Exam 3
January 26th, 2007
Math 242 Section 010
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Solutions
1) For the equations \( x = \tan^2(\theta) \quad y = \sec(\theta) \) where \( 0 \leq \theta < \frac{\pi}{2} \)

a) Calculate \( \frac{dy}{dx} \) using the parametric equations

\[
\frac{dy}{d\theta} = \sec \theta \tan \theta , \quad \frac{dx}{d\theta} = 2 \tan \theta \sec^2 \theta \quad \Rightarrow \\
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sec \theta \tan \theta}{2 \sec \theta \sec^2 \theta} = \frac{\tan \theta}{2 \sec \theta} = \frac{1}{2} \cos \theta
\]

b) Eliminate the parameter and get an equation in \( x \) and \( y \)

\( x = \tan^2 \theta , \quad y = \sec \theta \quad \Rightarrow \quad y^2 = \sec^2 \theta \quad \Rightarrow \\
\sqrt{x} - y^2 = \tan^2 \theta - \sec^2 \theta = -1 \quad \Rightarrow \quad x = y^2 - 1 , \quad y \in [1, \infty) \quad \Rightarrow \quad x \in [0, \infty) \)

\( (0,1) \)

\( (1,\sqrt{2}) \)

\( (x,y) \)

c) Graph the curve, indicating direction of increasing \( \theta \)
2) The following equations describe a curve called the 'Folium of Descartes'

\[ x = \frac{3 \cdot t}{1 + t^3} \quad y = \frac{3 \cdot t^2}{1 + t^3} \]

Write the equation of the tangent line at \( t = 1 \)

\[ \frac{dy}{dt} = \frac{6t(1 + t^3) - 3t^2(3t^2)}{(1 + t^3)^2} = \frac{6t + 6t - 9t^2}{(1 + t^3)^2} = \frac{6t + 3t}{(1 + t^3)^2} \]

\[ \frac{dx}{dt} = \frac{3(1 + t^3) - 3t(3t^2)}{(1 + t^3)^2} = \frac{3 + 3t^2 - 9t^2}{(1 + t^3)^2} = \frac{3 - 6t^2}{(1 + t^3)^2} \]

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t - 3t}{1 + t^3} = \frac{3 - 6t}{1 + t^3} \]

\[ \frac{dy}{dx} \bigg|_{t=1} = \frac{2 - 1}{1 - 2} = -1 \]

Need \( \rho \) at \( t = 1 \): \( x = \frac{3}{1 + 1} = \frac{3}{2} \), \( y = \frac{3}{1 + 1} = \frac{3}{2} \)

The tangent line has equation

\[ y - \frac{3}{2} = -(x - \frac{3}{2}) \Rightarrow y = -x + 3 \]
3) a) Graph \( r = 4 \cdot \cos(3 \cdot \theta) \) and \( r = 2 \) in the space at the bottom of the next page

b) On the graph, label the coordinates of the points of intersection of the two curves:

\[ 4 \cos 3 \theta = 2 \quad \Rightarrow \quad \cos 3 \theta = \frac{1}{2} \quad \Rightarrow \quad 3 \theta = \frac{\pi}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{9} \]

The graph is a 3-petaled rose. It displays a symmetry of sorts at the petals are evenly spaced. The "first" petal is symmetric across the x-axis, so the point of intersection below the x-axis is \( \theta = \frac{\pi}{9} \). All other petals are multiples of \( \frac{2\pi}{3} \) units away from the petal. So, \( (3, \frac{\pi}{9}), (0, \frac{2\pi}{3}), (2, \frac{13\pi}{9}) \) and \( (2, \frac{13\pi}{9}), (0, \frac{4\pi}{3}), (1, \frac{17\pi}{9}) \)

c) Calculate the area inside \( r = 4 \cdot \cos(3 \cdot \theta) \) but outside \( r = 2 \)

Because of the symmetry of the region, we only need to calculate the area in the upper half of the first petal and then multiply by 6. We calculate the area in the petal from \( 0 \) to \( \frac{\pi}{9} \) and then subtract the area in the circle.

\[
6 \left[ \int_0^{\frac{\pi}{9}} \frac{1}{2} [4 \cos 3 \theta]^2 \, d\theta - \frac{1}{2} (2)^2 \frac{\pi}{3} \right] = 3 \left[ \int_0^{\frac{\pi}{9}} 16 \cos^2 3 \theta \, d\theta - \frac{4\pi}{3} \right] =
\]

\[
= 24 \left[ 1 + \cos 6 \theta \right]_0^{\frac{\pi}{9}} - \frac{4\pi}{3} = 24 \left[ \theta + \frac{1}{6} \sin 6 \theta \right]_0^{\frac{\pi}{9}} - \frac{4\pi}{3} =
\]

\[
= 24 \left[ \frac{\pi}{9} + \frac{1}{6} \sin \left( \frac{2\pi}{3} \right) \right] - \frac{4\pi}{3} = \frac{8\pi}{3} - \frac{4\pi}{3} + 2\sqrt{3} = \frac{4\pi}{3} + 2\sqrt{3}
\]
d) For \( r = 4 \cdot \cos(3 \cdot \theta) \) find the equation of the tangent line at \( \theta = \frac{\pi}{3} \) and graph it with the curve.

\[
\frac{dy}{dx} = \frac{f'(\theta) \cos \theta + f(\theta) \sin \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}
\]

\[
f(\theta) = r(\theta) = 4 \cos \left( \frac{3\theta}{3} \right) = -4
\]

\[
f'(\theta) = r'(\theta) = -12 \sin \left( \frac{3\theta}{3} \right) \theta \frac{\pi}{3} = 0
\]

So,

\[
\frac{dy}{dx} \bigg| \frac{\pi}{3} = \frac{-4 \cos \frac{2\pi}{3}}{-4 \sin \frac{\pi}{3}} = -\cot \frac{\pi}{3} = -\frac{\sqrt{3}}{2}
\]

Polar points are \((-4, \frac{\pi}{3})\), \((-4, \frac{\pi}{3})\), \((-4, \frac{\pi}{3})\)

So the eqn is

\[
y + 2\sqrt{3} = -\frac{\sqrt{3}}{2} (x + 2)
\]
4) For the equation \( y^2 + 4 \cdot y + 8 \cdot x - 12 = 0 \)

a) put the equation in standard form and identify the curve

Complete the square for \( y: y^2 + 4y + 8x - 12 = 0 \) \( \Rightarrow y^2 + 4y + 4 + 8x - 12 = 4 \)

\[ (y + 2)^2 = -8x + 16 \Rightarrow (y + 2)^2 = -8(x - 2) \]  \( \text{Parabola w/ axis parallel to x-axis} \)

\[ 4p = -8 \Rightarrow p = -2 \]

focus \( (2 - 2, -2) = (0, -2) \)

directrix: \( x = 2 - (-2) = 4 \)

b) graph the curve and identify and label all significant points and references (i.e. focus or foci, vertex or vertices, directrix or asymptotes, etc.)
5) For the equation \( x^2 + 36 \cdot y = 9 \cdot y^2 + 72 \)

a) put the equation in standard form and identify the curve

\[
\begin{align*}
5) & \quad x^2 - 9y^2 + 36y = 72 \\
\Rightarrow & \quad x^2 - 9(y^2 - 4y + 4) = 72 - 36 \\
\Rightarrow & \quad x^2 - 9(y - 2)^2 = 36 \\
\Rightarrow & \quad \frac{x^2}{36} - \frac{(y - 2)^2}{4} = 1 \\
\text{hyperbola with transverse axis parallel to } x-axis
\end{align*}
\]

b) graph the curve and identify and label all associated points and references as in part b of question 4

![Graph of the hyperbola]

- Center \((0, 2)\)
- Vertices \((-2, 2), (2, 2)\)
- So foci \(B (\pm \sqrt{10}, 2)\)

\[
C^2 = a^2 + b^2 \Rightarrow C^2 = 40 \Rightarrow C = 2\sqrt{10}
\]
c) Look at the original equation in this problem (5). If the 72 on the right hand side of the equation is replaced with 36, what is the resulting equation in standard form? Does it have any solutions? If so, what are the resulting curve or curves?

Start at the stage labeled 0

\[ x^2 - 9(y^2 - 4y + 4) = 36 - 36 \] \[ \Rightarrow x^2 - 9(y - 2)^2 = 0 \] \[ \Rightarrow x^2 = 9(y - 2)^2 \]

It has an infinite # of solns. \[ \Rightarrow x = \pm 3(y - 2) \]

The curves are two intersecting lines.

6) If the sequence converges, find its limit

Assume the limit exists, then

\[ a_n = (\ln(n))^\frac{1}{n} \]

\[ \lim_{n \to \infty} \left[ \ln(n) \right]^\frac{1}{n} = 0 \] so proceed as follows.

\[ y = \lim_{n \to \infty} \left( \ln(n) \right)^\frac{1}{n} \begin{align*}
\Rightarrow \ln y &= \lim_{n \to \infty} \frac{1}{n} \ln(\ln(n)) \\
\begin{align*}
\Rightarrow \ln y &= \lim_{n \to \infty} \frac{\ln(\ln(n))}{n} = \frac{\infty}{\infty} \text{ so apply L' Hospital's} \\
(\text{Even though } n \text{ is discrete, we may treat it as continuous,} \\
\text{because all terms are continuous on the equivalent } f^n) \\
\Rightarrow \ln y &= \lim_{n \to \infty} \frac{\ln(\ln(n))}{n} = \lim_{n \to \infty} \frac{1}{n} = 0 \Rightarrow y = 1
\end{align*}
\end{align*}

So \[ \lim_{n \to \infty} \left[ \ln(n) \right]^\frac{1}{n} = 1 \] and the series converges.
7) For each series, determine if it converges or diverges. If possible, calculate the sum.

\[
\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)}
\]

**Apply partial fraction technique**

\[
l = A(n+1) + B(n-1)
\]

\[
n=1: A=\frac{1}{2} \quad n=-1: B=-\frac{1}{2}
\]

\[
\frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1}
\]

The series is a telescoping series, so we construct the nth term, seeing aside the \(\frac{1}{2}\) for now.

\[
S_n = (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \cdots + (\frac{1}{n-2} - \frac{1}{n}) + (\frac{1}{n-1} - \frac{1}{n+1})
\]

\[
S_n = 1 + \frac{1}{2} - (\frac{1}{n+1}) \Rightarrow \lim_{n \to \infty} \left( \frac{3}{2} - \frac{1}{n+1} \right) = \frac{3}{2}
\]

So, the sum is \(\frac{3}{2}\).

b) \[\sum_{n=2}^{\infty} \frac{T^n}{q^n} \] geometric series with \(|r| < 1\) so convergent and has a sum. We look at the series which starts \(0^0\).

\[
\sum_{n=0}^{\infty} \left( \frac{5}{9} \right)^n = \frac{1}{1 - \frac{5}{9}} = \frac{9}{4}
\]

The first 2 terms are 1, \(\frac{5}{9}\). So we subtract them to get the sum.

\[
\sum_{n=2}^{\infty} \left( \frac{5}{9} \right)^n = \frac{9}{4} - \frac{5}{9} - \frac{5}{9} = \frac{25}{36}
\]

c) \[\sum_{n=2}^{\infty} \frac{\ln(n)}{n} \] Apply the Integral Test

\[
\int_{2}^{\infty} x^{-2} \ln(x) \, dx = \lim_{b \to \infty} \left[ -x^{-1} \ln x \bigg|_{2}^{b} + \int_{2}^{b} x^{-2} \, dx \right] =
\]

Integrate by parts and apply techniques of improper integrals

\[
u = \ln x \quad dv = x^{-2} \, dx \]

\[du = \frac{1}{x} \, dx \quad V = -x^{-1} \]

\[
\lim_{b \to \infty} \left[ -\ln b + \frac{\ln 2}{2} \right] = \ln 2 + \frac{1}{2}
\]

(by L'Hopital's)

So the series converges.
d) \[\sum_{n=1}^{\infty} \frac{\sqrt{n} + \sqrt{n}}{\sqrt{n} + n}\]  
Limit comparison to \(\frac{1}{n}\)

\[
\lim_{n \to \infty} \left(\frac{\sqrt{n} + \sqrt{n}}{\sqrt{n^3} + n}\right) \cdot \frac{1}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^{3/2} + n^{3/4}}{\sqrt{n^3} + n} = \lim_{n \to \infty} \frac{n^{3/2} + n^{3/4}}{\sqrt{n^3} + n} \cdot \frac{\sqrt{n^3} + n}{\sqrt{n^3} + n} = 
\]

\[
= \lim_{n \to \infty} \frac{1 + n^{-1/4}}{\sqrt{1 + n^{-2}}} = 1 
\]
So the series diverges.

8) By using an infinite series write \(.35414141...\) as a fraction

\[.35414141\ldots = .35 + .00414141\ldots = \frac{35}{100} + \frac{41}{10^4} + \frac{41}{10^8} + \cdots \]

\[
= \frac{35}{100} + \frac{41}{10^4} \left(1 + \frac{1}{10^2} + \cdots\right) = \frac{35}{100} + \frac{41}{10^4} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n
\]

So the series has sum

\[
\frac{1}{1 - \frac{1}{100}} = \frac{100}{99} 
\]

plugging back in we get

\[
\frac{35}{100} + \frac{41}{10^4} \cdot \frac{100}{99} = \frac{35}{100} + \frac{41}{9900} = \frac{3465}{3900} + \frac{41}{9700} = \frac{3506}{9700} = \frac{3506}{9700}
\]

\[
= \frac{125}{4980}
\]
Extra Credit

Find whether the series converges or diverges

\[ \sum_{n=1}^{\infty} \frac{\tan \left( \frac{1}{n} \right)}{n} \]

Limit comparison to \( \frac{1}{n} \)

\[ \lim_{n \to \infty} \frac{\tan \left( \frac{1}{n} \right)}{\frac{1}{n}} = \frac{0}{0} \]

So, we apply L'Hopital's rule with our understanding that we could write a cs for containing all terms of the sequence.

\[ \lim_{n \to \infty} \frac{-\frac{1}{n^2} \sec^2 \frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n \to \infty} \sec^2 \frac{1}{n} = 1 \]

So the series diverges.
3b) The previous solution was more geometric. Here is one which is more mathematical.

\[ 4 \cos \theta = 2 \Rightarrow \cos \theta = \frac{1}{2} \]

Cosine is positive in quadrants I and III.
Also, every \( 2\pi \) we come back to the same \( \theta \).
So the above implies

\[ \theta = \frac{\pi}{3} + 2n\pi, \quad n = 0, 1, 2, \ldots \Rightarrow \]

\[ \Rightarrow \theta = \frac{\pi}{3} + \frac{2\pi n}{3} = \frac{\pi + 6n\pi}{9} = \frac{\pi(6n+1)}{9} \]

or

\[ \theta = \frac{5\pi}{3} + 2n\pi, \quad n = 0, 1, 2, \ldots \Rightarrow \]

\[ \Rightarrow \theta = \frac{5\pi}{3} + \frac{2\pi n}{3} = \frac{5\pi + 6n\pi}{9} = \frac{\pi(6n+5)}{9} \]

Each of the above generates 3 unique pts, for a total of 6.

\[ \theta = \frac{(6n+1)\pi}{9} \quad n = 0; \frac{\pi}{9}, \quad n = 1; \frac{7\pi}{9}, \quad n = 2; \frac{13\pi}{9} \]

\[ \theta = \frac{(6n+5)\pi}{9} \quad n = 0; \frac{5\pi}{9}, \quad n = 1; \frac{11\pi}{9}, \quad n = 2; \frac{17\pi}{9} \]

If \( n = 3 \) we start to get overlap. So the pts are

\[ (2, \frac{\pi}{9}), \quad (2, \frac{5\pi}{9}), \quad (2, \frac{7\pi}{9}), \quad (2, \frac{11\pi}{9}), \quad (2, \frac{13\pi}{9}), \quad (2, \frac{17\pi}{9}) \]