Maple Lab 3

L'Hospital's Rule

Maple's limit command can evaluate almost any limit we are likely to ever encounter. When evaluating limits using the limit command, l'Hopital's rule is applied automatically whenever it is needed. Simply using this command, however, does not help us develop an understanding of limits.

In addition, you should enter all the commands as they are in the lab in order to gain some familiarity with them. As always, more information about a command can be gained by accessing the built in Help.

This lab will use some features of the plots package. To have access to the commands contained in this package, you should enter the following as your first command.

>with(plots);

Example 1: \( \lim_{x \to 0} \frac{\arctan(3\,x)}{\arcsin(x)} \)

To enter the function and assign the limit to a label, enter the following commands:

> F := arctan(3*x)/arcsin(x):
L := Limit( F, x=0 ):
L;

When you first look at this problem, and note that \( \arcsin(0) = 0 \), it is difficult to know where to start. Most people just do not have the familiarity with the inverse trigonometric functions to be able to "see" the value of this limit. To begin to understand the problem, plot the function \( F(x) = \frac{\arctan(3\,x)}{\arcsin(x)} \) on an interval containing the origin. Because the arcsin function is defined only on \([-1,0) \cup (0,1]\), that is the largest domain on which this function is defined.

>p1 := plot( F, x=-1..-0.01, y=0..5 ):
p2 := plot( F, x=0.01..1, y=0..5 ):
display([p1,p2]);

You should see a plot which suggests that the value of the limit will be 3. Let's see if we can confirm this using properties of limits. Because the denominator converges to 0 as \( x \) approaches 0 there is a hope that l'Hospital's rule will apply. To check this, obtain the numerator and denominator and check their limits as \( x \to 0 \):

>f := numer(F):
Lf := Limit( f, x=0 ):
Lf = value( Lf ) ;
>g := denom(F):
Lg := Limit( g, x=0 ):
Lg = value( Lg ) ;

Because this limit has form \( 0/0 \), l'Hospital's rule can be applied. As stated before, Maple will automatically apply L'Hopital's rule in its calculations when the Limit command is used. To further our understanding however, let's apply L'Hopital's rule step by step and verify that it matches Maples calculation.

First, we need the derivatives of the numerator and denominator. We already have them assigned to the labels above, so the following
commands will get and display them for us:

\[ f_1 := \text{Diff}(f, x) : \]
\[ g_1 := \text{Diff}(g, x) : \]
\[ f_1 = \text{value}(f_1) ; \]
\[ g_1 = \text{value}(g_1) ; \]

Now, we'll assign their quotient to a new label.

\[ F_1 := \text{value}(f_1/g_1) : \]
\[ f_1/g_1 = F_1 ; \]

And, we'll assign the limit of \( F_1 \) to a new label.

\[ L_{1} := \text{Limit}(F_1, x = 0) : \]
\[ L_1 ; \]

And finally, we'll verify that the two limits are the same.

\[ \text{value}(L_1) ; \]
\[ \text{value}(L) ; \]

This is an excellent example of how much L'Hospital's rule can simplify limit calculations for us, or even allow us to find limits which we would not have been able to know otherwise.

**Example 2:** \( \lim_{n \to \infty} n^a e^{-n} \)

The first problem here is that \( n \) is an integer (this is by convention) so the functions are not continuous, hence not differentiable. In this case, it is not immediately obvious that L'Hospital's rule applies, but we will see that it does. To demonstrate this we will build a plot that shows it. To begin we define a continuous function in the same manner as our discrete function as follows:

The first command unassigns the previous labels so they are empty, the second command reloads the plots package.

\[ \text{restart: with(plots)} : \]

Now we define our functions:

\[ f_n := (n, a) \to (n^a) e^{-n} ; \]
\[ f_x := (x, a) \to (x^a) e^{-x} ; \]

Now, let's plot them.

\[ p_1 := \text{plot}( \text{seq}([n, f_n(n,2)], n=1..10) , \text{style} = \text{point}, \text{color} = \text{blue}, \text{view} = [0..10,0..5], \text{legend} = ["a=2, discrete points"] ) ; \]
\[ p_2 := \text{plot}( f_x(x,2), x=1..10, \text{color} = \text{cyan}, \text{legend} = ["a=2, continuous variable"] ) ; \]
\[ p_3 := \text{plot}( \text{seq}([n, f_n(n,10)], n=1..10) , \text{style} = \text{point}, \text{color} = \text{red}, \text{view} = [0..10,0..5], \text{legend} = ["a=10, discrete points"] ) ; \]
\[ p_4 := \text{plot}( f_x(x,10), x=1..10, \text{color} = \text{pink}, \text{legend} = ["a=10, continuous variable"] ) ; \]
You can experiment with the value of \( a \), making it larger and larger. To see the plot, you may need to enlarge the viewing window, or plot more values of \( x \). If you increase the value of \( x \), you will also need to increase the value of \( n \) in the sequence statement. Experiment and see how large must \( a \) get before the limit appears different from 0?

OK. The point of making these plots was to show that the discrete limit can be found by finding the limit of the continuous function. This is because, as the plots illustrate, the discrete function is a subset of the continuous function. Actually, it is a subsequence. We will learn about sequences in Chapter 12.

Let's apply L'Hopital's rule. First note that the function is not in the right form. Currently, the continuous function is an indeterminate form of type \( \infty \cdot 0 \). We need to rewrite it as type \( \frac{\infty}{\infty} \) or type \( \frac{0}{0} \). Fortunately, in this case, this is easy to do because we have the function \( \exp(-x) \). This is the same as the function \( 1/\exp(x) \).

Let's define new functions to be the numerator and denominator of our current function.

\[
> \text{fxn}:=x \rightarrow x^a; \\
> \text{fxd}:=x \rightarrow \exp(x);
\]

Now, just to be sure, let's verify that L'Hopital's rule applies.

\[
> \text{limit(fxn(x),x=infinity)}; \\
> \text{limit(fxd(x),x=infinity)};
\]

We see that the numerator and denominator both approach infinity as \( x \) approaches infinity.

We want to see if we can compute the limit for all \( a \), but we have a small problem. The power rule applies to the numerator, but what if \( a \) isn't an integer? Luckily for us, there is a function called the ceiling function which returns the next integer larger than any real number. We can apply it to \( a \) and assign the result to a label. (Note that we are assuming that \( a>0 \). If \( a<0 \), L'Hopital's rule would not apply.)

\[
> b:=\text{ceil}(a);
\]

Now we can differentiate the numerator \( b \) times. What will this do? Because \( b \) is larger than or equal to \( a \), it will ensure that the result of our differentiation is either a constant or a number raised to a negative exponent. Once raised to a negative exponent, the number is in the denominator.

\[
> \text{Dfxn}:=\text{diff(fxn(x),x$\text{b}$)};
\]

Uh-oh! The result looks really strange. Don't worry though. This is a function that we have never seen before, but it is the correct result.

Try entering some values for \( a \) to see what it does to \( \text{Dfxn} \).

\[
> a:=4; \text{Dfxn}; a:=3/5; \text{Dfxn}; a:='a';
\]
Don't forget to enter the last statement above, it unassigns a (removes the value 3/5 from it).

We also need to differentiate the denominator $b$ times. But the denominator is $\exp(x)$, so no matter how many times we differentiate it, the result is $\exp(x)$. Let's verify that with Maple.

```maple
>Dfxd:=diff(fxd(x),x$b);
```

Now, let's take the limit as $x$ goes to infinity. First we form our quotient by assigning it to a label and then use the limit function.

```maple
>Q:=Dfxn/Dfxd;
>limit(Q,x=\infty);
```

Notice that in the first example, we used Limit with a capital L and in the second example, we used it with a lowercase l. The two commands are slightly different. The capital command is called inert. It does not carry out its action until instructed to by the value command. This can be useful when using Maple to write reports and publications. This is also true for the Diff and diff commands as well as some other Maple commands.