The amount of time it takes for a chain of dominoes to fall varies as a function of the number of dominoes in the chain, the amount of space between the dominoes, the total distance the dominoes span and the shape of the domino arrangement. In this report, we have studied falling dominoes theoretically and experimentally in order to predict the fall time of any linear or nonlinear arrangement of falling dominoes.

1 Introduction

In certain arrangements, carbon monoxide molecules will interact with each other in such a way that they mimic the behavior of a falling arrangement of dominoes. During a power disruption of any size, the disruption affects any neighbors to the source of the disruption, and in the end, it may even affect a majority of the community in which it occurs. Modeling these type of dynamic interactions requires an understanding of a web of parts and is very related to a string of colliding dominoes.

A chain of dominoes has one external force acting on it, which is the force of gravity. In a cascading system the major internal force which causes the toppling of the line is the force of impact from one domino to the next. In our analysis of falling dominoes, we will position the first domino so that it is in a state of unstable equilibrium. With this setup, we can neglect all other external forces so it will be the only force acting on the dominoes. In our study, we have examined the system’s energy to develop a theoretical model for the toppling domino chain. We have also validated our theoretical models by comparing them to experimental data. Using both these mathematical and experimental techniques, we have developed a final model that predicts the toppling time of the line of dominoes.

2 Objectives

Studying topple time is the “big picture” of this project. In particular, we have completed the following challenges:
• Predict the topple time of an arrangement of dominoes—both linear and nonlinear, equal spacing and nonequal spacing.

• Given a fixed distance \( L \), construct a domino setup that minimizes topple time.

• Predict the topple time of other groups’ dominoes arrangement.

• Given a fixed distance \( L \) and \( N \) dominoes, construct a setup that minimizes topple time.

3 Theory

To begin creating our model, we first defined our parameters, variables and assumptions. We labeled the major dimensions of our system as in Figure 1, with \( w \) as the width of the domino, \( h \) as the height of the domino and \( t \) as the thickness of the domino. Additionally, we defined \( L \) as the distance from the front of one domino to the next domino, \( N \) as the number of dominoes and \( D \) as the total distance the dominoes span.

We made the following assumptions to simplify our cascading domino system:

• All of the dominoes are uniform in size and mass. In order to simplify our calculations, we assumed all the dominoes have the same dimensions and masses. In particular, we used the average height, width, thickness, and mass of the 60 dominoes we measured.

• Dominoes will not slip about their bottom edge, creating a pivot point which will remain in contact throughout its fall. Throughout the experiments, the dominoes fell on a sandpaper surface, which ensured that this assumption is true.

• Each domino will strike the successive domino once and only once and slide through the contact region (Stronge, Wave of Destabilizing). This allowed us to eliminate a double impact from one domino to the next which simplified our problem in its early stages.
• **Conservation of energy.** We assumed energy is conserved throughout the toppling of the line of dominoes. This allowed us to neglect frictional losses from dominoes sliding as well as losses to air resistance.

• **Conservation of angular momentum.** The angular velocity will be directly proportional one domino to the next through impact.

• **Initial angular velocity is ZERO.** The launching device that we built ensured that the initial angular velocity of the first domino was insignificant and no initial kinetic energy was added to the system.

To solve the problem of minimizing topple time, our team examined several different approaches. Each of these will later be compared to our experimental results to see which models the real system the best.

The first technique we tried was maximizing the initial torque of a single domino hitting the next domino in a two domino system. We simplified our system so that this technique would be applicable to an entire system regardless of the number of dominoes behind and in front of the two domino system. The cascading line is essentially a unlimited number of two object collisions.

The most important part of this process was defining the geometries of the system (see Figure 2). The force of gravity, \( mg \), acted directly at the center of mass of each of the two dominoes. In our system, \( h \) is the height of a single domino, \( t \) is the thickness, \( R \) is the initial distance from the point of contact to the pivot point, \( y \) is the initial height from the bottom of the domino and the angles were defined as follows:

\[
\theta = \cos(s/h) \tag{1}
\]

\[
\gamma = \tan(y/t) \tag{2}
\]

where \( y = h \sin(\theta) \).

![Figure 2: Domino Variables](image)

An equation for the moment imposed on domino 2 by domino 1 was created and modified so that it was only as a function of \( \theta \). The underlying equation for the moment on a rigid object is defined as
\[ M = RXF, \]  
where \( M \) is the moment in newton-meters, \( R \) is the distance from the force contact to the pivot point and \( F \) is the force in newtons. The RHS of the equation can be modified from a cross product into the form

\[ M = R \times F_R, \]

where \( F_R \) is the component of the force acting perpendicular to \( R \). This component of the force can be written as

\[ F_R = F \sin(\phi), \]

with the angle \( \phi = \frac{\pi}{2} - (\pi - \theta - \gamma) \). By taking the moment about the point A, we can solve for the moment that will be acting on the second domino by the first domino.

\[ \Sigma M_A = mg \sin(\theta) \frac{h}{2} - mg \cos(\theta) \frac{t}{2} - Fh \]

and can be simplified to

\[ F = \frac{mg(\sin(\theta)h - \cos(\theta)t)}{2h}. \]

We can now write the moment equation about the pivot point of the second domino.

\[ \Sigma M_B = RF \sin(\phi) \]

where \( R = \frac{h \sin(\theta)}{\sin(\gamma)} \) and \( \gamma = \tan(\frac{h \sin(\theta)}{t}) \). The moment equation now becomes

\[ \Sigma M_B = \frac{h \sin(\theta)}{\sin(\tan(\frac{h \sin(\theta)}{t}))} \frac{mg(h \sin(\theta) - t \cos(\theta))}{2h} \sin[\frac{\pi}{2} - (\pi - \theta - \tan(\frac{h \sin(\theta)}{t}))]. \]

We can find the maximum moment by plotting the function or by finding when the derivative is equal to zero. Solving for this gives us an angle of \( \theta_t = 1.37 \) radians = 78.5 degrees. Using (1) we can solve for \( s + t \) to be

\[ 0.867cm + 0.679cm = 1.55cm \]

to measure the front to front spacing of our domino line. This equal linear spacing will be tested in our experimental section to evaluate the theory.

The second technique we used to examine our spacing model was based upon the Stronge study in ‘The Domino Effect’ in which he was able to create a trend for the intrinsic angular speed of the domino cascade. They examined a set of linearly spaced dominoes and were able to solve for a natural propagation speed of the line. They created a set of trend lines for angular speed as a function of the ratio of the width over the height and the spacing over the height. The results of his study are shown in Figure 3.
Stronge used a different notation where $L$ is the height of the domino, $h$ is the thickness of the domino, and $\lambda$ is the spacing between dominoes. With his notation we solved for

$$\frac{h}{L} = \frac{6.79}{43.53} = .156$$

(11)

Using the fits for this ratio, we were able to predict the trend for our spacing ratio. We drew a vertical line to solve for $\lambda/h = 2.8$ which allowed us to solve for the spacing to be

$$1.9\text{cm} + .679\text{cm} = 2.579\text{cm}.$$ 

(12)

This constant spacing was also being tested for its validity in our experimental section.

The previous two approaches lead us to the following working theory. We analyzed each domino as it falls using our assumption of conservation of energy. After each domino has begun to fall, it looks like the domino in Figure 4.

As illustrated, it makes an angle, $\theta$, with the horizontal. Using the simple geometry of a
falling domino, as demonstrated by Figure 5, we determined that as the domino falls, its potential energy is

$$U = mg\left(\frac{t}{2}\cos \theta + \frac{h}{2}\sin \theta\right), \quad (13)$$

and its kinetic energy is

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}^2, \quad (14)$$

where $I$, the domino’s moment of inertia about its center of mass, and $v$, its tangential velocity at its center of mass, are as follows:

$$I = \frac{1}{12}m(h^2 + t^2) \quad (15)$$

$$v = \frac{d}{2} \dot{\theta}. \quad (16)$$
Using these values and solving once more for $T$, we get:

$$T = \frac{1}{6} m \dot{\theta}^2 (h^2 + t^2). \quad (17)$$

So, the total energy for the falling domino is $E = U + T$, or

$$E = mg \left( \frac{t}{2} \cos \theta + \frac{h}{2} \sin \theta \right) + \frac{1}{6} m \dot{\theta}^2 (h^2 + t^2). \quad (18)$$

Then, we worked backwards to see what energy the falling domino had just as it began to fall. As each domino (except for the first) begins to fall, as in Figure 6, $\theta_i = \frac{\pi}{2}$ is the angle that the edge of the domino makes with the horizontal. So, the domino’s initial potential energy is

$$U_i = mg \frac{h}{2}. \quad (19)$$

But, clearly, this equation is similar to the equation for the falling domino’s potential energy, since

$$U_i = mg \frac{h}{2} = mg \left( \frac{t}{2} \cos \theta_i + \frac{h}{2} \sin \theta_i \right) \quad (20)$$

when $\theta_i = \frac{\pi}{2}$.

As the first domino begins to fall, it looks like the left domino in Figure 7, where $\theta_i$ is the angle that the edge of the domino makes with the horizontal and $d = \sqrt{h^2 + t^2}$. Obviously, the domino’s initial potential energy is $U_i = mg \frac{d^2}{2}$, or

$$U_i = mg \frac{\sqrt{h^2 + t^2}}{2}. \quad (21)$$

But, since $\cos \theta_i = \frac{t}{d}$ and $\sin \theta_i = \frac{h}{d}$, this equation is also similar to the equation for the falling domino’s potential energy,

$$U_i = mg \frac{\sqrt{h^2 + t^2}}{2} = mg \left( \frac{t^2}{2d} + \frac{h^2}{2d} \right) = mg \left( \frac{t}{2} \cos \theta_i + \frac{h}{2} \sin \theta_i \right) \quad (22)$$

by simple geometry.
Next, we realized that the kinetic energy was exactly the same for any domino as it begins to fall, namely
\[
T_i = \frac{1}{2}mv_i^2 + \frac{1}{2}I\dot{\theta}_i^2 \tag{23}
\]
where \( I \), the domino’s moment of inertia about its center of mass, is the same as above and \( v_i \), its tangential velocity at its center of mass, is as follows
\[
v_i = \frac{d}{2}\dot{\theta}_i. \tag{24}
\]
Plugging \( v_i \) and \( I \) into the equation, it is easy to see that the domino’s kinetic energy is
\[
T_i = \frac{1}{6}m\dot{\theta}_i^2(h^2 + t^2). \tag{25}
\]
However, one of our assumptions is that \( \dot{\theta}_i \) is negligible for the first domino. So, Equation 13 only holds for the first domino assuming \( \dot{\theta}_i = 0 \) for the first domino.

Subsequently, we combined our theory and found that for any domino, the total initial energy, \( E_i = U_i + T_i \), and so
\[
E_i = mg\left(\frac{t}{2}\cos\theta_i + \frac{h}{2}\sin\theta_i\right) + \frac{1}{6}m\dot{\theta}_i^2(h^2 + t^2). \tag{26}
\]
Finally, we applied our assumption that energy is conserved in the system, so that for each domino, \( E_i = E \). Solving this equation for \( \dot{\theta} \), we got a separable first order differential equation:
\[
\dot{\theta} = -\sqrt{\theta_i^2 + \frac{3g}{h^2 + t^2}(h(sin\theta_i - sin\theta) - t(cos\theta_i - cos\theta)).} \tag{27}
\]
The above equation applies to the first domino with \( \theta_i \) as it was measured in our experiments.

Figure 8: Spacing between the Current Domino and the Next Domino

It applies with \( \theta_i = \frac{\pi}{2} \) for any domino after that. In either case, \( \theta \) varies from \( \theta_i \) to \( \theta_f \), where \( \theta_f \) is the angle at which the domino touches the next domino, that is
\[
\theta_f = \arccos\frac{l}{h}. \tag{28}
\]
where $l$ is the spacing between the current domino and the next domino, as shown in Figure 8.

In order to use Equation (27) it became essential that we know the initial angular velocity of each domino. To obtain this we assumed that the initial angular velocity of a domino is equal to the final velocity of the previous domino multiplied by a scaling term $e$. The scaling term $e$ is a dimensionless number that is a function of the ratio of domino spacing to domino height, so $e = e(l/h)$. Therefore, $\dot{\theta}_2 = e\dot{\theta}_1$. To determine the function for $e$, we looked at our experimental results and found the number for $e$ that yielded the most accurate time predictions. After obtaining $e$ from 31 different experiments, a graph of $e$ vs. $l/h$ was constructed (see Figure 9).

![Figure 9: Experimental Scaling Terms](image)

There appeared to be a quadratic trend to the data so we fitted a second order polynomial to the data with moderate success, $R^2 = 0.7027$. The equation for $e$ we found was:

$$e = -2.9098(l/h)^2 + 3.4744l/h - 0.3987. \quad (29)$$

To predict the fall time for any domino arrangements, we created a Maple worksheet (See Appendix A) that determines fall time of a set of dominoes based on the number of dominoes and the spacing between them. Once the relevant spacings are input, $e$ is calculated from our fit, and then the time it takes for each domino to get from its starting position to the next domino (or the ground in the case of the final domino) is determined and added together to obtain the total fall time.

A supplemental theory was generated so that our linear theory could also predict curved domino cascade setups. Using AutoCAD we input the given curved setup from an overhead
view so that it was possible to see the width and thickness of each domino. The correct dimensional values of each domino gave an exact position of each in space while in the upright position. The position of each domino is defined by the spacing and angle from the previous domino. The position of the first domino is relative and the rest of the dominoes can be set up accordingly.

The positives of using a CAD tool in our experiments allows us to input accurate lengths and angles and get an overall picture of the setup. Our linear calculation used the average spacing between dominoes to find the angle of collision and our velocity transfer coefficient. To predict the dynamics of a linear string of dominoes, we used the distance from front to front of each domino as the spacing. For our curved experiments, we tried using several different distances from one domino to the next. With the CAD setup (See Figure 10) we could easily find the distance between each center to center, contact points and corner to corner using either closest or farthest corners. Each of these spacing methods was used to predict the fall time and compared. For each of the three measurement techniques we tried using individual spacing as well as average spacing in our model to predict the time.

![Figure 10: CAD Setup for Spacing and Angle Measurements](image)

After calculating the predictive model time we found that the actual time appeared to be a factor related to the total rotational angle of the dominoes. We generated a non-dimensional scaling factoring based upon the sum of the absolute value of the relative angle changes. It was also important for our curve theory to be continuous with our linear theory so that when the angle of change was zero the scaling factor would be one and not affect the result. This resulted in the following equation:

\[
t \times (1 + \frac{\Sigma |\theta|}{360}),
\]

where \(t\) is the predicted time from our model and \(\Sigma |\theta|\) is the sum of the absolute values of the angles of the dominoes as measured parallel to the previous domino. This scaling mechanism allows the time predicted to be a continuous function when moving from linear
to nonlinear strings of dominoes. Thus, the fall time for a linear string of dominoes remains the same when \( \theta \) is 0 and the time will increase when dominoes are not parallel.

4 Experiments and Results

After measuring the masses and dimensions of the dominoes, we found that all of the dominoes are not perfectly uniform in size and mass. We took measurements of 60 dominoes and the masses ranged from 3.5g to 5.2g. Twelve dominoes closest to the average mass of 4.56g were chosen to perform our series of experiments. This way the assumption that all the dominoes are uniform can still be used for our model. Figure 11 shows the average domino dimensions, which will be used in our calculations.

![Figure 11: Dimensions of an Average Domino](image)

Our experimental method needed to be repeatable so that our results were accurate and precise. We built a stand that would rotate a screw to slowly tilt the first domino into a position of unstable equilibrium. The setup is as shown in Figure 12.

![Figure 12: Launching device](image)

In our experiments, we started by studying the fall time and initial angle of a single domino. The initial angle was 80.25 degrees and the fall time was 0.3153 seconds. Photogate 1 recorded the start time and photogate 2 recorded the stop time. We took data of an array
of 6, 9, and 12 dominoes with spacing of 2.0cm, 2.5cm, and 3.0cm. The results are plotted in Figures 13 and 14.

We noted that the photogates provided were incapable of measuring spacing less than 1.5cm. The domino requires a spacing of twice its thickness to provide sufficient spacing in order to reach its toggle point. That distance is $2 \times t = 2 \times 0.679\text{cm} = 1.358\text{cm}$, and that leaves only 1.4mm of clearance for the photogate’s beam thus making the experiment not very repeatable. The first domino was in the way of the beam before it reached its unstable equilibrium position. Therefore, we chose to start with spacing of 2.0cm.
It took many trials and much practice before getting consistent data. Setting up the row, the photogates and initiating the fall proved to be very time consuming. If one of the trials seemed to be an outlier compared to other tests, we removed it from our data section.

After much experimentation using photogates, we determined that the results relied heavily on how the first photogate (measuring start time) was positioned. So, we decided to use a high-speed camera to record the fall time of dominoes. We modified our launching device a little bit to adapt to the change in equipment. Our team first glued sandpaper on a piece of wooden board. The sandpaper is completely attached to the board to achieve a flat surface. We then taped a ruler on top of the sandpaper for placement of dominoes. The launching device is then clamped to the board. Figure 15 illustrates the new setup.

We noticed that there was a lot of room for error when deciding the time the domino leaves the screw, so we sharpened the end of the screw, so that the tip is pointy. With the pointy tip, it’s easier to tell when the domino leaves the screw. There was also a problem with dark colored dominoes because they did not reflect light as well as the lighter colored ones, so we eliminated the darker ones from our experiments.

In setting up the software, we used 1000 frames/second frame rate, shutter speed of 1/5000 second, and a resolution of 512x512. The video of falling dominoes was captured under these settings. A sample frame is shown in Figure 16.

Figure 15: New Setup with Modified Launching Device

Figure 16: Sample Frame taken with High Speed Camera

The next step was to review the video to determine the time the first domino left the screw and the time the last domino hits the ground. We paused the video near the point
when the first domino was leaving the screw. We then changed the frame rate to the slowest setting and zoomed in to view movements in “pixel” details. Therefore, we were able to determine the instant when the first domino left the screw. After we recorded the start time, we fast-forwarded the video to the instant before the last domino hit the board. We wanted to find the first frame that the last domino became parallel to the board. We had to be careful, because much of the time the last domino bounced against the board at least once. We then record the first instant the last domino was parallel to the board as the stop time. The difference of the start time and stop time gave us the fall time that we are interested in.

We started to run experiments after we became familiar with the new setup and equipment. We experimented with equally-spaced dominoes with different total distances: 5cm, 10cm, 15cm, 20cm and 25cm. For each total distance, we varied the number of dominoes. The following is the formula we used to calculate the distance between dominoes:

\[
\text{spacing} = \frac{(D - 0.7\text{cm})}{(N - 1)}
\]

where \(D\) is the fixed total distance, 0.7cm is the thickness of a domino, and \(N\) is number of dominoes.

We stopped experimenting after the 25cm distance because the lens of the high speed camera could not capture distances larger than 25cm. With no zoom option available on the camera, we were constrained by the distance the camera could be backed up from the domino line. The data is organized into plots in Figures 17 and 18.

![Figure 17: Time vs Spacing between Dominoes](image)

In our experiments with nonlinear spacing, we used the high speed camera with the same lens, frame rate, shutter speed and resolution as before. We also used the same launching device. We first began our experiments by analyzing the time it took for two nonparallel dominoes to fall. We made sure to set up the two dominoes so that the camera could record the first one as it began to fall and the second one as it hit the ground. Again, we paused the video to determine the instant when the first domino left the screw and the then again paused
it to find the first frame that the second domino was parallel to the horizontal. Finally, we used these time values to find the total time it took for the two dominoes to topple. Using the coordinate system in Figure 19, we experimented with $x$ values of 2cm and 3cm, $y$ values of 0, $w/4 = 0.539\text{cm}$, and $w/2 = 1.078\text{cm}$, and $\theta$ values 0°, 30° and 60° to determine the relationship between fall time and domino orientation. In a few of the trials, the second domino would not topple away from the first domino. It was either too far away for the first domino to reach it, or the angle caused it to fall towards the first domino. The data for the trials where the second domino did topple away from the first is organized in Figure 20.
From this data no explicit trend is obvious. This is due to the many sources or error involved, mainly the non-uniformity of the dominoes. However, in general time increases with both $y$ and $\theta$.

Next, we set up experiments with curves of dominoes. Again, we made sure that although the curves were nonlinear, the first domino was recorded as it began to fall and the last domino in the arrangement was recorded as it hit the ground. To organize our experiments with more than two dominoes, we set up a different coordinate system with the spacing and angle from one domino to the next as in Figure 21. Using these coordinates, we tested four semicircular domino arrangements and two undulating domino arrangements. The semicircular arrangements had the spacings and angles between consecutive dominoes and fall times in Figure 22. The undulating arrangements looked similar to the one in Figure 23.
<table>
<thead>
<tr>
<th>Trial #1</th>
<th>Domino</th>
<th>Spacing(cm)</th>
<th>Angle(°)</th>
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<tbody>
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<td>2</td>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>1.8</td>
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<td>4</td>
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</tr>
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<td>6</td>
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</tr>
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<tr>
<td>Total Fall Time(s)</td>
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</tr>
<tr>
<td>Total Fall Time(s)</td>
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</table>

Figure 22: Experimental Setup for Semicircular Domino Arrangements
Again, we calculated the dominoes’ fall time by pausing the video to determine the time the first domino began to fall as well as the time at which the last domino hit the ground. The spacings and angles between consecutive dominoes in the arrangements as well as their fall times are included in Figure 24.

5 Experimental and Theoretical Comparison

We examined our data to see how the two theoretical spacings would perform at minimizing the topple time. Using the maximum-torque method predicted spacing, we looked at the experimental results and found that the cascade time was not close to the minimum time for that number of dominoes. In addition, we decided that the initial torque is not directly related to the minimal cascade time for an entire set of dominoes. This method would also be extremely difficult to proceed with, because it would not be easy to predict the time from the impact force.

We also tried testing the distance predicted from the Stronge method in our experiments.
We found that the time exceeded the minimum experimental time. We can also understand that the intrinsic angular velocity may not be directly related to the maximum linear cascading velocity. In addition, Stronge’s technique does not consider nonconstant spacing. This type of setup may perhaps be the best way to minimize the topple time over a distance.

We then tested our working theoretical model against experimental data. Using collapse time data for different spacings and number dominoes we were able to find how accurate our model was as well as the areas of strength and weakness. A trend for 7, 8 and 9 dominoes is shown in Figure 25.

![Experimental vs. Theoretical Times as a function of Spacing](image)

Figure 25: Experimental vs. Theoretical Times as a function of Spacing

Our data tested spacings of 5, 10, 15, 20 and 25 centimeters at equal spacings ranging from the minimum number of dominoes to the maximum. Using our calculation model with the third order collision constant function we were able to compare the time values. Over a series of 30 different trials we found the following about the percent error: $AverageError = 10.65$, $MinimumError = 1.69$, $MaximumError = 32.00$.

Looking at Figure 26, we can see that our predictions were not very accurate at the two centimeter spacing mark. This can be attributed to several experimental factors. There was difficulty finding the exact departure of the first domino from the screw in the camera because it was so close to the next domino. In addition, it was difficult for the experimenter to see when exactly to stop turning the screw as to not give the first domino any kinetic energy. It also was not easy to get the first domino into its unstable position because it was so close to the second domino.

The plot shows that our model predictions were very good right around three centimeters. We would speculate that in this setup the dominoes are in a good balance of potential and kinetic energy. There was enough room to tilt the first domino into its unstable position and the transfer of energy was smooth.
The accuracy decreased after about four centimeters and we can relate this mostly to physical errors in our model. Because the dominoes are just over four centimeters tall, the collision occurs below the center of mass of the next domino. This collision naturally pushes the domino’s bottom edge forward creating sliding. Our model assumes that each domino rotates perfectly about its front axis and neglects all friction. These extreme spacings show how the model not as accurate at large distances.

Since our group mainly tested equal spacings, it was necessary to try our model against unequal spacings. Using several different groups’ original spacings we executed our program to compare the predicted time to the experimental time.

We first tried defining each spacing in the cascading line so that the final angle would be specific to each collision. We were able to find velocity transfer constants at for each collision based on the spacing and angle of collision of the previous domino to the next domino. Although this produced positive results, we tried averaging the spacing across the domino line to get an overall constant to be used at every collision. The error in predicted time was consistently smaller with the average spacing as opposed to the individual previous spacing (See Figure 27).

We can analyze this result to further understand why it is so. For each of our experiments, a single velocity transfer coefficient was found over the chain of dominoes. Because the constant was not specific to an individual spacing and collision, the coefficient would be more accurate over a line of dominoes. Therefore using an averaged coefficient would be more accurate to predict an average spacing experiment.

During one of the challenges, predict the topple time of other groups’ dominoes arrangement, we were able to test our model against more types of domino setups. Two were evenly spaced domino chains and two were not, and we got a good overall evaluation of the performance of our model. Our average error was 4.405 percent which was the best overall among all the groups (See Figure 28).

We then go on to compare our nonlinear theoretical model to our nonlinear experimental...
data. We predicted the times of two semicircular trials and both wave curve trials. Our average error for the original theoretical model was 35.53%, and average error for the scaled model was 8.14% (See Figure 29). The scaled model was much more accurate so we picked it as our final model. This also demonstrated that our scaling factor worked well. The higher percent error was due to the increase in uncertainties in the nonlinear curve arrangement. There were some unknown physical properties that we did not account for. For example, dominoes arranged at varying angles are more likely not to follow the set of assumptions that we made for our model. The domino collisions were less uniform in that they were no longer perpendicular to the surfaces of impact. Because of this dynamic interaction, a domino that is not hit through its center of mass is prone to change from its original angle before striking the successive domino to a new angle. In essence, the arrangement of dominoes can change during the set of collisions and each domino can tip, slip and spin. Therefore, the 8.14% error in nonparallel setup versus 4.405% error in setup is reasonable.
6 Conclusion

The dynamic analysis and prediction of a cascading line of dominoes is a very difficult. By first simplifying the problem, we set up a number of assumptions to help us proceed into developing a model. We experimentally and theoretically studied a single falling domino released from its unstable position. Our team developed an energy equation based upon the balance of kinetic and potential energy. After understanding the topple of a single domino, we expanded this model to a system of dominoes cascading through collisions.

Our team first used experimental data to create a mathematical prediction of time of a linear chain as a function of spacing and number of dominoes. This theory was inaccurate at nonconstant spacing so we developed a new model using a velocity transfer coefficient from one domino to the next. This coefficient was a function of a dimensionless parameter of spacing/height and was found using experimental data.

The next step was to expand the theory so that we could predict falling time for curved arrangements. An additional scaling constant was used to account for the increase in falling time due to collisions at various angles. This constant enabled our mathematical model to be continuous across both linear and curved arrangements.

The evolution of our theory developed to account for numerous domino arrangements. Our methods have been very successful in determining the fall time of a chain of dominoes.

7 Annotated Bibliography

Banks looks to examine mainly the ‘wave motion’ that is associated with cascading dominos. First, an individual slender object is studied with a frictionless axle at the base that is free to rotate, neglecting air resistance. Conservation of energy yields the relation between potential and kinetic energy of the system. The equation is modified into a form that will find the time for the pole to fall over. He adds that it is necessary to give a nudge to create a small angular velocity on the object, which is part of the equation.

He similarly examines a wave of dominos spaced evenly apart. He makes the assumption that “only one domino affects the next one in the row” and the preceding three or four leaning dominos are neglected. After some manipulation he forms the equation that gives the time of fall of the wave.

Our initial model will be similar to that of Banks, but we hope to expand to include all dominos that have any dynamic effect on the system. Still, his model is a good place to start in that he makes several rational simplifications to create a rough model.


This paper will be very useful as a preliminary guide when investigating the mechanics of falling dominos. The author uses the following assumptions:

- All dominos are identical.
- The dominos are parallel to each other and equally spaced.
- There is no sliding of the dominos on the plane and each domino remains in contact with the plane at least along one edge.
- Each domino rotates in one direction only.
- There are no external friction losses.
- The dominos are perfectly rigid with no energy loss at contact.
- The time of propagation for each domino is the same, i.e., there is no interaction between dominos other than initial contact.

Based on these assumptions one can get a model for the motion of falling dominos. This article has a few problems for our project. It does not present a clear model for the fall time of dominos, and in comparison to other papers this one is oversimplifying the problem with their assumptions. In the early stages oversimplification is useful to help us grasp the subject. But over time many assumptions here would need to be replace with better ones. For example instead of assuming the dominos are rigid, assume they have a specific coefficient of restitution.


Molecule cascades are similar to an array of falling dominos: one molecule causes the subsequent motion of another resulted in a cascade of motion. The scanning tunneling microscope is used to study the physical and functional properties of molecule cascades. The
hopping rate was assumed to be a function of temperature, isotope, and local environment. The study found that the hopping rate of carbon monoxide molecule cascades is independent of temperature below 6 kelvin. Logic gates such as AND gate, OR gate, were employed in the study. A dominoes cascades model was used to perform mechanical computation. Topple state is represented by the binary number 0 and untoppled state is represented by the binary number 1. Dominoes were set in patterns to perform logic gates operations. The resulted carbon monoxides molecule cascades experienced a range of variation in propagation times. This result deviated from the dominoes cascades as they are usually deterministic with predictable timing.


While this article deals mainly with other factors that were believed to cause the fallen tree patterns, it briefly deals with the domino effect that one falling tree can have on other trees around it. In the study, samples of three different forests were surveyed. Using the collected data, Lin and his colleagues determined that 15% of all fallen trees had been a part of a domino effect, and evidence of this effect was found in two of the three surveyed areas. In these regions, trees that had recently died and fallen were more likely to be in an open area than living trees. However, just because the phrase ‘domino effect’ is used so frequently, it rarely means that one falling tree hit another, causing it to fall. Usually, other factors, such as tree diseases and wind damage, cause trees around a fallen tree to fall down as well, like falling dominoes. As in previous studies, the observation and analysis of the patterns of fallen trees indicated that considering “domino effects” may improve the understanding that Lin and his colleagues have of the phenomenon (24-25).

This source is helpful to demonstrate the relevance of falling dominoes to natural phenomena, but it will probably not be beneficial until the end of the semester, in case we want to apply our model of falling dominoes to this and other real world situations.


In this article the authors attempt to explain the physical events that caused the disintegration of the Larson B ice shelf. The setup of the collapsing ice is very similar to that of our domino problem. In their study, they implied that the ice shelf fragments were narrower in the along-flow direction than they were thick. This would allow the fragments to rotate easily about their bases in the direction of the cascade because they are slightly unstable when upright.

The initial cause of the break up may be linked to global warming which melted crevasses deep into the shelf. This created many unstable ice elements that eventually capsized under a number of forces. The individual sections were acted on by a downward gravity force as well as an upward buoyancy forced caused by the difference in densities of the ice and ocean water. These forces created a moment about the fragment which made contact forces to the sections on either side. Eventually some of the fragments turned over completely which created enough gaps for the fragments to fall in a cascading manner.
We can draw many similarities of this natural phenomenon compared to that of our domino experiment. We will be using similar slender objects that will rotate along the same plane. However, our model will begin its cascade at one end and move linearly along the system. In the ice shelf model fragments could fall at any place in the line of dominoes and create a much more complicated system. Also, these fragments could rotate about almost any location and are not limited to the bottom edge or center of mass.


This begins with an analysis of a wobbling domino. Which is not particularly useful at the moment, but McGeer and Palmer also analyze the mechanics of a falling pencil. Their models are quite convincing when looking at their graphs of calculated vs. measured data. This paper may be useful in the later parts of the semester when sliding friction may be a possibility, but at the moment we do not stand to gain a lot from it.


An investigation into a networked system and how a disturbance at one node can propagate through part or all of the structure is performed here. This paper illustrates the commonality of the domino effect and how it can apply to power grids. This paper could be a tool to rationalize the importance of understanding the wave effects of falling dominoes.


Shaw examines a simple setup of a linear set of N evenly spaced dominoes. The objective was to measure the elapsed time of falling N dominoes. The dominoes are placed on sandpaper to reduce slipping and photo gates are used to measure the falling of the first and last domino. The first domino is placed in an unstable position and released such that it has a small initial angular velocity. Energy conservation played an important role in the falling of dominoes. “The model presented uses energy conservation between collisions to yield the instantaneous angular velocities and conservation of angular momentum when a collision occurs provides a new set of angular velocities after contact.” (Shaw 640)

Shaw plotted the experimental values of time $T_N$ versus the number of dominoes $N$ ranging from 1 to 20. Shaw first acquired the time $t_i$ for domino $N$ to move from its vertical position to contact domino $N + 1$. He then summed the individual $t_i$ starting from $i = 2$ to $i = N$ to obtain the total elapse time. The $T_N$ versus $N$ curve showed that when $N$ is greater than 6, the curve is almost linear. This means that as domino 1 begins to fall, only domino 2 to domino 6 made major contributions to the total energy.

Shaw used a method in which he would first focus on two dominoes at a time- the domino falling and causing the impact and the domino that is getting hit. He sets up a geometric relationship between these two dominoes
\[
\sin(\theta_{N-1} - \theta_N) = \frac{\cos(\theta_N)}{b} - \frac{a}{b} \quad (32)
\]

which relates the angle of the first domino to the second domino, including the dimensions of
the height and thickness. (The angle is measured from the vertical in a clockwise direction.)
Taking the time derivative of the equation, the angular velocity of the second domino is
related to the first domino.

\[
\dot{\theta}_{N-1} = [1 - \frac{(c \sin \theta_N)}{b} \cos(\theta_{N-1} - \theta_N) \dot{\theta}_N] \dot{\theta}_N \quad (33)
\]

The initial energy of the entire system \(E\) is found using the addition of the potential and
kinetic energy at \(t = 0\). This takes into account any kinetic energy (such as the first domino
having an initial velocity) and energy is assumed to be conserved throughout the system.
Therefore, in his system, the collapsing dominoes will use this initial energy causing them
all to fall. The model also takes into account the fact that for every domino the final angle
may not be 90 degrees.

\[
E = \left(\frac{1}{2}\right)mg(b \sum_{i=1}^{N} \cos \theta_i + a \sum_{i=1}^{N} \sin \theta_i) + \left(\frac{1}{2}\right)I \sum_{i=1}^{N} \dot{\theta}_i^2. \quad (34)
\]

This equation can be manipulated into a form that can solve for \(\theta_N\) based off of several
coefficients. It is possible to discretize this equation and write the angle of domino \(N\) as
a function of the time step \(dt\). A numeric solution would define this time step so that
the change in the angle of a domino from \(\theta_0\) to \(\theta_1\) divided by the time would be equal to
the angular velocity. The angular velocity of the second domino just after it is struck by
the previous domino is equal to that of the first domino because of conservation of angular
momentum.

This numerical solution would similarly be used to move along the line of dominoes in
the system. The total time would be equal to the time for each domino to impact the next
domino plus the time for the last domino to strike the ground.

One of the conclusions that Shaw draws from his experiment is that his theory only works
for \(N < 6\) because when \(N > 6\), the curve is nearly linear. We saw similar results in our
experimental data and concur that this is true. This phenomena occurs because when \(N\) is
between 1 and 6 the energy of the system increases, but after \(N\) greater than 6, the energy
stays essentially the same. Because his theory only works for less than 6 dominoes we are
reluctant to go through his approach and create the simulation. However, we are able to get
some good ideas about the overall dynamics of the system.


This paper deals directly with falling dominoes and the angular velocity of each domino
compared to the previous falling domino. It assumes that the dominoes are identical, and
evenly spaced on a “rough, level surface” (199). Stronge has many interesting ideas about
the modeling of falling dominoes. For example, he determines a ratio for the angular velocity at the beginning of one collision vs. the angular velocity at the beginning of the one before it. At the end of his paper, Stronge summarizes his five biggest conclusions:

1. There is a natural propagation speed for the collisions if the domino thickness is at least half the distance between two dominoes.
2. As the number of dominoes that have fallen increase, the propagation speed should approach this “intrinsic speed” (206).
3. For dominoes that are grouped closer together, momentum is not conserved from collision to collision.
4. If there is a lot of friction, a domino can reverse direction during a collision.
5. While dominoes arranged farther from one another have only one collision at a time, dominoes arranged closer together have multiple collisions, so that each new collision builds upon the last.

This source is very useful right now as a complicated example of the factors we should take into consideration when forming our initial model. It will be incredibly useful in the future when we are ready to consider a system that loses kinetic energy or a system where the dominoes slide or a system where a domino hits another domino and then reverses direction.


This article assumes all of the conclusions proven in the previous article (see above). It does, however, take the model one step further. It analyzes an enormous row of fallen dominoes, all leaning on the currently falling domino. It also neglects friction as well as the coefficient of restitution. Instead of just having a ratio for successive angular velocities, Stronge and Shu calculate the ratio of the kinetic energy for the currently falling domino vs. the kinetic energy that the first domino had while it was falling. The most groundbreaking part of this paper is the idea that when the distance between dominoes is large, “there are some multiple collisions and some single collisions” (162). Also, when the distance between dominoes is small and one domino falls on another, there is either “multiple collisions” or the impact between dominoes is “sliding” (162). This really demonstrates to me how complicated it is to tell how fast linear, evenly spaced dominoes fall. Additionally, Stronge and Shu experimentally test their theory to determine how accurate their model is. It is very clever that they used small rectangular blocks as opposed to actual dominoes in all of their experiments, since dominoes tend to differ slightly in their dimensions. Also, although their model assumed the coefficient of restitution was negligible, when in fact, it was measured at at least one half in each experiment, their “cooperative group theory” model still very closely predicts their experimental data (161).

This article should be even more helpful in the future than the previous article. Although the model is much more complex than the previous model, it is very accurate, and if we could improve it even a little, we would have a very good understanding of the time it takes linear, evenly spaced dominoes to fall.

Watts studied the large cascades, called global cascades. An example of global cascade is the cascading failure of power transmission grid in the western United States in August 10, 1996. Global cascades seldom occur and are extremely difficult to predict. Network infrastructure is composed of nodes. It is found that the more traffic connected to the node, the more likely it will trigger a cascade when compared to an average connected node. Moreover, initial failures also increase the probability of subsequent failures. The model studied in the journal was a model of motivation named “Binary decisions with externalities.” This model studied the decision made by an individual depend solely on the decisions of others in this model. The propagation of cascades is limited by the global connectivity of the network when the network of interpersonal influences is fairly sparse. Another behavior is when the network of interpersonal influences if fairly dense, the propagation of cascades is limited by the stability of the individual nodes.

8 Appendix

```plaintext
> restart;
> t1:=1;
> t2:=1.1;
> t3:=1.2;
> e1,2:=2.9960*(114.3533+3.4764*(114.3533)-0.3987;
> e2,3:=2.9960*(114.3533+3.4764*(114.3533)-0.3987;
> D1:=sqrt((3*sqrt(e2,3)*sqrt(k^2+t^2)-k*sin(a)-t*cos(a))/(k^2+t^2));
> af := arccos(1/k);
> int(1/e2,a=0..af);
> t:=e2;
> h:=e,3533);
> q:=90;
> af:=evalf(subs(a=0,D1));
> t1:=(int(1/D1,a=0.3/100+Pl..af));
> Math for the next domain.
> abot:=1.2*2*3.77086077;
> ab:=P1/2;
> af:=arccos(1/k);
> D2:=sqrt((abot^2+3*y^2)+k*sin(a)-t*cos(a))/(k^2+t^2);
> t:=int(3/D2,a=0..af);
> Math for the next domain.
> abtot=-1.2*2*3.77086077;
> ab:=P1/2;
> af:=arccos(1/k);
> D3:=sqrt((abot^2+3*y^2)+k*sin(a)-t*cos(a))/(k^2+t^2);
> t:=int(1/D3,a=0..af);
> t:=t2:t3;
```

Figure 30: Theoretical and Experimental Error Comparison