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Sign Name: Answer Key

All work must be shown to receive credit. Try to do all your work on this paper and clearly indicate your final answer. If you need additional paper, ask for it. All paper used must be turned in with your exam.

No Calculators may be used. All cellular phones and/or electronic devices of any kind must be turned off and put away. You should check your phone now. If your phone rings during the exam, you may be asked to leave.
1. Find all points on the curve $x^2y - xy^2 = 2$ where the tangent line is horizontal or vertical. 

**Hint:** Think carefully about what it means for the tangent line to be vertical. What derivative must be zero?

$$2xy + x^2 \frac{dy}{dx} - y^2 - 2xy \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

**Horizontal:**

$$\frac{dy}{dx} = 0 = \frac{y(y - 2x)}{x(x - 2y)} \Rightarrow y = 0 \text{ or } (y - 2x) = 0 \text{ but } y = 0 \text{ is not on the curve.} \Rightarrow y = 2x$$

$$x^2(2x) - x(y^2) = 2 \Rightarrow 2x^3 - yx^2 = 2 \Rightarrow -2x^3 = -1 \Rightarrow x = -1 \Rightarrow y = -2$$

**Vertical:**

$$\frac{dx}{dy} = 0 = x = 0 \text{ or } (x - 2y) = 0 \text{ but } x = 0 \text{ not on the curve.} \Rightarrow x = 2y$$

$$y^3 - 2y^2 = 2 \Rightarrow 2y^3 = 2 \Rightarrow y = 1 \Rightarrow x = 2$$

**Answer:** $\rho = (-1, -2)$ horizontal, $\rho = (2, 1)$ vertical.

2. For the following, find $\frac{dy}{dx}$.

a. $x^3 + y^3 = 1 \Rightarrow \frac{d}{dx}(x^3 + y^3) = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2}$

b. $\tan(x + y) = \frac{y}{1 + x^2} \Rightarrow \sec^2(x + y)(1 + \frac{dy}{dx}) = \frac{dy}{dx} + \frac{-2xy}{(1 + x^2)^2}$

$$= \frac{dy}{dx} = -\sec^2(x + y) \frac{\frac{2xy}{(1 + x^2)^2}}{\sec^2(x + y) - \frac{1}{1 + x^2}} = -\frac{(1 + x^2)^2 \sec^2(x + y) - 2xy}{(1 + x^2)^2 \sec^2(x + y) - (1 + x^2)}$$

c. $y = \cos^{-1}(e^{2x})$

$$y' = \frac{-1}{\sqrt{1 - e^{4x}}} \cdot 2e^{2x} = -\frac{2e^{2x}}{\sqrt{1 - e^{4x}}}$$
e. $y = \frac{(x^4 + 1)^2 \sin^{-1}(x) \ln(x)}{\sqrt{x^4 + 2}} \Rightarrow \ln y = \ln \left( \frac{(x^4 + 1)^2 \sin^{-1}(x) \ln(x)}{\sqrt{x^4 + 2}} \right)$

$\Rightarrow \frac{y'}{y} = \frac{8x^3 \sin^{-1}(x) \ln(x)}{x^4 + 1} + \frac{1}{\sin^{-1}(x) \sqrt{1-x^2}} + \frac{1}{x \ln(x)} - \frac{x}{x^4 + 2}$

$\Rightarrow y' = \left( \frac{(x^4 + 1)^2 \sin^{-1}(x) \ln(x)}{\sqrt{x^4 + 2}} \right) \left( \frac{8x^3}{x^4 + 1} + \frac{1}{\sin^{-1}(x) \sqrt{1-x^2}} + \frac{1}{x \ln(x)} - \frac{x}{x^4 + 2} \right)$

3. A roast turkey is taken from the oven when its temperature has reached $185^\circ F$ and is placed on a table in a room where the temperature is $75^\circ F$.

a. If the temperature of the turkey is $150^\circ F$ after half an hour what is the temperature after 45 minutes?

b. When will the turkey have cooled to $100^\circ F$?

\[
\frac{dT}{dt} = k(T - T_o) = k(T - 75) \quad \text{let } y = T - 75 \Rightarrow \frac{dy}{dt} = \frac{dy}{dt} = ky
\]

$\Rightarrow y = y_0 e^{kt}$

we have $T_0 = 185 \Rightarrow y_0 = 110 \quad \therefore y = 110 e^{kt}$

Need $k$: $75 = 110 e^{k \cdot 30} \Rightarrow \frac{1}{30} \ln \frac{75}{110} = k \Rightarrow \frac{1}{30} \ln \frac{15}{22} = k$

a. After 45 minutes, $y = 110 e^{\frac{45}{30} \ln \left( \frac{15}{22} \right)} = 110 \left( \frac{15}{22} \right)^{\frac{3}{2}} \Rightarrow$

$T = 110 \left( \frac{15}{22} \right)^{\frac{3}{2}} + 75$

b. $25 = 110 e^{\frac{5}{30} \ln \left( \frac{75}{110} \right)} \Rightarrow \ln \frac{25}{110} = \frac{5}{30} \ln \left( \frac{15}{22} \right) \Rightarrow$

$\Rightarrow t = \frac{30 \ln \frac{5}{\ln \left( \frac{15}{22} \right)}}{\ln \left( \frac{15}{22} \right)}$
4. Each edge of a variable cube is increasing at a rate of 3 inches per second. How fast is the volume of the cube increasing when an edge is 12 inches long?

\[
\frac{dl}{dt} = 3 \quad \Rightarrow \quad V = l^3 \Rightarrow \frac{dV}{dt} = 3l^2 \frac{dl}{dt} \Rightarrow
\]

\[
\Rightarrow \frac{dV}{dt} = 3 \cdot 144 \cdot 3 = 9 \cdot 144 = 1296
\]

5. We assume that an oil spill is being cleaned up by deploying bacteria that consume the oil at 4 cubic feet per hour. The oil spill itself is modeled in the form of a very thin cylinder whose height is the thickness of the oil slick. When the thickness of the slick is .001 feet, the cylinder is 500 feet in diameter. If the height is decreasing at .0005 feet per hour, at what rate is the area of the slick changing?

\[
\frac{dV}{dt} = -4 \quad \frac{dh}{dt} = -0.0005
\]

\[
V = bh \Rightarrow \frac{dV}{dt} = \frac{db}{dt} h + b \frac{dh}{dt} \Rightarrow -4 = \frac{db}{dt} (.001) + b (-0.0005)
\]

(b is area of the base)

\[
\Rightarrow \frac{db}{dt} = \frac{-4 + 0.0005b}{0.001} = \frac{-4 + 0.0005(62,500 \pi)}{0.001}
\]

Find b = \pi r^2 \Rightarrow b = \pi (250)^2 = 62,500 \pi
6. Use a linear approximation (i.e. differentials) to approximate the following.

\( a. \) \((2.001)^5\)

Let \( y = x^5 + 4x^3 \) \( \Rightarrow \) \( y' = 5x^4 \) \( \Rightarrow \) \( 2 \) \( y' = 5 \cdot 16 = 80 \)

\( dy = f'(x)dx \) \( \Rightarrow \) \( dx = \frac{.001}{80} \) \( \Rightarrow \) \( dy = 80(.001) = .08 \)

and \((2.001)^5 \approx 2^5 + .08 = 32.08\)

\( b. \) \( \tan 44^\circ \)

Let \( y = \tan \theta \Rightarrow y' = \sec^2 \theta \) \( \Rightarrow \) \( x = \frac{\pi}{4} \) \( y' = 2 \)

\( dy = f'(x)dx = 2dx \) \( \Rightarrow \) \( dx = \frac{\pi}{180} + t \) \( \Rightarrow \) \( dy = \frac{-\pi}{90} \)

and \( \tan (44^\circ) = \tan (\frac{44^\circ}{180}) \approx \tan (\frac{\pi}{4}) - \frac{-\pi}{90} = 1 - \frac{\pi}{90} \)

7. For the following two functions, find and list all critical points. Then using these, find the maximum and minimum values of the function on the indicated intervals.

\( a. \) \( f(x) = 2x^3 - 3x^2 - 12x + 1, \) \([-2, 3]\)

\( f'(x) = 6x^2 - 6x - 12 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \)

\( \Rightarrow \) \( \text{C. P.'s: } x = 2, \) \( x = -1 \)

\( f(2) = -16 - 12 + 24 + 1 = -3 \), \( f(-1) = -2 - 3 + 12 + 1 = 8 \); \( f(3) = 54 - 27 - 36 + 1 = -8 \)

\( \max@(-1, 3), \min @(2, 19) \)

\( b. \) \( f(t) = 2 \cos t + \sin 2t, \) \([0, \frac{\pi}{2}]\)

\( f'(t) = -2 \sin t + 2 \cos 2t \Rightarrow -\sin t + \cos 2t = 0 \Rightarrow \)

\( -\sin t + (1 - 2 \sin^2 t) = 0 \Rightarrow 2 \sin^2 t + \sin t - 1 = 0 \Rightarrow \)

\( \sin t = \frac{-1 \pm \sqrt{1 + 8}}{2} = \frac{-1 \pm \frac{3}{2}}{2} \Rightarrow \sin t = \frac{1}{2} \); \( \sin t = -1 \)

\( \Rightarrow t = \frac{\pi}{6} + 2 \pi n \); \( \frac{5\pi}{6} + 2 \pi n \) \( \Rightarrow t = \frac{3\pi}{2} + 2 \pi n \)

\( f(0) = 2; \) \( f(\frac{\pi}{2}) = 0 \); \( f(\frac{3\pi}{2}) = 2 \cos (\frac{3\pi}{2}) + \sin (\frac{3\pi}{2}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} \)

\( \min @ (\frac{\pi}{6}, 0), \max @ (\frac{5\pi}{6}, \frac{3\pi}{2}) \)
8. Use the Mean Value Theorem to answer the questions below.

a. Verify that the function satisfies the hypotheses of the Theorem on the given interval. Then, using the interval for a and b, find all numbers c that satisfy the conclusion.

\[ f(x) = x^3 + x - 1, \quad [0, 2] \]

\[ f'(x) = 3x^2 + 1 \]

by MVT \[ 3x^2 + 1 = \frac{f(2) - f(0)}{2 - 0} = \frac{9 + 1}{2} = 5 \]

\[ \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm \frac{2}{\sqrt{3}} \]

\[ \Rightarrow \boxed{c = \frac{2}{\sqrt{3}}} \]

b. Show that the equation \( 2x - 1 - \sin x = 0 \) has exactly one real root.

Let \( f(x) = 2x - 1 - \sin x \)

then \( f'(x) = 2 - \cos x \)

Applying the I.V.T. to \( f(0) = -1 < 0 \) and \( f(\frac{\pi}{2}) = \pi > 0 \)

\[ \Rightarrow \exists \text{ at least one real root} \]

\[ \therefore f'(x) = 2 - \cos x \geq 1 \quad \forall x \in \mathbb{R} \]

the function is always increasing, \( \Rightarrow \) there is only one.