1 Introduction
Starch is a long chain of sugar molecules called a polysaccharide [1]. Starches differ by the percentage amounts of amylose and amylopectin, the sugars that form the starch [2]. During heating of a starch, these sugars break down, or lose order, which allows for water absorption and therefore swelling of the granule [2]. This process, called gelatinization occurs between 58°C and 75°C [7].

Our investigations will be focusing on mathematically modeling the gelatinization front of potatoes and rice. Our first goal is to know when and what parts of starches have gelatinized during the heating (cooking) process. By increasing the level of understanding of the gelatinization process it is possible to optimize industrial production of these types of foods while controlling quality and stability [6]. This in turn will lead to a more efficiently produced final product with improved taste and texture [5].

Previous research in this area has been extensive and results have varied. Chi Kai shows a linear relationship between time and water uptake during the gelatinization process [4]. Sharpe [3], Landman, and McGuinness [5] show a nonlinear relationship in their results. Our preliminary results and heat transfer model suggest a nonlinear relationship. Ogawa et al. show that water penetrates into rice unequally during heating [8]. This indicates a possibility that gelatinization does not move with a uniform front. Our early experiments show that this front, in potatoes, is somewhat uniform, and our model is based on the assumption that the front is uniform. More experimentation and modeling is necessary to determine the actual shape of the front.

2 Mathematical Model
2.1 Introduction
A number of factors must be considered in order to model the gelatinization in a potato during the heating (cooking) process. Based on our literature review and experimentation we have developed several assumptions that will allow us to mathematically describe the loss of order during the cooking process. These assumptions will more easily allow us to derive a
mathematical representation of the gelatinization process without drastically changing the solutions. Assumptions:

- The potato is spherical. We will form the potatoes into spheres before experimentation. This is necessary for us to mathematically model the gelatinization front.

- The level of gelatinization can be determined by measuring or modeling the temperature inside the potato. This allows us to solve the heat equation on a sphere and model the isotherm as it moves through the potato to measure the loss of order.

- The potato is homogeneous throughout. This will be accomplished by skinning the potato before experimentation. This allows for an easier calculation of the heat flux into the potato.

- The gelatinization moves on a uniform front. In other words, the geometry of the gelatinization front evolves axis-symmetrically. Some of the literature states that this is not true for starches, but our experiments have shown that this assumption holds.

- Little swelling occurs during the gelatinization process and we therefore consider the radius constant during cooking. Our experimentation shows this and we have hypothesized that it may be due to the fact that the water present in the raw potato helps stimulate the gelatinization of the cells.

- Based on initial experimentation, we notice that the boundary between gelatinized to ungelatinized starch is relatively small in consideration to the size of the potato. Please see Figure 1.

2.2 Theoretical

Based on our assumption that gelatinization can be determined by the temperature of the potato (63°C indicates full gelatinization), we can model the isotherm inside the potato as the gelatinization front. This isotherm is determined by solving the heat equation. The general heat equation is

$$\frac{\partial U}{\partial t} = \alpha^2 \nabla^2 U.$$  \hspace{1cm} (1)

Assuming that our potatoes are spherical allows us to use the heat equation on a sphere as our model. Using the variable transformations:

$$x = r \sin(\phi) \cos(\theta),$$ \hspace{1cm} (2)

$$y = r \sin(\phi) \sin(\theta),$$ \hspace{1cm} (3)

$$z = r \cos(\phi),$$ \hspace{1cm} (4)

we obtain the heat equation in spherical coordinates:

$$U_t = \frac{1}{r^2} \alpha^2 \left( 2r U_r + r^2 U_{rr} + \frac{\cos(\phi)}{\sin(\phi)} U_\phi + U_{\phi\phi} + \frac{1}{\sin^2(\phi)} U_{\theta\theta} \right).$$  \hspace{1cm} (5)
Next, by applying our initial assumption that this phenomena is axis-symmetric (independent of $(\theta)$ and $(\phi)$), we have
\[
\frac{\partial U}{\partial \phi} = \frac{\partial U}{\partial \theta} = 0.
\]
(6)

We can now simplify (5) to
\[
U_t = \alpha^2 \left( \frac{2U_r}{r} + U_{rr} \right).
\]
(7)

2.3 Boundary Conditions

Based on the temperature of our bath and the assumption that the outer layer of the potato is instantly the temperature of the bath once submerged, we have the boundary condition
\[
U(c, t) = 75^\circ C.
\]
(8)

After taking the temperature readings of the center of a potato during experimentation, we found the center to be at room temperature (approximately 21.2$^\circ$C). This gives us the following boundary condition:
\[
U(0, 0) = 21.2^\circ C.
\]
(9)

2.4 Non-Dimensional Model

Making our variables dimensionless gives us the following:
\[
\bar{t} = \left( \frac{t}{\alpha^2} \right), \quad (10)
\]
\[
\bar{r} = \left( \frac{r}{c} \right), \quad (11)
\]
\[
\bar{U} = \left( \frac{U - U_{\text{initial}}}{U_{\text{final}} - U_{\text{initial}}} \right), \quad (12)
\]
\[
\bar{U} = \left( \frac{U - U_{\text{initial}}}{U_{\text{final}} - U_{\text{initial}}} \right). \quad (13)
\]

When we apply our dimensionless variables to (7), we have our dimensionless heat equation on a sphere:
\[
\bar{U}_t = \frac{2\bar{U}_r}{\bar{r}} + \bar{U}_{\bar{r}\bar{r}}.
\]
(14)

With boundary conditions:
\[
U(0, 0) = 0 \quad (15)
\]
\[
U(1, t) = 1 \quad (16)
\]
2.5 Solving the Equation

We will use the separation of variables technique to solve (14). To do this we assume $U$ is in the form (Note: from here on we assume all variables are dimensionless for simplicity)

$$U(r, t) = R(r)T(t).$$  \hfill (17)

By taking partial derivatives, we have

$$U_t = RT', \quad U_r = R'T, \quad U_{rr} = R''T.$$  \hfill (18), (19), (20)

We plug (18), (19), and (20) into (17), thus:

$$RT' = \left(\frac{2}{r}\right) R''T + R'T.$$  \hfill (21)

Dividing by $U$ gives

$$\frac{T'}{T} = \left(\frac{2}{r}\right) \frac{R''}{R} + \frac{R'}{R}.$$  \hfill (22)

Let the eigenvalues be $k$, where $k$ is nonnegative. We therefore define our constant equal to $-k^2$. If our constant were positive it would yield solutions that are exponential, which is not desirable in our case. Therefore,

$$\frac{T'}{T} = -k^2,$$  \hfill (23)

$$\left(\frac{2}{r}\right) \frac{R''}{R} + \frac{R'}{R} = -k^2,$$  \hfill (24)

or,

$$T' + k^2 T = 0,$$  \hfill (25)

$$R'' + \frac{2}{r} R' + k^2 R = 0.$$  \hfill (26)

Notice that (25) is an ODE yielding solutions in the form

$$T(t) = e^{(-k^2 t)}.$$  \hfill (27)

Equation (26) is not quite a Bessel equation of order zero because of the $k^2$. We can absorb $k^2$ by a change of variable. If we scale $r$ as $t = \lambda r$, where $\lambda$ is to be determined. Then $\frac{d^2}{dr^2} = \left(\frac{d}{dt}\right) \left(\frac{d}{dr}\right) \lambda \frac{d}{dr}$ and (26) becomes, after division by $\lambda$,

$$t R'' + R' + \frac{k^2}{\lambda^2} t R = 0.$$  \hfill (28)

Thus, $k^2$ can be absorbed by choosing $\lambda = k$. Notice (28) then becomes a Bessel equation of order zero with general solution $R(t) = AJ_0(t) + BY_0(t)$. Therefore, we have

$$R(t) = AJ_0(t) + BY_0(t) = AJ_0(kr) + BY_0(kr).$$  \hfill (29)
a general solution of (26). $J_0$ and $Y_0$ represent the Bessel functions of the first and second kind, respectively. Both are of order zero. So we have

$$R(r) = \begin{cases} AJ_0(kr) + BY_0(kr) & k \neq 0 \\ C + D \ln(r) & k = 0 \end{cases}$$

(30)

$$T(t) = \begin{cases} Ee^{-k^2t} & k \neq 0 \\ F & k = 0 \end{cases}$$

(31)

Since $Y_0(0)$ and $\ln(0)$ are unbounded and do not exist at the origin, respectively, these terms must be eliminated. Thus the coefficients of these terms, $G$ and $H$, are equal to zero. Plugging these solutions into equation (17) yields

$$U(r, t) = (C + D \ln(r))F + (AJ_0(kr) + BY_0(kr))e^{-k^2t}.$$  

(32)

Simplifying and absorbing constants leaves us with

$$U(r, t) = G + P J_0(kr)e^{-k^2t}.$$  

(33)

By our boundary condition (15), $R'(0) = 0$. Notice that the solution for $R$ when $k = 0$ $R'(0)$ does not exist, so we will only be considering when $k > 0$. We can now simplify equation (33):

$$U(r, t) = G + P J_0(kr)e^{-k^2t}.$$  

(34)

Applying the next boundary condition that $U(1, t) = 1$ leaves us with:

$$U(1, t) = 1 = G + P J_0(kc)e^{-k^2t}.$$  

(35)

From this we gather that $G = 1$ and $P J_0(k) = 0$. We choose to keep the $P$ term because we need the Bessel function and exponent term in our final answer. Also, choosing positive roots of the Bessel function to satisfy $k_n = z_n$ where $z_n$ is the $n^{th}$ root of the Bessel function $J_0$. We are left with the series:

$$U(r, t) = 1 + \sum_{n=1}^{\infty} P_n J_0(z_n r) e^{-z_n^2 t}.$$  

(36)

where the $P_n$’s are the series arbitrary constants with the appropriate weight function $w(r) = r$ and are equal to:

$$P_n = \frac{\langle 1, J_0(z_n r) \rangle}{\langle J_0(z_n r), J_0(z_n r) \rangle} = \frac{\int_{0}^{1} F(r) J_0(z_n r) r dr}{\int_{0}^{1} (J_0(z_n r))^2 r dr}$$

(37)

Simplified, $P_n$ is equal to

$$P_n = -\frac{2}{z_n J_1(z_n)}.$$  

(38)

The final solution to our heat equation solved on a sphere with all of our assumptions and initial conditions after plugging the $P_n$’s in is equal to:

$$U(r, t) = 1 - 2 \sum_{n=1}^{\infty} \frac{1}{z_n J_1(z_n)} J_0(z_n r) e^{-z_n^2 t}.$$  

(39)
This Equation will be used to model the gelatinization front in potatoes. By substituting in the non-dimensionalized variables we can solve for temperature at a certain radius within the potato.

3 Experimental

3.1 Experiment 1

This experiment was relatively simple, and designed to determine whether or not the red potatoes used in the experiment absorb water during gelatinization. This information is valuable because, if it can be proved that the potatoes do not swell during cooking, it can be assumed in the model. Please see Figure 2 for experimental set-up.

3.2 Procedure

1. Construct the potato-cooker in the following manner. Fill large aluminum tank about half-way with tap water. Attach heating element to the side of the tank. Place small bucket into larger tank and fill with tap water until the water level in the bucket is greater than the water level in the tank. Turn on the heater and heat until water temperature is to desired temperature (75 Celsius).

2. Record the initial masses and volumes of the potatoes. Water-displacement method was used to determine volumes.

3. Once the bath has reached a temperature of 75 Celsius put the potato samples into the water bath to cook. Once fully cooked, remove potato samples.

4. Record final data and compare to initial data.

3.3 Discussion

After comparing the initial and final data for the experiment, the conclusion could be made that the potatoes swelling was negligible. The specimen which swelled the most was the first sample. It had an initial volume of 81-mL and a final volume of 85-mL. This is a 5-mL increase in volume which is a volume increase of 4.7%. Please see Figure 3 for all experimental results.

3.4 Experiment 2

The goal of this experiment was to be able to qualify the gelatinization of a potato visually. The potato samples of similar properties will be cooked at the same temperature for different intervals of time. They will then be cross-sectioned and treated with iodine to visualize the process of gelatinization in a potato. Iodine is used because it attaches to gelatinized starch but not ungelatinized starch. If applied to a potato it should produce a sharp front where the gelatinized and ungelatinized starches meet.
3.5 Procedure

1. Construct the potato-cooker in the following manner. Fill large aluminum tank about half-way with tap water. Attach heating element to the side of the tank. Place small bucket into larger tank and fill with tap water until the water level in the bucket is greater than the water level in the tank. Turn on the heater and heat until water temperature is to desired temperature (75 Celsius).

2. Carve potatoes into spheres of the same radius.

3. Once the water in the bucket has heated to the desired temperature, place the potato spheres into the bucket. Remove the potatoes at the desired interval (10 minutes).

4. Slice the potato spheres through the circumference.

5. Put on latex gloves and wet end of finger with iodine. Rub the iodine onto the potato cross-sections. Wait a couple of minutes and rinse the potato with water.

6. Take pictures of the potato cross-sections for analyzation.

3.6 Discussion

As seen in figure 7, the gelatinization front can be viewed as the barrier between the part of the potato stained purple (gelatinized) with iodine and the part of the potato which remains ungelatinized. As time progressed the gelatinization front moved closer to the center of the potato. Matlab was used to analyze the pictures of the potato cross-sections and determine the degree of gelatinization. This data will be used to check our model once the $\alpha$ value for our red potatoes is established in the next experiment.

3.7 Experiment 3

In this experiment the $\alpha$ value will be determined by taking the temperature at the center of a potato sphere with respect to time as the potato cooks. If this data is then graphed and linearized, the result is a constant $\alpha$. $\alpha$ is used in our model as the coefficient of thermal diffusivity. A graph of just the experimental data before it was compared to the theoretical data for $\alpha$ determination can be found in figure 4.

3.8 Procedure

1. Set up the apparatus as in the two previous experiments.

2. Carve the potatoes into spheres of equal radii.

3. Drill a hole to the center of a potato sphere. Push isotherm into the center of the potato. Seal the hole with petroleum jelly.

4. Connect the isotherm to the thermometer.
5. Repeat 3 for every potato used.

6. Once the water has reached the desired temperature, place the potatoes into the bucket.

7. Record temperature at the center of the potato every on minute.

8. Cook potatoes in water until the isotherm reads above the temperature of gelatinization of the potatoes.

9. Remove the potatoes from the water.

10. Repeat if necessary.

3.9 Discussion

After the data was analyzed, a good alpha value was determined for our model. This alpha value was then plugged into the mathematical model and the output was close to the data taken from the experiment.

4 Analysis and Results

Using Maple, a study of the behavior of our equation was completed. Non-dimensional plots of both Temperature vs. Time and isotherms of our 63 degree gelatinization front were constructed. Please see figures 5 and 6 for the theoretical simulation of the temperature at the center and the gelatinization isotherm, respectively. Using maple it is possible to enter the initial water, potato, and gelatinization temperature, along with radius, $\alpha$, and time that the potato is cooking. Please see appendix for a detailed Maple code. We can clearly see the movement of the gelatinization front thought the potato as time increases in the Isotherm Figure. After analyzing the gelatinization front in the cut potatoes. It is easy to tell that the front moves in a more or less uniform fashion towards the center of the potato. In figure 7, the dark areas are the gelatinized regions, while the lighter areas are the ungelatinized regions. Using a matlab pixel counter, we were able to directly determine the ratio of black to white pixels, and from there determine the percentage of gelatinization in the potato. The matlab code used can be found in figure 8. Since the value of thermal diffusivity for the red potato varies greatly with temperature, and varies slightly with position within the potato, we felt that a good average value of $\alpha$ was needed. To determine this value, we first plotted the non-dimensionalized experimental data. Next, we ran a maple code to export data that corresponded to the experimental data that we had previously plotted. By adjusting the thermal diffusivity we were able to, in essence, achieve a best fit curve for our experimental data. The value of $\alpha$ that gave us this best fit curve has been said to be our ”average” thermal diffusivity and is equal to $0.027 \text{cm}^2/\text{s}$. The plot showing how our experimental and theoretical values compare for the center of the potato during cooking can be seen in figure 9. Using this thermal diffusivity value, we were able to go back to our derived equation. By knowing the gelatinization temperature, the thermal diffusivity, and the time cooked, we were able to back out the value for the radius of gelatinization. Using this radius we compared the circular area of the gelatinized to ungelatinized portion of the potato. These
values for the percentage of gelatinization in the potato were then compared to the actual percentage of gelatinization that was found through the matlab program. A graph showing how our expected gelatinization curve fits our experimentally observed gelatinization curve can be seen in figure 10.

5 Conclusion

It is clear that, as the mathematical model depicted, the gelatinization front moved through the potato in a more or less uniform manor. We can say that this model represented the physical phenomena that is happening inside the potato well, however, more variables need to be taken into account in order to more accurately represent the phenomena of starch gelatinization inside a potato. The reason that our theoretical curves do not perfectly match the experimental curve is because of error in our mathematical model. By incorporating water absorption into our model, we believe this will increase the amount of gelatinization per time, and therefore yield better accuracy of the model.

5.1 Theoretical and Experimental Strengths and Weaknesses

Some of the strengths and weaknesses to our mathematical model include the following:

5.2 Strengths

- A solid and reasonably accurate model was created for finding out the percent of gelatinization inside a potato during cooking as a function of time.

- We were able to take into consideration during the problem, $U_{water}$, $U_{potato}$, density, thermal conductivity, and specific heat.

- There is no free convection term, which, at high temperature, can greatly influence the conduction of heat into a submerged sphere. At high temperature, convection within a fluid is increased. This increased convection leads to increased heat transfer within the system. As heat transfer due to convection is increased, a mathematical approximation for this phenomena must be present.

5.3 Weaknesses

- The skin of the potato was removed, simplifying the heat transfer problem.

- There is no attempt to include water absorption into the mathematical model. Although from our experimental data, we can reasonably say that no water was absorbed during cooking, new volumetric and mass measurements may lead this project into a direction where water absorption plays an important role.

Some of the strengths and weaknesses in our experimental model include the following:
5.4 Strengths

- A low amount of water circulation was achieved, reducing the effect of forced convection due to the water heater during cooking.

- A good visual guide for the amount of starch gelatinization was employed.

5.5 Weaknesses

- It is not clear that the temperature of the water inside of our holding tank was exactly the same as the temperature of the water going through the heater.

- The iodine method for detecting starch gelatinization is not very precise, and a good way of quantifying the iodine method results is needed.

- Only one experimental trial was run, therefore giving us no insight as to the error involved in this process. More trials need to be undertaken to have an idea of reliability.

6 Future Work

For the upcoming people working on this project we wish to display a few points we feel would be a good starting point to build on our project.

- Find a better way to use the iodine method results, and run more experimental trials. Both of these will reduce error and increase reliability of our methods.

- At low temperatures a gelatinization football occurs. A good understanding of the basis for this football could lead to new insights into the gelatinization of starches.

- Meet with the food scientists and investigate incorporating lipids and emulsifiers into the mathematical model.

- Relax some of the assumptions that simplify the model to produce a more accurate model. The first relaxed assumption should be the water absorption assumption.

7 Acknowledgments

- Professor Rossi

- Lauren Rossi

- Math 512 students

- Food Science students
References


8 Appendix A: Maple Code

> restart:
> with(plots):

Warning, the name changecoords has been redefined <br>
> alpha:=.034103; c:=2.25;
t[water]:=(70-20)/(70-20); t[potato]:=(20-20)/(70-20);
t[gelat]:=(60-20)/(70-20);

\[ \alpha := 0.034103 \]
\[ c := 2.25 \]
\[ t_{\text{water}} := 1 \]
\[ t_{\text{potato}} := 0 \]
\[ t_{\text{gelat}} := \frac{4}{5} \]

> Ubar:=t[water]+2*(t[potato]-t[water])*sum(1/(BesselJZeros(0,n)*BesselJ(1,BesselJZeros(0,n)))*BesselJ(0,BesselJZeros(0,n)*rbar)*exp(-(BesselJZeros(0,n)^2*timebar)),n=1..infinity);

\[ Ubar := 1 - 2 \sum_{n=1}^{\infty} \frac{\text{BesselJ}(0,\text{BesselJZeros}(0,n)rbar)e^{-(\text{BesselJZeros}(0,n))^2\text{timebar}}}{\text{BesselJ}(1,\text{BesselJZeros}(0,n))\text{BesselJ}(0,\text{BesselJZeros}(0,n))} \]

> G:=0: <br>for i from 0 to 30 do
<br>ealvf(subs(rbar=0, timebar=G, Ubar)): <br>G:=G+1/30; <br>end do;
0.0
$G := 1/30$
0.0010732260
$G := 1/15$
0.0444663900
$G := 1/10$
0.1516448866
$G := 2/15$
0.2773494276
$G := 1/6$
0.3956104036
$G := 1/5$
0.4985131394
$G := \frac{7}{30}$
0.5853284300
$G := \frac{4}{15}$
0.6576307858
$G := 3/10$
0.7175129306
$G := 1/3$
0.7669888310
$G := \frac{11}{30}$
0.8078237420
$G := 2/5$
0.8415112266
$G := \frac{13}{30}$
0.8692966738
$G := \frac{7}{15}$
0.8922120766
$G := 1/2$
0.9111102839
$G := \frac{8}{15}$
0.9266952643
$G := \frac{17}{30}$
0.9395477944
$G := 3/5$
0.9501469085
$G := \frac{19}{30}$
0.9588876821
$G := 2/3$
0.9660959330
$G := \frac{7}{15}$
0.9720403573
$G := \frac{11}{15}$
0.9769425418
$G := \frac{23}{30}$
0.980852228
$G := 4/5$
0.9843190977
$G := \frac{5}{6}$
0.9870684418
$G := \frac{13}{15}$
0.9893357414
$G := \frac{3}{10}$
0.991205137
$G := \frac{9}{10}$
0.9927474575
$G := \frac{14}{15}$
0.9940190511
$G := \frac{15}{16}$
0.9950676953
$G := \frac{29}{30}$
0.9973357414
$G := 1$
0.9993357414
$G := \frac{31}{30}$

> plot(eval(Ubar, rbar=0), timebar=0..1,
> labeldirections=[HORIZONTAL, VERTICAL], labels=["Non-Dimensional
> Time","Non-Dimensional Temperature"], title="Temperature vs Time
> for Potato Center: r=0");
\[ P := t[\text{gelat}] = t[\text{water}] + 2(t[\text{potato}] - t[\text{water}]) \times \sum \frac{1}{BesselJZeros(0,n) \times BesselJ(1,BesselJZeros(0,n))} \times BesselJ(0,BesselJZeros(0,n) \times rbar) \times \exp(-BesselJZeros(0,n)^2 \times \text{timebar}), \quad n=1..\infty \]
Figure 1: Diagram representing Potato slice while it is being cooked.

Figure 2: Experimental Set-up Diagram

Figure 3: Experimental Data
Figure 4: Experimental Heat Increase in the Potato Center
Figure 5: Theoretical Temperature Change in Center of Potato with Time
Figure 6: Theoretical Isotherm for 63 Degrees Celsius

Figure 7: Gelatinization in potato for various times
Function

\[
\text{Function } [b] = \text{percentwhite}(a)
\]
\[
[iw, jw] = \text{find}(a(:, :, 1) > 128);
\]
\[
b = \text{length}(iw) / \text{prod(size}(a(:, :, 1)));\]

Program

\[
a = \text{imread(’imagename.jpg’)};
\]
\[
\text{image(a) } \% \text{(shows picture)}
\]
\[
\text{percentwhite(a)};
\]

Figure 8: Matlab Code used in Determining the Percentage of Gelatinization

Figure 9: The above theoretical data was manipulated to find the best fit curve, giving us the best fitting \( \alpha \) value for our experiment.
Figure 10: Theoretical and Experimental Gelatinization Curves.