Final Exam
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Math 243 Section 456
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Print Name: Answer Key

Academic Honesty Policy: Students at the University are expected to be honest and forthright in their academic endeavors. To falsify the results of one's research, to steal the words or ideas of another, to cheat on an examination, or to allow another to commit an act of academic dishonesty corrupts the essential process by which knowledge is advanced. It is the official policy of the University of Delaware that all acts or attempted acts of alleged academic dishonesty be reported to the Dean of Students Office.

By signing below, I acknowledge that I have read the above and that I have neither given nor received assistance on this examination.

Sign Name: Answer Key

All work must be shown to receive credit. Try to do all your work on this paper and clearly indicate your final answer. If you need additional paper, ask for it. All paper used must be turned in with your exam.

No Calculators may be used. All cellular phones and/or electronic devices of any kind must be turned off and put away. CHECK YOUR PHONE NOW!! If your phone rings during the exam, you may be asked to leave.
1. Show that the midpoint of the line segment from \( P_1(x_1, y_1, z_1) \) to \( P_2(x_2, y_2, z_2) \) is 
\[
\overrightarrow{\text{M}} = \frac{1}{2} \overrightarrow{P_1P_2} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)
\]
\[
\overrightarrow{P_1P_2} = \left( x_2 - x_1, y_2 - y_1, z_2 - z_1 \right)
\]
\[
\overrightarrow{\text{M}} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)
\]
\[
\overrightarrow{P_1P_2} = \left( x_2 - x_1, y_2 - y_1, z_2 - z_1 \right)
\]
\[
\overrightarrow{\text{M}} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)
\]

2. For the curve \( r(t) = \langle t^3, 1 + (\sin t - t \cos t), t + (\cos t + t \sin t) \rangle \)

a. find the arc length on the interval \( 0 \leq t \leq \pi \) and 
b. find the general curvature

a. \[ \vec{r}'(t) = \langle 3t^2, \cos t - t \sin t, \sin t + t \cos t \rangle \]
\[ \left\| \vec{r}'(t) \right\| = \sqrt{9t^4 + 1 + \cos^2 t} = \sqrt{1 + 9t^4} \]
\[ \int_{0}^{\pi} \left\| \vec{r}'(t) \right\| dt = \frac{\sqrt{1 + \pi^4}}{3} \]

b. \[ \vec{r}''(t) = \langle 6t, -\sin t - \cos t, \cos t + t \sin t \rangle \]
\[ \left\| \vec{r}''(t) \right\| = \sqrt{36t^2 + 1 + \sin^2 t + \cos^2 t} = \sqrt{36t^2 + 1} \]
\[ \kappa = \frac{\left\| \vec{r}'(t) \right\| \left\| \vec{r}''(t) \right\| - \left( \vec{r}'(t) \cdot \vec{r}''(t) \right)}{\left\| \vec{r}'(t) \right\|^3} = \frac{1}{5t} \]
3. Find an equation for the curve of intersection of \( z^2 + y^2 = 4 \) and \( z = x^2 \) and sketch the curve.

\[
x = 2 \cos \theta, \quad y = 2 \sin \theta
\]

\[
z = 4 \cos^2 \theta
\]

\[
\vec{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta, 4 \cos^2 \theta \rangle
\]
4. Calculate the following limits.

a. \( \lim_{i \to \infty} \left( t \mathbf{i} + \frac{t^2 - 4}{t^2 + 2t} \mathbf{j} + \frac{1}{t} \mathbf{k} \right) = \langle 2, 2, \frac{1}{2} \rangle \)

b. \( \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 (\cos^2 \theta + \sin^2 \theta)}{r^2} = \lim_{r \to 0} r (\cos^2 \theta + \sin^2 \theta) = 0 \)

c. \( \lim_{(x,y) \to (3,3)} \frac{2x^2 + 3y}{2 + xy + 6y^2} = 36 + 3 + 12 + 2y = 75 \)

d. \( \lim_{(x,y,z) \to (0,0,0)} \frac{x^3 y^2}{x^2 + y^2 + z^2} \) (Hint: Convert to spherical.)

\[ \lim_{\rho \to 0} \rho^3 (\sin^2 \theta \cos \theta \cos \phi \sin \phi) = 0 \]
5. Write an equation of the tangent plane and the normal line to \( z = x^2 + xy + 3y^2 \) at the point \((1, 1, 5)\).

\[
F(x, y, z) = x^2 + xy + 3y^2 - z = 0; \quad \nabla F = \langle 2x + y, x + 6y, -1 \rangle
\]

\[
\nabla F(1, 1, 5) = \langle 3, 7, -1 \rangle; \quad s = 1 + 1 + 3 = 5
\]

Plane: \( 3(x-1) + 7(y-1) - (z-5) = 0 \)

Line: \( \vec{r}(t) = \langle 3, 7, -1 \rangle + t \langle 1, 1, 5 \rangle \)

6. Find the indicated derivatives.

a. \( \frac{\partial w}{\partial r}; \ w = \sqrt{4 - 2x^2 - 2y^2}; \ x = r \cos \theta; \ y = r \sin \theta \)

\[
\frac{\partial w}{\partial r} = \frac{-2x}{\sqrt{4 - 2x^2 - 2y^2}}; \quad \frac{\partial w}{\partial \theta} = \frac{-2y}{\sqrt{4 - 2x^2 - 2y^2}}
\]

b. \( \frac{\partial w}{\partial z}; \ z^2 + y^2 + x^2 + 6xz - 8w^2 = 5 \)

\[
F(x, y, z, w) = x^2 + y^2 + z^2 + 6xz - 8w^2
\]

\[
\begin{align*}
F_x &= 2x + 6z \\
F_y &= 2y + 6z \\
F_z &= 2x - 6w \\
F_w &= 6x - 16w
\end{align*}
\]

\[
\frac{\partial w}{\partial x} = -\left( \frac{2x + 6z}{6x - 16w} \right) = \frac{2x + 6z}{16w - 6x}
\]
7. Find and classify all critical points of \( f(x, y) = x^3y + 12x^2 - 8y \).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2y + 24x \\
\frac{\partial f}{\partial y} &= x^3 - 8 \\
\frac{\partial f}{\partial x}(2, y) &= 0 \Rightarrow x^3 - 8 = 0 \Rightarrow x = 2 \\
\frac{\partial f}{\partial x}(2, y) &= 0 \Rightarrow 12y + 48 = 0 \Rightarrow y = -4 \\
\frac{\partial f}{\partial x} &= 6xy + 24 \\
\frac{\partial^2 f}{\partial y \partial x} &= 0 \\
\frac{\partial^2 f}{\partial x \partial y} &= 3x^2 \\
D &= -9x^2 \\
&= f(2, -4) \text{ is a saddle point} \\
f(2, -4) &= -32 + 48 + 32 = 48 \\
(2, -4, 48) \text{ is a saddle point}.
\end{align*}
\]
8. Find the maximum value of \( f(x, y, z) = xyz \) subject to \( x^2 + 2y^2 + 3z^2 = 6 \).

\[ \nabla f(x, y, z) = \langle yz, xz, xy \rangle \quad \nabla g(x, y, z) = \langle 2x, 4y, 6z \rangle \]

\[ yz = \lambda xz \quad \text{(1)} \]
\[ xz = \lambda xy \quad \text{(2)} \]
\[ xy = \lambda 6z \quad \text{(3)} \]
\[ x^2 + 2y^2 + 3z^2 = 6 \quad \text{(4)} \]

By symmetry, we see \((0,0,\pm\sqrt{2})\), \((\pm\sqrt{2},0,0)\).

\[ 2x^2 = 4y^2 \implies x = \pm 2y \quad \text{or} \quad \Theta = 4y^2 + 3z^2 = 6 \quad \text{(5)} \]

\[ y = 2z \implies 0 = \lambda (4y^2 - 6z^2) \implies 4y^2 = 6z^2 \quad \text{(6)} \]

\[ \lambda = 0 \quad \Rightarrow \quad y = 0, z = 0 \quad \text{or} \quad x = 0 \]

\[ \lambda = 0 \quad \Rightarrow \quad x = 0, y = 0 \quad \Rightarrow \quad x = y = z = 0 \]

\[ y = 4 \quad \Rightarrow \quad y = \pm 1 \quad \Rightarrow \quad x = \pm \sqrt{2} \]

\( \text{Thus} \quad (\pm \sqrt{2}, \pm 1, \pm \sqrt{2}) \), \((0,0,\pm\sqrt{2})\), \((\pm\sqrt{2},0,0)\).

\[ \text{max occurs at} \quad (\pm \sqrt{2}, \pm 1, \pm \sqrt{2}) \]

\[ f(\pm \sqrt{2}, \pm 1, \pm \sqrt{2}) = \frac{2}{\sqrt{3}} \]
9. Do the following

a. Find the volume bounded by $z = x^2 + 3y^2$ and $x = 0, y = 0, y = x, z = 0$.

$$\iint_D x^2 + 3y^2 \, dy \, dx = \int_0^1 x^2 y + y^3 \bigg|_0^1 \, dx = 2\int_0^1 x^2 \, dx = \frac{1}{2}$$

b. \( \iint_D x^2 y^2 \, dA \); \( D = \{(x,y)|0 \leq x \leq 2, -x \leq y \leq x\} \)

$$\int_0^2 \int_{-x}^x x^2 y^2 \, dy \, dx = \int_0^2 \frac{x^3}{3} y^3 \bigg|_{-x}^x \, dx = \frac{2}{3} \int_0^2 x^6 \, dx = \frac{256}{21}$$

c. \( \iint_D y \cos(x^2) \, dx \, dy \)

$$\int_0^9 \int_0^{\sqrt{x}} y \cos(x^2) \, dy \, dx = \frac{1}{2} \int_0^9 x \cos(x^2) \, dx = \frac{1}{2} \sin(x^2) \bigg|_0^9 = \frac{1}{2} \sin(9)$$
d. $\int \int_{R} \cos (x^2 + y^2) \ dA; \quad R \text{ is the region that lies to the left of the y-axis between the}
\text{circles } x^2 + y^2 = 1 \text{ and } x^2 + y^2 = 4$

\[ \int_{0}^{2\pi} \int_{1}^{2} r \cos r^2 \ dr \ d\theta = \frac{\pi}{4} \sin (r) \bigg|_{1}^{2} = \frac{\pi}{4} (\sin (4) - \sin (1)) \]

\[ \int_{0}^{2\pi} \int_{0}^{1} r^3 \sin \theta \cos \theta \ dr \ d\theta = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{1} r^2 \sin \theta \cos \theta \ dr \ d\theta =
\frac{1}{2} \int_{0}^{\frac{\pi}{2}} \left[ \int_{0}^{1} r^2 \sin \theta \cos \theta \ dr \right] d\theta = \frac{1}{2} \int_{0}^{\frac{\pi}{2}} \left[ \frac{1}{3} r^3 \right]_{0}^{1} \sin \theta \cos \theta \ d\theta =\frac{1}{3} \int_{0}^{\frac{\pi}{2}} \sin \theta \cos \theta \ d\theta = \frac{1}{4} \cdot \frac{4}{2} = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6} \]

f. $\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} \frac{1}{\sqrt{x^2+y^2}} \ dx \ dy$

\[ \int_{0}^{\frac{\pi}{2}} \int_{0}^{1} r^2 \sin \phi \ d\phi \ d\theta = \]

\[ \frac{\pi}{2} \left( -\cos \phi \right) \left|_{0}^{\frac{\pi}{2}} \right. \int_{0}^{1} r^2 \ sin \phi \ d\phi = \frac{\pi}{2} \left( 1 - \frac{1}{2} \right) \left( \frac{\pi}{2} \right) = \frac{\pi^2}{8} \]

\[ \frac{4}{3} \pi \left( 1 - \frac{1}{2} \right) \left( \frac{\pi}{2} \right) = \frac{4}{3} \pi \left( \frac{\pi}{2} \left( 1 - \frac{1}{2} \right) \right) = \frac{4}{3} \pi \left( \frac{\pi}{2} \right) = \frac{4}{3} \pi^2 \]

\[ \int_{0}^{\frac{\pi}{2}} \int_{0}^{1} r^2 \sin \phi \ d\phi \ d\theta = \]

\[ \frac{\pi}{2} \left( -\cos \phi \right) \left|_{0}^{\frac{\pi}{2}} \int_{0}^{1} r^2 \ sin \phi \ d\phi = \frac{\pi}{2} \left( 1 - \frac{1}{2} \right) \left( \frac{\pi}{2} \right) = \frac{\pi^2}{8} \]

\[ \frac{4}{3} \pi \left( 1 - \frac{1}{2} \right) \left( \frac{\pi}{2} \right) = \frac{4}{3} \pi \left( \frac{\pi}{2} \left( 1 - \frac{1}{2} \right) \right) = \frac{4}{3} \pi \left( \frac{\pi}{2} \right) = \frac{4}{3} \pi^2 \]
10. Evaluate the following:

$$\int_C \tan y \, dx + x \sec^2 y \, dy$$

where $C$ is given by beginning and ending at (2,3) by going around

$$(x - 4)^2 + (y - 3)^2 = 4$$

Field is conservative, so $\int_C \nabla \phi \cdot dr = 0$. 


11. Verify Green's Theorem by performing the line integral and the integral over the region.

\[ \oint_C xy \, dx + x^2y^3 \, dy \]

C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

**On C_1**

\[ \begin{align*}
\gamma &= 0 \Rightarrow x_1 = 0 \\
\gamma &= 0 \Rightarrow dy = 0
\end{align*} \]

\[ \int_0^1 y^2 \, dy = \frac{1}{3} \]

**On C_2**

\[ \begin{align*}
\gamma &= x_1 \Rightarrow dx = 0 \\
\gamma &= 0 \Rightarrow dy = 2dx
\end{align*} \]

\[ \int_0^1 2x^3 + x^2y^3 \, dx = 0 \]

\[ \begin{align*}
\oint_C = 20 - \frac{10}{3} = \frac{20}{3}
\end{align*} \]

Green's Theorem

\[ \int_0^1 2x \, dy - x^2 \, dx = \int_0^1 xy^2 - xy_0^2 \, dx = 8x^2 - \frac{2}{3} = \frac{4}{3} \]

\[ \sqrt{3} \]

11
12. **Extra Credit:** Use the Divergence Theorem to evaluate

\[ \iiint_S \mathbf{F} \cdot d\mathbf{S} \]

\[ \mathbf{F}(x, y, z) = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j} + yz^2 \mathbf{k} \]

\( S \) is the surface of box bounded by the planes \( x = 0, x = 1, y = 0, y = 1, z = 0, z = 2 \)

\[ \nabla \cdot \mathbf{F} = e^x \sin y - e^x \sin y + 2yz = 2yz \]

\[ \iiint_B 2yz \, dz \, dy \, dx = 2 \cdot \frac{1}{2} \cdot 2 = 2 \]