The evolution of Kirchhoff elliptic vortices

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A Kirchhoff elliptic vortex is a two-dimensional elliptical region of uniform vorticity embedded in an inviscid, incompressible, and irrotational fluid. By using analytic theory and contour dynamics simulations, we describe the evolution of perturbed Kirchhoff vortices by decomposing solutions into constituent linear eigenmodes. With small amplitude perturbations, we find excellent agreement between the short time dynamics and the predictions of linear analytic theory. Elliptical vortices must have aspect ratios less than \( a/b = 3 \) to be completely stable. At late times, unstable perturbations evolve to states consisting of filaments surrounding and connecting one or more separate vortex core regions. Even modes have two different evolution paths accessible to them, dependent on the initial phase. Ellipses can first fission into more than one separate region when \( a/b = 6.046 \), from the negative branch \( m = 4 \) mode. Increasing the perturbation amplitude can result in nonlinear instability, while the perturbation is still small relative to any vortex dimension. For the lowest \( m \) modes, we quantify the transition from linear to nonlinear behavior. © 2008 American Institute of Physics. [DOI: 10.1063/1.2912991]

I. INTRODUCTION

A. Motivation

The subject of vortex dynamics has been studied for almost 150 years, due to the central role of vorticity in fluid dynamics and turbulence.1 Experiments and simulations have established that nominally two-dimensional (2D) vortices can emerge from wakes, laminar flows, and structureless turbulent initial conditions. The vorticity structures which emerge from unstable initial conditions are often elliptical, such as the states which evolve from 2D vortex strips or filaments.7,8 Elliptical vortices can also be produced from circular ones as a result of merger9,10 or background strain and shear flows.11–13 Instabilities associated with ellipticity can be an important mechanism for entrstrophe cascade and are associated with chaotic evolutions; so a comprehensive understanding of the dynamics of elliptical vortices is of general interest.

The earliest work done on waves and modes on a 2D vortex used a highly idealized “vortex patch” model of a bounded region of uniform vorticity embedded in an inviscid, incompressible, and irrotational fluid. Although a realistic fluid vortex will have smooth rather than discontinuous transitions from its interior to the exterior irrotational region, a vortex patch of the same shape is a reasonable approximation which is much easier to model. Simulations have also shown that vorticity gradients can be steepened by background strain, and this mechanism provides one way in which patchlike vorticity distributions can naturally form.13

In this paper, we describe the dynamics of 2D elliptical vortex patches by using analytic theory and numerical simulations and with an emphasis on the modes of the system. One motivation for returning to this subject, which was described 115 years ago as already a ”somewhat ancient matter,”14 is to quantify the boundary between linear and nonlinear dynamics in this system. A second motivation is that many of the phenomena that will be described below have been observed in recent laboratory experiments conducted by us. These experiments will be the subject of a future paper.

B. Survey of previous results

Kirchhoff showed that isolated 2D vortex patch ellipses of any aspect ratio \( a/b \) are exact solutions to the nonlinear Euler equations in 1876.15 The associated flow is nonsteady, with the ellipse performing a steady rotation about its center. There are only a few such exact solutions, and as a consequence, elliptical vortex patches are often referred to as “Kirchhoff elliptic vortices.”

Kelvin showed that circular vortex patches sustain waves which vary as \( \cos(m\theta) \) on the edge.16 These waves are linearly stable and propagate in the direction of the rotation of the fluid but with a lower angular velocity. These are commonly known as Kelvin \( m \) waves or modes on a circular vortex patch,17 or as “Kelvin vortex waves” if it is needed to distinguish them from the “Kelvin tidal waves” occurring in geophysics. The Kirchhoff ellipse is a continuation of the linear Kelvin \( m=2 \) mode to finite amplitude.

In 1893, Love analyzed the linear stability of Kirchhoff vortices.14 The perturbations used were extensions of Kelvin’s perturbations onto an elliptical geometry. Love found that at large ellipse aspect ratios, these become unstable, with the \( m=3 \) perturbation unstable at \( a/b > 3.0 \), the \( m=4 \) unstable at \( a/b > 4.61 \), and so on. He also obtained analytic expressions for the oscillation frequencies and growth rates. The complete solutions to the linearized equations of motion were calculated by Guo et al.18

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Love’s approach to calculate stability was linear and appears to be applicable only to the case where the vortex is elliptical. Burbea and Landau used a conformal mapping method to derive the linear stability of all of the finite amplitude Kelvin $m$ waves. Exact agreement with Love’s results for the onset of instability was obtained as a subset of their work. They also numerically calculated the oscillation frequencies and growth rates and found agreement with Love’s results within the precision of their calculations.

Subsequent analytic work studied the Kirchhoff vortex’s nonlinear stability by using the complete Euler equations. Among other results, it was established that a Kirchhoff vortex is nonlinearly stable when $a/b < 3$, the same region where it is linearly stable. Nonlinear stability in the context of this analytic work refers to regularity results, which show that the difference between the perturbed elliptical vortex and the unperturbed vortex continuously depends on the initial conditions. Thus, given a finite interval in time and a desired error tolerance, one can determine a finite bound on initial perturbations, such that the perturbation will remain close to the unperturbed system within the specified tolerance. In this paper, we establish that this bound is very small in many elementary circumstances and explore what happens beyond this tolerance, in what we refer to as the nonlinear (in amplitude) regime.

Dritschel and Kamm studied the stability and energetics of vortex patches. Dritschel found conjugate states between elliptical vortices and corresponding members of the family of corotating vortices and postulated that a transition was possible between these states. Kamm pointed out that the solution path of a $m=4$ perturbation bifurcates away from the elliptical solution, while $m=3$ and 5 perturbation solutions only diverge away. Other workers showed that contours which are initially smooth remain so, indicating that filamentation commonly observed in unstable ellipses will not lead to a cusp in the contour in finite time.

Turning now to simulation studies, in 1978, Deem and Zabusky pioneered the use of contour dynamics (CD) simulations to study the finite amplitude $m \geq 3$ Kelvin waves or “V states” in their terminology. CD simulations, which integrate the complete Euler equations forward in time in an efficient fashion, are ideal for investigating the linear and nonlinear dynamics of vortex patches. Among their results was that the largest wave amplitude possible is limited by the onset of filamentation. This phenomenon, which occurs when the hyperbolic points of the comoving streamfunction penetrate the vortex contour, was subsequently studied in more detail by Polvani et al.

Dritschel used CD simulations to investigate the nonlinear evolution of elliptical (and annular) vortex patches. In a survey over $m$ and $a/b$, he applied $m = 2 \rightarrow 6$ perturbations onto ellipses with aspect ratios $a/b = n/1$, with $n$ as an integer between 1 and 6. Dritschel concluded that the ellipse is nonlinearly stable when $a/b < 3$. At $a/b = 5$ and above, he found nonlinear instability for all $m \geq 3$. He also showed that the postulate of Ref. 21 that ellipses could transition to corotating vortices and vice versa was correct. Several examples of late times states were presented, establishing that unstable elliptical vortices evolve at late times into states consisting of filaments surrounding and connecting one or more separate vortex core regions.

C. Organization of this paper

The goal of this paper is to explicate the linear and nonlinear dynamics of the Kirchhoff vortex beyond what is currently known and to quantify the boundaries between these two regimes. In Sec. II, we present the theoretical background needed for our study, beginning with the equations of motion of 2D inviscid and incompressible fluids, and the associated conserved quantities. We then present the Kirchhoff vortex solution and Love’s linear stability results. Lastly, we describe our CD simulation and analysis routines.

The complete parameter space of initial conditions is four dimensional in aspect ratio $a/b$, mode number $m$, mode amplitude $\alpha_m$, and mode phase $\beta_m$. Our strategy to describe this space is to scan in one variable while holding the other parameters fixed. In Sec. III, we discuss the evolution of perturbations with the initial amplitudes in the linear regime (small). Separate subsections then describe the evolution of modes with increasingly higher $m$ number. We compare our simulation results with predictions from linear theory and also characterize the highly nonlinear states that evolve at late times from unstable ellipses.

In Sec. IV, we examine the effect of increasing the initial perturbation amplitude on the ellipse’s stability and its nonlinear evolution. We are able to make limited comparisons with earlier studies where large perturbations were also used.

In Sec. V, we state our conclusions and discuss their possible relevance to fluid vortices with more realistic characteristics. Our MATLAB routines for CD simulations and analysis can be downloaded from the Electronic Physics Auxiliary Publication Service (EPAPS).

II. THEORETICAL ASPECTS

A. Equations of motion

Our governing equations are those of an inviscid, incompressible fluid with constant density $\rho = 1$ described by the 2D Euler equations,

$$\nabla \cdot \mathbf{u} = 0, \hspace{1cm} (1a)$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} = - \nabla p, \hspace{1cm} (1b)$$

where $\mathbf{u} = [u, v]^T$ is the velocity field and $p$ is the pressure. The first expression is a continuity equation describing incompressibility. The second equation is a statement of momentum conservation. While there are situations where one might work with primitive variables $\mathbf{u}$ and $p$ when studying flows dominated by compact regions of vorticity, there are distinct advantages to transforming Eq. (1b) into a vorticity-streamfunction formulation. To do so, one takes its curl so that one has
\[ \zeta_i + (\mathbf{u} \cdot \nabla)\zeta = 0, \] (2)

where \( \zeta \) is the \( \hat{k} \) component of the vorticity field \( \nabla \times \mathbf{u} \). Thus, vorticity moves as a material element in the flow field. Since the flow field is incompressible, there is a streamfunction \( \psi \), such that

\[ u = \psi_y, \quad v = -\psi_x. \] (3)

We can rewrite Eqs. (1a) and (1b) as

\[ \zeta_i + J(\zeta, \psi) = 0, \] (4a)

\[ \nabla^2 \psi = -\zeta, \] (4b)

where \( J(\zeta, \psi) \) is the Jacobian with respect to the standard basis,

\[ J(\zeta, \psi) = \frac{\partial(\zeta, \psi)}{\partial(x, y)} = \zeta_x \psi_y - \zeta_y \psi_x. \] (5)

These same equations also arise in the quasigeostrophic model where vertical variations are neglected. For \( R^2 \), it is possible to transform \( \zeta \) back into \( \psi \) by using Green’s function,

\[ \psi(x_0) = \frac{1}{4\pi} \int \int \zeta(x) \log(|x_0 - x|^2) dx. \] (6)

We can extend this concept by transforming the position \( (x, y) \) into a complex variable \( z = x + iy \) and defining a complex valued potential \( w = \phi + i\psi \), where \( \nabla \phi = \mathbf{u} \) and \( \psi \) is the streamfunction. Then, \( w' = dw/dz = u - i\psi \) and

\[ w(z_0) = -\frac{i}{2\pi} \int \int \zeta(x) \log(z_0 - z) dx. \] (7)

This compact expression relating the vorticity to the flow field is central to the CD algorithm used in this paper. We explore this further in Sec. II E.

### B. Conserved quantities

2D vortex patches in an inviscid flow have several conserved quantities which are useful for constraining evolution and determining stability. These include the vortex area \( A \), vorticity \( \zeta \), and the total circulation \( \Gamma = \int \zeta(x) dx \). Three moments are also conserved, the centroids

\[ \bar{x} = \frac{\int \int \zeta(x) x dx}{\Gamma}, \quad \bar{y} = \frac{\int \int \zeta(x) y dx}{\Gamma}, \] (8)

and the angular momentum

\[ Q = \int \int \zeta(x)(x^2 + y^2) dx. \] (9)

The kinetic energy of the unbounded system is infinite, and so researchers work with an excess energy defined in a form such as

\[ T = \frac{1}{2} \int \int \zeta(x) \psi(x) dx. \] (10)

### C. Kirchhoff elliptic vortex and its stability

Kelvin showed that circular vortex patches sustain linear waves which vary as \( \cos(m\theta) \) on the edge and derived their frequency \( \omega_m \).

\[ \omega_m = \frac{\zeta}{2} \left[ m - 1 + \left( \frac{R_0}{R_w} \right)^2 \right]. \] (11)

Here, \( \zeta \) is the uniform vorticity of the patch, \( R_0 \) is the patch radius, and \( R_w \) is the radius of a circular boundary within which the vortex is centered. Positive vorticity results in counterclockwise rotation of the waves, in the same direction but slower than the fluid.

Kirchhoff had provided the nonlinear continuation of the \( m=2 \) Kelvin wave to finite amplitude with his earlier proof that isolated elliptic vortex patches of any aspect ratio \( a/b \) are exact solutions of the complete Euler equations. The ellipse has a steady rotation about its center with a rotation frequency \( \Omega \) given by

\[ \Omega = \frac{\zeta ab}{(a + b)^2}. \] (12)

Love analyzed the linear stability of a perturbed Kirchhoff vortex by requiring continuity of the pressure and component velocities on the ellipse boundary, and that this surface always contain the same particles. He used Cartesian coordinates to describe the interior and cylindrical elliptical coordinates \( (\xi, \eta) \) for the exterior. The two coordinate systems are related by \( x = c \cos \xi \cos \eta \) and \( y = c \sin \xi \sin \eta \), with \( c^2 = a^2 - b^2 \). The Jacobian for the change of variables from Cartesian to elliptical coordinates is

\[ J(\xi, \eta) = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{c^2}{2} [\cosh(2\xi) - \cos(2\eta)]. \] (13)

The elliptical boundary of an unperturbed patch is specified by \( \xi = \xi_0 \), which is related to the aspect ratio by \( \xi_0 = \tanh^{-1}(b/a) \). In his stability analysis, Love considered a perturbed boundary \( \xi = \xi_0 + \delta \xi \) and discarded higher order terms as they arose. He estimated the departure from the undisturbed ellipse by the ratio of the normal displacement on a point on the perturbed ellipse to the central perpendicular on the tangent at that point. This ratio is proportional to the quantity

\[ q(\eta, t) = J(\xi, \eta) \delta\xi(\eta, t) = J_0(\eta) \delta\xi(\eta, t), \] (14)

where \( J_0(\eta) = a^2 \sin^2 \eta + b^2 \cos^2 \eta = h_0^{-2} \) in Love’s notation, and the assumption has been made that the perturbation does not depend on \( \xi \).

A disturbance at any time \( t_0 \) can be expressed as a sum over \( m \)-number perturbations in the forms
The condition for stability of an ellipse is that all the values \( \lambda_m \) are positive. Instability is never indicated for the \( m=1 \) and \( 2 \) perturbations by Eq. (17), since \( \lambda_1^2 = \Omega^2 \) and \( \lambda_2^2 = 0 \) for all aspect ratios. The \( m \geq 3 \) perturbations have \( \lambda_m^2 \) become negative above a particular aspect ratio. This occurs at \( a/b = 3 \) for the \( m=3 \) mode, and larger \( m \) perturbations are unstable at progressively higher aspect ratios.

In Fig. 1, we plot for the first four unstable modes the predictions of Eq. (17) for \( \lambda_m \) as oscillation rates (dashed lines) and linear growth rates (solid lines). To remove the dependence on \( \zeta \), we plot the dimensionless quantity \( \lambda_m T_R \). The Kirchhoff ellipse rotation period \( T_R \) is given by

\[
T_R = \frac{2\pi}{\Omega} = \frac{2\pi(a+b)^2}{\zeta ab} = \frac{2\pi(y+1)^2}{\zeta \gamma},
\]

where \( \gamma = a/b \) is the ellipse aspect ratio.

D. Linear solutions of the perturbed vortex

The complete linear solutions to perturbations of the form of Eq. (15) on a Kirchhoff vortex were calculated by Guo et al.\(^{18} \)

Defining the quantities

\[
\mu_m^+ = \frac{\zeta}{2} \left( \frac{\gamma - 1}{\gamma + 1} \right)^m + \left[ \frac{2m\gamma}{(\gamma+1)^2} - 1 \right],
\]

(19a)

\[
\mu_m^- = \frac{\zeta}{2} \left( \frac{\gamma - 1}{\gamma + 1} \right)^m - \left[ \frac{2m\gamma}{(\gamma+1)^2} - 1 \right],
\]

(19b)

we have \( \lambda_m = \sqrt{\mu_m^+ \mu_m^-} \).

For unstable \( m \) perturbations, where \( \mu_m^- > 0 \), the general solution for each \( m \) component is

\[
q(\eta, t) = \left[ A_m^0 \cos(\lambda_m t) - \frac{\mu_m^+}{\lambda_m} B_m^0 \sin(\lambda_m t) \right] \cos(m\eta) + \left[ B_m^0 \cos(\lambda_m t) + \frac{\lambda_m}{\mu_m^+} A_m^0 \sin(\lambda_m t) \right] \sin(m\eta).
\]

(20)

Unstable perturbations have two eigenmodes where, by using the form of Eq. (15b), the phases are constant with time but their amplitudes \( \alpha_m \) vary as \( e^{\pm \lambda_m \omega^2} \). Exponentially growing and decaying eigenmodes, respectively, are given by the conditions

\[
m\beta_m^- = -\tan^{-1}\left( \sqrt{\frac{\mu_m^-}{\mu_m^+}} \right) \leftrightarrow A_m^0 \frac{\beta_m^-}{\mu_m^+} = -B_m^0 \lambda_m.
\]

(21a)

\[
m\beta_m^+ = +\tan^{-1}\left( \sqrt{\frac{\mu_m^-}{\mu_m^+}} \right) \leftrightarrow A_m^0 \frac{\beta_m^+}{\mu_m^+} = B_m^0 \lambda_m.
\]

(21b)

In Fig. 2, we plot as lines the phase \( m\beta_m^\pm \) predicted by Eq. (21a) for the first four growing eigenmodes.

For stable \( m \) perturbations, where \( \mu_m^- < 0 \), the general solution is

\[
q(\eta, t) = \left[ A_m^0 \cos(\lambda_m t) - \frac{\mu_m^+}{\lambda_m} B_m^0 \sin(\lambda_m t) \right] \cos(m\eta) + \left[ B_m^0 \cos(\lambda_m t) + \frac{\lambda_m}{\mu_m^+} A_m^0 \sin(\lambda_m t) \right] \sin(m\eta).
\]

(22)

With stable oscillations, the amplitudes and phase are modulated with time by an angular frequency of \( \lambda_m \) due to the cosine and sine terms. These modulations are required for conservation of conserved quantities during an oscillation.
E. Contour dynamics simulations

Deem and Zabusky are credited with being the first to utilize the CD method to simulate the evolution of a patch of vorticity or a configuration of patches, and it has since become a popular and effective tool for inviscid simulations of discrete patches of vorticity. The algorithm itself is a direct application of Green’s theorem. If \( D \) is a bounded patch of vorticity with strength \( \zeta \) and a piecewise smooth boundary \( \partial D \), then the velocity at some point \( z_0 = x_0 + iy_0 \) outside of \( D \) can be expressed through the complex potential [see Eq. (7)],

\[
\mathbf{w} = \frac{-i \zeta}{2 \pi} \int \frac{1}{z - z_0} \, dx \, dy.
\]

By applying Green’s theorem, we can reduce the dimensionality from two to one,

\[
\mathbf{w} = \frac{-i \zeta}{2 \pi} \oint \log(z - z_0) \, dz.
\]

Since the singularity in \( \log(z - z_0) \) is outside of \( D \), Cauchy’s integral theorem asserts that

\[
\oint \log(z - z_0) \, dz = 0.
\]

The imaginary part of Eq. (25) can be substituted into Eq. (24), so that

\[
\mathbf{w} = \frac{-i \zeta}{2 \pi} \oint \log(z - z_0) \, dy \, i dx
\]

Taking the complex conjugate of both sides,

\[
\mathbf{w} = \frac{\zeta}{2 \pi} \oint \log(z - z_0) \, dx \, idy.
\]

A further transformation yields the equivalent expression,

\[
\mathbf{w} = \frac{\zeta}{4 \pi} \oint \frac{z_0 - z}{z_0 - z} \, d\mathcal{L}.
\]

The CD simulation method is simply the temporal integration of material points along the boundary of a patch of vorticity by using the velocity field specified by Eq. (28). The self-induction term is excluded because it contributes zero in the weak limit. We used a variable-order Adams–Bashforth–Moulton PECE solver as implemented in MATLAB to do the numerical integration.

When creating an initial condition for simulations, we used 500 material points evenly spaced in \( \eta \), which concentrates nodes at points of greatest curvature. When continuing simulations to highly nonlinear late time states, we would periodically adjust the spacing of the points to avoid too great or small distances between them.

Two fitting routines are used to determine the mode content of a perturbed contour. In one, we determine the displacements \( \delta \) from an unperturbed ellipse at the same ellipse angle \( \phi \). Displacements are determined by finding the points on the perturbed contour which intersect lines perpendicular to the unperturbed contour. We then do a nonlinear least-squares fit \( q(\eta) \) to \( \sum_n \alpha_n \cos[m(n - \beta_n)] \).

We also perform nonlinear least-squares fits where a fit contour is constructed of superpositions of Love perturbations as defined by Eq. (15b), and \( a, b, m, \alpha_m, \beta_m, \) and \( \phi \) are varied to minimize the sum of the squares of the distances to the nearest contour points. This fitter allows us to follow the mode evolution to a larger amplitude. With small perturbation amplitudes, no differences are seen between the two fitting approaches.

III. EVOLUTION FROM LINEAR PERTURBATIONS

In this section, we discuss the simulated evolution of Kirchhoff ellipses which have been seeded with small amplitude Love \( m \) perturbations. Different \( m \) numbers are considered in different subsections. Later, in Sec. IV, we examine how things change when the perturbation amplitudes are increased.

Many of the equations from analytic theory do not depend on the absolute scale of an ellipse, but the perturbation definitions are exceptions to this. In this paper, we use the convention that the initial ellipse semimajor axis \( a = 1 \), which has the advantage that the two perturbation amplitudes of Eqs. (15b) and (16) will be very close in value (i.e., \( \alpha_m = \alpha'_m \)) when fit to the same boundary.

To set the scale, note that when a perturbation has phase \( \beta_m = 0 \), the absolute value of the displacement along the direction of the minor axis will be \( \alpha_m \), and that along the major axis will be \( \alpha_m / b \). Nonlinear effects can begin to be seen at
amplitudes as low as $\alpha_m=2 \times 10^{-4}$, so in this section, we work with seed amplitudes at or below this value.

Analytic theory predicts stability for linear $m=1$ and 2 perturbations through Eq. (17). Geometrically, a linear $m=1$ perturbation is a displacement of the unperturbed ellipse. A linear $m=2$ perturbation corresponds to elliptic displacement; i.e., it yields an unperturbed ellipse with a rotation phase $\phi$ and aspect ratio $a/b$ slightly different from that of the base state. One would not consider these to be modes of the system in either case. We find that simulations of ellipses seeded with linear $m=1$ and 2 perturbations do not result in stable oscillations or growth of these perturbations, which supports this view.

### A. Unstable $m=3$ perturbations

The $m=3$ mode is the first to become unstable as the aspect ratio is increased, with linear theory predicting instability when $a/b > 3$. With small perturbations, we find excellent agreement between the various predictions of linear theory and the ellipse dynamics simulated by our CD code.

Figure 3 shows the phase and amplitude versus time of two $\alpha_3(0)=10^{-4}$ perturbations on an $a/b=3.5$ ellipse (solid symbols). We normalize the simulation time in all figures by rotation period $T_R$. The noise floor of our mode fitter is $\sim 5 \times 10^{-7}$. The phases at $t=0$ were set to $\beta^+_3$ and $\beta^-_3$, the predicted values for growing and decaying eigenmodes, respectively. The associated lines are the predictions of Eqs. (20) and (21).

A growing eigenmode emerges from the seeded decaying eigenmode and has amplitude $\alpha_3=10^{-6}$ at $t=1.5T_R$ in Fig. 3(b). Unstable vortices always eventually destabilize in the simulations, with the details depending on what perturbations are present and on the aspect ratio, which governs growth rates. We find that modes can be seeded from modes of other $m$ number; this may be due to nonlinear couplings between the modes. Numerical noise is also present. A $m=3$ will grow from noise to $\alpha_3=10^{-6}$ in approximately eight rotation periods on an unperturbed $a/b=3.5$ ellipse, where $\lambda_3T_R=3.04$.

To compare linear growth rates with theory, we seed small amplitude perturbations with phases of $\beta_3(0)=\beta_3^\pm$ and fit the resultant amplitude growth to an exponential. The measured growth rates are plotted as solid squares in Fig. 1, and these have uncertainties similar to the symbol sizes. No discrepancies greater than our measurement precision between theory and simulation rates are seen. The aspect ratio where the ellipse becomes $m=3$ unstable has previously been shown to agree with linear theory using numerical calculations.

To compare the eigenmode phases with theory, we seed over a range of phases and fit to find where the phase of the growing mode is stationary. Some measurements are plotted in Fig. 2 as solid squares. No discrepancies with linear theory have been found.

More generally, any initial $m=3$ perturbation on an unstable ellipse can be decomposed into a superposition of growing and decaying eigenmodes. As these evolve in time, simulations demonstrate that the growing mode comes to dominate. (See Fig. 4 below for a plot of this taking place with $m=4$ perturbations.) This is consistent with an observation based on nonlinear analytic theory that the dynamics will be determined by the fastest growing mode for the corresponding linearized equation.

For a given perturbation, the transformation $\alpha_m \rightarrow -\alpha_m$ or, equivalently, $\beta_m \rightarrow \beta_m + \pi/m$ reverses the sign of the displacements at each point. For the $m=3$ and the higher $m$ odd modes, the effect of this is a rotation of the evolving eigenmode by $180^\circ$ on the base ellipse, which due to the ellipse’s rotational symmetry is dynamically unimportant. However, as will be discussed below, a bifurcation in the evolution path is produced by this transformation with the even modes.

The highly nonlinear states to which the unstable modes grow at long times were first comprehensively surveyed in Ref. 29. The inset of Fig. 3(b) shows an exponentially grow-
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B. Stable m=3 perturbations

Figure 3 also shows the phase and amplitude versus time of a stable perturbation on an a/b = 2.5 ellipse with initial conditions \( \alpha_3 = 10^{-4} \) and \( \beta_3 = 0 \) (open circles). The perturbation rotates along the surface of the ellipse in the same direction as the fluid and ellipse rotation (counterclockwise for positive vorticity). Modulations are present in the \( \alpha_3(t) \) and \( \beta_3(t) \) curves, while \( \alpha_3 \) and \( \beta_3 \) do not vary with time. The predictions of Eq. (22) for linear perturbations are plotted in Fig. 3 as lines. It is useful to quantify amplitude modulations with \( \alpha_m = \max[\alpha_m(t)]/\min[\alpha_m(t)] \). For small perturbations, linear theory gives \( \alpha_m^\text{mod} = \lambda_m/\mu_m \).

The analytic predictions for the stable mode oscillation frequency \( \lambda_m \) given by Eq. (17) can be compared to the simulation results by fitting evolving perturbed ellipse boundaries to Eq. (22). Some sample normalized frequencies are plotted as open squares in Fig. 1, and as with the unstable growth rates, excellent agreement is seen between linear theory and simulation results.

C. Linear m=4 perturbations

The \( m=4 \) mode is predicted to become unstable when \( a/b = 4.612 \). The \( m=4 \) mode oscillation frequencies, growth rates, and unstable eigenmode phases have been determined from CD simulations in the same fashion as was done with the \( m=3 \) mode, and the results plotted as circles in Figs. 1 and 2.

For the unstable \( m=4 \) and the higher even modes, the sign/phase of the perturbation causes a bifurcation into two branches of possible evolutions, although linear properties such as growth rates are unaffected. Figure 4 shows the effect in a plot of the phase versus time of \( m=4 \) perturbations with seed amplitude \( \alpha_4 = 2 \times 10^{-4} \) and initial seed phases spaced by \( \Delta \beta = \pi/48 \).

For initial \( \alpha_4 \) positive and phase \( \beta_4 = \beta_4^+ \), the seed perturbation causes the points near the major and minor axes to have a positive displacement (in the radial direction). At late times, this saturates into a double-spiral structure with a near-circular core surrounded by two winding filaments (bottom right insets of Fig. 4 and 5). Within a range of initial phases of \( \pm \pi/8 \) centered about \( \beta_4^+ \), the same growing “positive” \( m=4 \) mode branch comes to dominate.

However, if the initial \( \alpha_4 \) is negative or alternatively the initial seed phase is shifted by \( \pi/4 \), the points near the axes have a negative radial displacement and at large amplitude, the “negative” \( m=4 \) mode branch produces two vortex regions at the ends with the midsection pinched in (top figure half and inset of Fig. 4). We distinguish between the two mode branches with the notation \( m = \pm 4 \). Despite their topological differences, the two branches have very symmetric \( \beta_4(t) \) curves. The \( m=-4 \) branch emerges from seed perturbations with phases within a range of \( \pm \pi/8 \) centered about \( \beta_4^+ + \pi/4 \).

At the aspect ratio \( a/b = 5.5 \) of Fig. 4, the \( m=-4 \) mode branch saturates with the ellipse midsection reaching a minimum value and then increasing with time. At larger aspect ratios, the mode can instead result in a fission of the perturbed ellipse into two separate vortex core regions connected by a thin filament (left inset of Fig. 5). A convenient way to characterize pinching in or out of the midsection is with a plot of \( r_{\text{min}} \), the minimum radial value of any point on the ellipse boundary.

Figure 5 plots \( r_{\text{min}}(t) \) for \( m=-4 \) perturbations with their initial phase centered on the mode branch, at \( \beta_4 = \beta_4^+ + \pi/4 \). The seed amplitude was \( \alpha_4 = 2 \times 10^{-4} \). Aspect ratios were varied from \( a/b = 4.3 \rightarrow 6.6 \) in 0.1 increments. The top four curves with \( a/b = 4.3 \rightarrow 4.6 \) are flat, and the perturbation stably rotates without growth, as expected since the theory prediction for \( m=4 \) instability is \( a/b > 4.6116 \). Once instability begins, three different regimes occur as \( a/b \) is increased: filamentation, oscillations, and fission.

1. Filamentation

The eight aspect ratio curves from \( a/b = 4.7 \rightarrow 5.4 \) show growth of the \( m=-4 \), but the shrinking of the midsection saturates at a finite value after about one rotation period, and then expands beyond its initial value of \( b \). From a mode perspective, the phase of the \( m=-4 \) mode smoothly transitions from the negative to the positive mode branch range. The final state in this aspect ratio range is, therefore, the double-spiral filamentation state of the positive branch, an example of which is inset in the bottom right.

Overman and Zabusky were among the first to use CD simulations to investigate the merger of two corotating sym-
metric vortices. The elliptical state resulting from merger whose evolution is plotted in their Figs. 7–9 is a nice example of a perturbed Kirchhoff vortex in what we are here labeling the filamentation regime.

2. Oscillations

The six aspect ratio curves from $a/b = 5.5 \rightarrow 6.0$ show the midsection shrinkage saturating, but the phase and mode amplitude return to the seeded value, such that there are stable oscillations of the $m = -4$ mode. The final state will be a large amplitude $m = 3$ because this will eventually grow from noise and not saturate. The aspect ratio boundary at which the transition from filamentation to saturating $m = -4$ oscillations takes place is, from simulations, $a/b = 5.435$.

3. Fission

Above $a/b = 6.044$, the unstable $m = -4$ does not saturate and there is a fission of the ellipse into two equal vortex core regions connected by a filament (left inset of Fig. 5). The details of the late time states depend on the ellipse aspect ratio. Figure 6 shows three additional examples of the late time states, here resulting from $m = -4$ perturbations with initial amplitude $\alpha_4 = 2 \times 10^{-5}$ and phase $\beta_4 = \beta_4^0 + \pi/4$.

In Fig. 6(a), a late time state for $a/b = 6.046$, slightly above where fission first begins, is shown. It is very similar to the lowest possible angular momentum equilibrium state of two corotating vortices, which features bilateral symmetry and pearlike shapes. Dritschel has shown that with near-inviscid and bounded flows, there are aspect ratios where Kirchhoff ellipses are identical, except topologically, with equilibria of $N$ corotating vortices. For $N = 2$, two such pairings occur when $a/b = 6.0423$ and 6.0459, which are very close to where we observe fission to result in near-equilibrium corotating states.

Fission at higher aspect ratios also results from $m = -4$ perturbations, but the states which emerge at late times are increasingly far from the equilibrium states of two corotating vortices. This is consistent with Dritschel’s calculations which indicate that as $a/b$ is increased, there is an increasing energy mismatch between the ellipse and the corotating equilibrium states (see Fig. 15 of Ref. 29).

Figures 6(b) and 6(c) illustrate some trends as $a/b$ is increased. At intermediate times, the vortex “rolls up” rather than symmetrically pinches off at its center [$r = 0.98T_R$ contour of Fig. 6(b)]. The vorticity between the two cores at the ends is then stretched out until it becomes a thin filament ($t = 1.27T_R$ contour). At the late times, the vortex cores shapes tend to be greatly perturbed ellipses, rather than circular or pearlike shapes. With increasing aspect ratio, greater amounts of vorticity end up in filamentary structures separate from the cores.

D. Linear $m \geq 5$ perturbations

The behavior of the higher order modes is very similar to that of the $m = 3$ and 4 modes. The $m = 5$ and 6 mode oscillation frequencies, growth rates, and unstable eigenmode phases have been measured from CD simulations and plotted in Figs. 1 and 2.

The details of the late time state of an unstable mode, such as the number and sizes of the vortex core regions and filaments, are dependent on the aspect ratio, the mode number $m$, and the branch if it is an even mode. Figure 7 shows late time states of seeded $m = 5$, 6, 9, and ±10 modes. The seed amplitude was $\alpha_m = 2 \times 10^{-5}$ and the initial phases were
unstable mode becomes well represented by
a maximum growth rate of
apply it to a highly anisotropic ellipse
late time outcomes:
still linear are also plotted for
branches. Intermediate mode amplitudes which are large but
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wavelengths roughly eight times the width of the strip.7,8 If
results agree with Lord Rayleigh’s analysis of infinite strips of
m
+ for positive branches and \( \beta'_m + \pi/m \) for negative
branches. Intermediate mode amplitudes which are large but
still linear are also plotted for \( m=5 \) and \( \pm 6 \). There are three
late time outcomes:

1. Unstable odd modes evolve at late times to a state with
one filament ejected from an end and \((m-1)/2\) separate
twist core regions connected by filaments. Figures 7(a)
and 7(d) show the \( m=5 \) and 9 modes.
2. Even positive branch modes evolve to a state with
\( m/2-1 \) separate regions and two filaments, one from
each end. Figures 7(b) and 7(e) show the \( m=+6 \) and
+10 mode branches.
3. Even negative branch modes evolve to a state with \( m/2 \)
separate regions and no end filaments, as shown in Figs.
7(c) and 7(f) for the \( m=-6 \) and \(-10 \) mode branches.

These outcomes naturally follow from the numbers and po-
sitions of the antinodes of the initial perturbation. Negative
radial displacements move in and at large amplitude connect
with each other, thus pinching off separate vorticity regions.
With the \( m=4 \) mode, there is a range in aspect ratio,
4.61 < \( a/b < 6.04 \), where the mode is unstable but a \( m=-4 \)
seed does not result in fission. No such range is seen in
simulations of the higher order even modes. As long as the
even mode is unstable, negative branch seeds produce \( m/2 \)
regions and positive branch seeds produce \( m/2-1 \) regions,
for all aspect ratios.

In Fig. 8, we plot the normalized predicted growth rates
from Eq. (17) out to \( a/b=26 \). The growth rate of the most
unstable mode becomes well represented by \( \lambda_m T_R \)
=1.253(\( a/b \)), which is plotted (offset dashed line). This
result agrees with Lord Rayleigh’s analysis of infinite strips of
vorticity where the most unstable mode was found to be at
wavelengths roughly eight times the width of the strip.7,8 If
we use the maximum growth rate for an infinite thin strip and
apply it to a highly anisotropic ellipse \((a/b \to \infty)\), it predicts
a maximum growth rate of \( \lambda_m T_R = 1.264(\a/b) \) in this limit.

As \( a/b \) increases, more modes become unstable and
the late time states become increasingly sensitive to the
initial perturbations and numerical noise. The insets show
three \( a/b=20 \) ellipses at \( t=0.6T_R \), where we seeded all of the
\( m=3 \to 20 \) modes with extremely small amplitudes
(\( \alpha_m = 10^{-8} \)) and random initial phases. Three different evolu-
tions ensued, with contributions from the \( m=9 \) and 10 modes
dominating.

IV. EVOLUTION FROM NONLINEAR PERTURBATIONS

As Sec. III showed, the predictions of linear theory agree
extremely well with CD simulations of Kirchhoff ellipses
seeded with small perturbations. There are compelling rea-
sons to investigate the effects of increasing the perturbation
amplitude into the nonlinear regime. In physical fluid flows,
where noncircular vortices would most likely to be created
by noisy processes, such as background flows or merger, one
would expect the emergent elliptical vortices to have large
deviations from the exact Kirchhoff solution and, thus, small
seed amplitudes would be the exception rather than the rule.
This expectation is corroborated by simulations of Legras
et al.19 done for circular vortices driven to large aspect ratio
by applied background flows.

Another motivation to study nonlinear dynamics is that a
survey by Dritschel of CD simulations of seeded Kirchhoff
vortices showed several discrepancies with linear theory.29
Our results suggest that these can be explained by the fact
that although the perturbations were small compared to any
dimension of the vortex, they were large enough (as quanti-
fied by \( \alpha_m \)) to be in the nonlinear regime. Unfortunately,
because values of \( a \) were not reported in Dritschel’s paper, it
is not possible to exactly reproduce his initial conditions.
Additionally, his initial conditions were slightly different as
he used linearized perturbations similar to Eq. (16). How-
ever, where relevant below, we will compare our simulation
results with his.

A. Nonlinear instability

Allowing the amplitude to vary results in a new degree of
freedom, and as a result the initial amplitude \( \alpha_m \) phase
\( \beta_m \), mode number \( m \), and aspect ratio \( a/b \) are required to
specify an initial condition with a simulation mode. We find,
not surprisingly, that increasing the amplitude of a Love \( m \)
perturbation above a linear threshold results in the dynamics
becoming a function of the amplitude. At large amplitudes,
nonlinear instability can occur for modes which are linearly
stable.

Figure 9 shows the transition to nonlinear instability of a
\( m=3 \) perturbation seeded on an \( a/b=2.99 \) ellipse with initial
phase angle \( \beta_m=0 \). At small initial amplitudes, the \( m=3 \)
perturbation stably oscillates with its mode rotation frequency
independent of mode amplitude. At larger amplitudes, there
begins to be an oscillation frequency dependence, and the
mode amplitude modulation increases above the linear value
of \( \alpha_3^{\text{mod}} = \lambda_3/\mu_3^2 \). Ultimately, above initial amplitudes \( \alpha_3 \)
=0.0049, the mode is unstable and grows without limit to a
filamented state.
Similar amplitude dependence effects are seen with the \( m=+4 \) mode, but differences are seen with the negative branch. The large amplitude \( m=-4 \) has amplitude modulations which are smaller than its linear value, \( \alpha_m^{\text{mod}} < \lambda_1/\mu_1 \), instead of larger. Additionally, the large amplitude \( m=-4 \) does not necessarily grow without limit if the \( m=4 \) mode is unstable, as will be quantified in Sec. IV C below.

B. Nonlinear stability boundaries for \( m=3 \) and \( 4 \)

To quantify the effect of perturbation amplitude on stability, we generated nonlinear stability curves by bracketing the aspect ratios at which stability and instability are seen, for a range of amplitudes. To apply the largest possible amplitude seeds, we used the initial phase \( \beta_m=\pi/2m \). This is the phase where the tips of the ellipse, which with the Love \( m \) perturbations become unphysical sooner, have the smallest displacements.

The insets of Fig. 10 show the largest stable perturbations we can apply by using this seed phase. The perturbations are plotted at times when \( \beta_3=\pi/2 \) and when \( \beta_4=0 \), along with the unperturbed base ellipse. With larger mode amplitudes than these, the modes are immediately unstable to filamentation at the tips without any mode growth required. Because the amplitude of the stable oscillations as quantified by \( \alpha_m \) varies with time, as shown in Fig. 9, the amplitude we use to characterize the stability boundary is the maximum value, which occurs at phases close to \( \beta_m=0 \).

Figure 10 shows the stability boundary of the \( m=3 \) and 4 modes. Increasing the mode amplitude on a Kirchhoff ellipse to the maximum value possible without filamentation reduces the \( m=3 \) stability point from \( a/b=3.0 \) to \( \sim 2.8 \) and the \( m=4 \) from \( a/b=4.61 \) to \( \sim 3.8 \). With the \( m=4 \), the stability boundary is measurably shifted at amplitudes as low as \( \alpha_4=10^{-4} \) and so, one can conclude that the threshold for nonlinear amplitude effects is quite small.

We see similar reductions of stability with the higher order modes. Here, the measurement precision possible is reduced because simulation run times are limited, typically to less than one rotation period, due to growth of the unstable lower order modes. As one example of nonlinear instability, the predicted onset of linear instability for the \( m=-4 \) is at \( a/b=6.197 \). We observe the \( m=5 \) to be stable when the maximum mode amplitude \( \alpha_5=10^{-3} \) and unstable when \( \alpha_5=10^{-3} \).

In Dritschel’s survey, which as discussed above used perturbations different from ours, a large amplitude \( m=4 \) seed was found to be stable on an \( a/b=4 \) ellipse and unstable when \( a/b=5 \). These results are consistent with the \( m=4 \) stability boundary plotted in Fig. 10.

However, the survey also reported that a \( m=5 \) mode seed was unstable on an \( a/b=4 \) ellipse, and a \( m=6 \) seed was unstable on an \( a/b=5 \) ellipse. These aspect ratios are far below the values required for linear instability of \( a/b=6.197 \) and 7.774, respectively. With perturbations in the form of Eq. (15) with those modes, we were not able to achieve nonlinear instability shifts of the same magnitude.

C. Nonlinear fission boundary of \( m=-4 \)

In Sec. III C, we showed that the \( m=-4 \) mode can cause fission, and that with a linearly perturbed Kirchhoff ellipse, this occurs when \( a/b \approx 6.046 \). Here, we quantify the effect of increasing the seed \( m=-4 \) mode amplitude on fission. Perturbations of the form of Eq. (15) becomes unphysical when the amplitude is increased above \( \alpha_4=0.01 \). The slopes at the perturbed ellipse tips become discontinuous, and then the perturbed boundary loops at the end become double valued at particular values of \( \eta \).

While it is straightforward to numerically calculate stationary perturbed states, here we take the alternative approach of analytically continuing the Love perturbation to achieve larger seed mode amplitudes. We used the following procedure to modify the perturbation:
The equilibrium corotating vortices, based on consideration of the value for where an unperturbed Kirchhoff ellipse in a specific direction takes place. The line at which fission first takes place due to the Love perturbation is successful in producing filaments surrounding and connecting one or more separate vortex core regions. Even modes have two different evolution paths and, thus, late time states accessible to them, depending on the sign/phase of the initial displacement. Kirchhoff ellipses first fission into two separate vortex core regions when $a/b$ is increased beyond 6.044, due to the $m=−4$ mode.

Increasing the amplitude of a perturbation beyond a linear threshold can result in nonlinear instability. The transition to nonlinear dynamics occurs at small mode amplitudes, when the perturbations are still quite small relative to any vortex dimension. We presented stability curves from simulations quantifying this transition for the $m=3$ and $4$ modes. The onset of fission from the $m=−4$ mode is likewise changed as the perturbation amplitude is increased.

We presented two simple analytical approaches for creating seeds with large mode amplitudes. Above $\alpha_m \sim 5 \times 10^{-2}$, filamentation results with no growth of the mode required. Numerical methods of creating large amplitude stationary states should be able to probe the greatly perturbed regime in more detail.

In physical flows, vortex distributions with extremely sharp edges such as patches are rarely seen compared to those with smoother edges. Statistically, this is not surprising since a vortex patch is a singular distribution solution. Moreover, if an undisturbed patch was to form, its discontinuous edge would be smoothed by viscous transport at a rate dependent on the viscosity of the flow.\(^\text{13}\)

It is, therefore, worthwhile to consider how our results for the vortex patch would be affected by a smoother edge. One change is that the filamentation of Love $m$ waves will take place at lower mode amplitude than with patches because fluid elements more quickly become resonant with the waves. For the same reason, the Kirchhoff ellipse will no longer be a stationary solution once its aspect ratio exceeds a value which depends on the vorticity distribution. Simulations indicate that even a modest smoothing of the edges tends to result in filamentation which will drive elliptical vortices to aspect ratios $a/b<3$ within a rotation period or so.\(^\text{30}\)

As a consequence, one might expect that the phenomena discussed in this paper will be important in physical flows only when $\lambda_m/T_R$ is large enough to affect the vortex within a time $T_R=1$. Laboratory experiments and vortex method simulations are currently underway to determine how the growth rates of Love’s instability vary as a function of the vortex distribution. Our conclusions will be the subject of a future paper.

V. CONCLUSIONS AND DISCUSSION

By using analytic theory and CD simulations, we have investigated the evolution of perturbed Kirchhoff vortices by decomposing solutions into constituent linear eigenmodes. An initial condition with a single perturbation is completely specified by four parameters: the ellipse aspect ratio $a/b$, the perturbation’s mode number $m$, phase angle $\beta_m$, and amplitude $\alpha_m$.

With small perturbations, we find excellent agreement between the early-time dynamics and the various predictions of linear analytic theory. Elliptical vortices with small aspect ratios are stable, with the $m=3$ becoming unstable at $a/b \geq 3$, the $m=4$ becoming unstable at $a/b \geq 4.61$, etc.

At late times, the unstable vortices evolve to states consisting of filaments surrounding and connecting one or more separate vortex core regions. Even modes have two different evolution paths and, thus, late time states accessible to them, depending on the sign/phase of the initial displacement. Kirchhoff ellipses first fission into two separate vortex core regions when $a/b$ is increased beyond 6.044, due to the $m=−4$ mode.

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Once a circular vortex has a finite edge, its Kelvin $m$ modes are also susceptible to the nonlinear extension of the filamentation resonance, beat-wave damping, which results in the modes scattering downward in $m$. With Kelvin modes, the downscattering rates become comparable to those of damping from filamentation at large mode amplitude. It is reasonable to conjecture that stable Love’s instability was the mechanism in both cases and speculated that the appearance of secondary vorticity structures in shock wave experiments might also be related to it.

Finally, physical flows are always bounded, and boundaries will modify the equilibria and mode properties of vortices contained within. It is straightforward to extend the CD approach to investigate the evolution of vortex patches within circular or more complex domains.

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41See EPAPS Document No. E-PHFLE6-20-063804 for MATLAB routines for CD simulations and analysis. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.