{3}a$	$6a-7\frac{2}{3}a$	$7a-7\frac{2}{3}a$	$\frac{5}{9}a$

Since $\frac{4}{3}a>\frac{19}{18}a>\frac{5}{9}a$ one should order either 4 or 5 magazines.

4.5−1 Since both businesses have the same mean $\mu=150$, the one with the steadier income will be the first business with a lower standard deviation of \$30 (versus \$50).

4.5−3 $p(x)=\begin{cases}(|x-3|+1)/28 & x=-3,-2,-1,0,1,2,3\\ 0 & \text{otherwise}\end{cases}$

$Var(X)=E((X-E(X))^2)=E(X^2)-E(X)^2$

$p(k)$	$\frac{7}{28}$	$\frac{6}{28}$	$\frac{5}{28}$	$\frac{4}{28}$	$\frac{3}{28}$	$\frac{2}{28}$	$\frac{1}{28}$
k	−3	−2	−1	0	1	2	3
k^2	9	4	1	0	1	4	9

$\Rightarrow E(X)=-1, \quad E(X^2)=4$

$\Rightarrow Var(X)=4-(-1)^2=\underline{\underline{3}}$

4.5-5 $P(X=i)=\frac{1}{N}$ for $i=1,\dots,N$

$$\Rightarrow E(X)=\sum_{i=1}^{N} i\cdot\frac{1}{N}=\frac{N\cdot(N+1)}{2}\cdot\frac{1}{N}=\underline{\underline{\frac{N+1}{2}}}$$

$$E(X^2)=\sum_{i=1}^{N} i^2\frac{1}{N}=\frac{N(N+1)(2N+1)}{6}\frac{1}{N}=\frac{(N+1)(2N+1)}{6}=\frac{2N^2+3N+1}{6}$$

$$\Rightarrow Var(X)=E(X^2)-E(X)^2=\frac{2N^2+3N+1}{6}-\frac{N^2+2N+1}{4}$$

$$=\underline{\underline{\frac{N^2-1}{12}}}$$

$$\Rightarrow \sigma_X=\underline{\underline{\sqrt{\frac{N^2-1}{12}}}}$$

4.5-7 $E(X)=1$, $E(X(X-2))=3 \Rightarrow E(X^2)=3+2E(X)=5$

$$Var(-3X+5)=(-3)^2\,Var(X)=9\,(E(X^2)-E(X)^2)=9(5-1)=\underline{\underline{36}}$$

4.6-1 X_i = TV sets sold in store i $(i=1,2)$

$E(X_1)=13$ $\sigma_{X_1}=5$

$E(X_2)=7$ $\sigma_{X_2}=4$

The standardized variables are

$$X_1^*=\frac{X_1-E(X_1)}{\sigma_{X_1}}=\frac{10-13}{5}=-\frac{3}{5}=-0.6$$

$$X_2^*=\frac{X_2-E(X_2)}{\sigma_{X_2}}=\frac{6-7}{4}=-\frac{1}{4}=-0.25$$

Since $X_1^*<X_2^*$ he should hire the second salesperson.