

Altea's PassPort™ System: Electrical and Thermal Model

MATH 512

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Introduction:

The PassPort™ System (Figure 1) produced by Altea Therapeutics allows transdermal drug delivery by burning small holes in the outer layer of the skin. This will be accomplished by supplying a voltage to an array of filaments (Figure 2) to achieve a desired temperature profile. Our goal is to design a voltage profile that will be delivered to the array of stainless steel filaments such that the filaments achieve an optimal temperature profile without exceeding 700 degrees Fahrenheit in 2 milliseconds. An optimal temperature profile has not been provided by Altea and for this reason we must speculate as to what characteristics they would like in their “optimal” temperature profile. The maximum temperature was determined by the amount of energy required to vaporize the outer layer of skin without penetrating far enough to reach the nerves. We begin by researching Joule heating which governs the energy generation term in our mathematical model. We then formulate a simple mathematical model based on Joule heating and several forms of heat loss. Our mathematical model requires several variables that we must investigate. Some of these can be acquired from the data provided by Altea; however, the coefficient of convection, an important quantity related to the cooling of the filaments, must be found experimentally. Once a basic mathematical model is formulated, the first step is to scale the model by non-dimensionalizing the variables. This reveals the relative importance of the various heat loss terms. If any of the terms are shown to be insignificant, they can be eliminated from the model. Once the model is simplified, we solve it for particular scenarios and compare our analytical solution to experimental data. The difficulty of this project lies in the complexity of the mathematical model. To obtain an accurate solution the model must account for three different modes of heat transfer, one of which requires spatial variation. Furthermore, several variables in the model that are often assumed to be constants vary greatly during the process because of the wide range of temperatures encountered by the filaments. Our ability to successfully account for each of the modes of heat transfer and to obtain reasonably accurate values for certain variables will ultimately determine the qualitative utility of our model.

Background:



Figure 1: PassPort™ System

Altea Therapeutics is a private company based in Georgia that specializes in medical technology. It has designed the PassPort™ system, as a noninvasive drug delivery system for the short or long term delivery of peptides, proteins, small-molecule drugs, genes and vaccines. Before this system, the delivery of medicine through the skin was limited to lipid-soluble drugs with relatively small molecular weights. This restriction was due to the properties of the stratum corneum, the outmost layer of the skin. Since the stratum corneum is about a 15 μ m waterproof layer of dead skin, it keeps out harmful bacteria as well as beneficial drugs. The PassPort™ system was designed to allow a greater variety of drugs to be delivered through the skin.

The PassPort™ system works for a wide range of drugs through poration of the stratum corneum. A hand-held reusable battery-driven activator heats an array of filaments, which in turn creates micropores of the skin by vaporizing microscopic amounts of dead cells from the stratum corneum. A patch containing medicine is then placed over the micropores, which are deep enough to allow the entry of the drugs into the skin, but shallow enough that the pain receptors in the dermis, the layer of skin below the stratum corneum, are unaffected. So the whole method is painless.

Since the filaments are measured in micrometers, the production of the array must be precise. First, a copper rectangle is electroplated onto a stainless steel #304 rectangle of the same size. Then some of the copper is etched away, so a zigzagging “river” of stainless steel is left in the middle of the copper and steel-layered rectangle. The copper spokes that remain are called traces. Then a triple YAG laser is used to cut out some pieces of the stainless steel in the "river" so that only the stainless steel filaments remain.

The circuitry of the PassPort™ system involves 80 stainless steel filaments (Figure 2) in parallel, a voltage source with a resistance of about 4 ohms, and a 10 Farad capacitor. The copper traces act as the anode and cathode of this system and are connected by the filaments, which close the circuit.

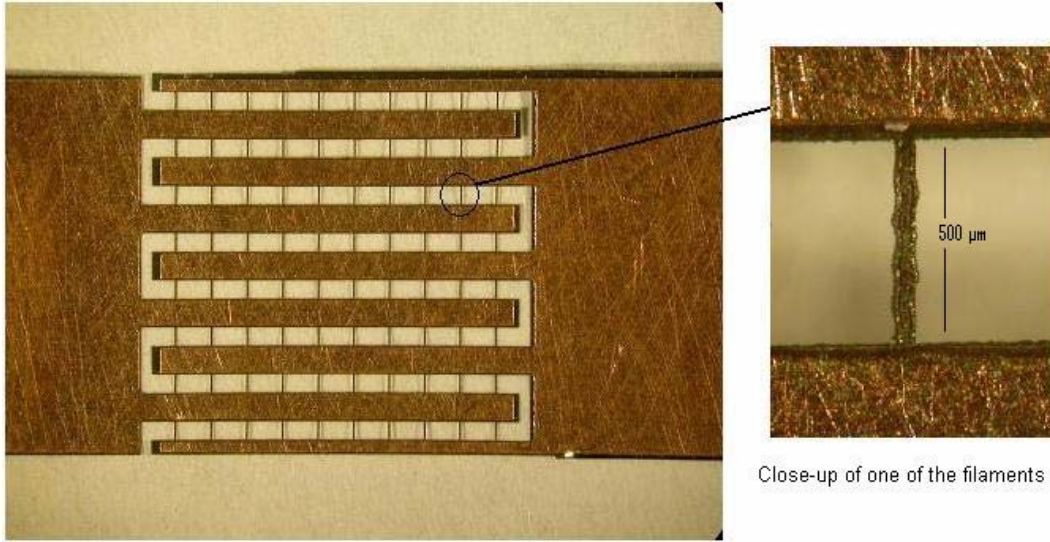


Figure 2: The array of filaments on the PassPort™ system

The 80 filaments, each with dimensions of 500μm by 50μm by 15μm, are made out of stainless steel and are held in place by copper traces. The array is heated for 2 milliseconds with a 1 second cooling time and has a maximum current of 1000 amps.

More information can be found on Altea's website at <http://www.alteatherapeutics.com>.

Developing the Model:

We are developing a mathematical model that represents the electrical and thermal properties of Altea's PassPort™ system. Since electrical conductivity, $\sigma(T)$, which depends on resistance, $R(T)$, is included in both the electrical and thermal models, we arrive at a coupled system.

In the following coupled system, the temperature with respect to time will be our focus. For initial analysis and simplicity, we will assume no spatial dependence.

$$V(t') = I(t')R(T) \quad (1)$$

$$mc_p \frac{dT}{dt'} = I(t')^2 R(T) - (q_{conv} + q_{rad} + q_{cond}) \quad (2)$$

Note that Equation (1) is Ohm's Law where $V(t')$ is voltage, $I(t')$ is current, and $R(T)$ is resistance, and Equation (2) is based on energy conservation and energy transfer per unit time. Therefore the units for each term in Equation (2) shall be energy per time or Joules/second.

Since $R(T) = \frac{L}{\sigma(T)A}$ where L is the length of the material and A is the cross-sectional area of the material, we can rewrite Equation (1) as

$$V(t') = \frac{I(t')L}{\sigma(T)A}. \quad (3)$$

Since Equation (3) is easy to manipulate, we can solve for current and substitute it into Equation (2). This results in the following:

$$mc_p \frac{dT}{dt'} = \frac{V(t')^2 \sigma(T)A}{L} - (q_{conv} + q_{rad} + q_{cond}). \quad (4)$$

Going from left to right, this model says that the energy storage with respect to time in the system is equal to the heating of the system due to the Joule heating phenomenon minus the energy losses of the system. These losses include energy losses due to natural convection to the surrounding air, radiation losses to other objects, and finally conductive losses to the copper anode and cathode. These losses will be discussed in more detail below.

The left hand side of Equation (4)

$$mc_p \frac{dT}{dt'} \quad (5)$$

is the energy storage term in the system, where m is mass, c_p is specific heat, and $\frac{dT}{dt'}$ is the temperature differential with respect to time.

The first term on the right hand side of Equation (4)

$$\frac{V(t')^2 \sigma(T)A}{L} \quad (6)$$

is the only energy generation term present in our model. It models the heat input per unit time as a function of time and temperature. This heat is generated by the phenomenon of Joule heating and is governed by Joule's Law,

$$Q = Pt = I^2 Rt = I^2 \frac{L}{\sigma A} t. \quad (7)$$

Joule's Law describes the amount of heat, Q, generated when a current, I, flows through a conductor of length, L, cross sectional area, A, and conductivity, σ , for a time, t. The data provided by Altea indicates that resistance is an exponential growth function;

however, for simplicity, we approximate resistance, and therefore conductivity, to have a linear relationship with temperature.

The next three terms in Equation (4) model the various modes in which heat is dissipated from the filament. The first heat loss term governs the heat that is dissipated due to natural convection to the surrounding air. The equation is as follows:

$$q_{conv} = hA_s(T_s - T_\infty) \quad (8)$$

where h is the coefficient of convection, T_s is the temperature of the surface of the filament, A_s is the surface area, and T_∞ is the temperature of the ambient air.

The next heat dissipation term models the energy lost due to radiation to the surroundings,

$$q_{rad} = \varepsilon\sigma_{SB}A_s(T_s^4 - T_{sur}^4). \quad (9)$$

In this case ε is the emissivity of the material, A_s is the surface area, σ_{SB} is Stefan-Boltzmann's constant ($\sigma_{SB} = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$), T_s is the temperature of the surface of the filament, and T_{sur} is the temperature of the surrounding objects which can be approximated by the temperature of the ambient air. Note that the Stefan-Boltzmann's constant is usually denoted by σ , but in order to avoid confusion with conductivity, $\sigma(T)$, we have adjusted the notation to σ_{SB} .

The final heat dissipation term takes into account the conduction of heat from the ends of the filaments to the copper electrodes. This term has not been fully explored and is expected to be the most difficult heat loss term to model mathematically because it requires spatial dependencies. In addition, the energy losses modeled by this term are expected to be significant because of the large temperature gradient between the copper electrodes and the steel filaments and because of the massive heat capacity of the copper electrodes relative to the filaments. This was confirmed visually in our experiment with metal blocks and nichrome wire. It was clear that the wire temperature close to the blocks was significantly lower than the middle of the wire.

To account for spatial variation we can model conduction using

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + \frac{|\nabla \psi|^2}{r(T)} \quad (10)$$

where ρ is density, ψ is electric potential and $r(T)$ is resistivity. Note that the heat losses are not including in the PDE because they can be incorporated into the boundary conditions. Because of time restrictions, we will not incorporate this into our mathematical model.

Scaling the Model:

The purpose of scaling a model is to identify the relative importance of various effects. The first step is to non-dimensionalize our variables in order to reduce parameter space.

$$t = \frac{t'}{t_p} \quad (11)$$

$$U = \frac{T - T_\infty}{T_m - T_\infty} \quad (12)$$

The scaling variable for time, t_p , is the time elapsed during a voltage pulse. The scaling variables for temperature are T_m , the maximum temperature, and T_∞ , the temperature of the ambient air. We can now rewrite the functions in our model using the non-dimensional variables.

$$\sigma(T) = \sigma_o f\left(\frac{T - T_\infty}{T_m - T_\infty}\right) \quad (13)$$

$$V(t')^2 = V_m^2 g\left(\frac{t'}{t_p}\right) \quad (14)$$

σ_o is the conductivity of the material at room temperature, so $\sigma(T_\infty) = \sigma_o$ and V_m is the maximum voltage reached in the pulse.

Now that we have non-dimensionalized the variables, we can plug them back into a simplified version of Equation (4). In this simplification we will ignore cooling effects from conduction. The simplified scaled model is as follows:

$$U_t = Pf(U)g(t) - BU - B_r \left[\left(U + \left(\frac{T_\infty}{T_m - T_\infty} \right) \right)^4 - \left(\frac{T_\infty}{T_m - T_\infty} \right)^4 \right] \quad (15)$$

$$P = \frac{t_p V_m^2}{mc_p (T_m - T_\infty) R_o} \quad (16)$$

$$B = \frac{hA_s t_p}{mc_p} \quad (17)$$

$$B_r = \frac{\varepsilon \sigma_{SB} A_s t_p (T_m - T_\infty)^3}{mc_p} \quad (18)$$

An initial condition is required to solve this equation,

$$T(0) = T_{\infty} \quad (19)$$

$$U(0) = 0. \quad (20)$$

To show the relevance of radiation in comparison to convection, the radiation term is scaled so that it is in a non-dimensionalized form (Equation (18)). The emissivity of a body is a property that measures how efficiently it radiates its energy. This value can range between 0 and 1 and is strongly dependent on the surface material and finish. For most stainless steels in the appropriate temperature range, the values of emissivity range from about .17 to .28 [4]. When comparing B_r to the B of convection, we see that their ratio shows us their relative significance. The ratio of B to B_r is as follows:

$$h : \varepsilon \sigma_{SB} (T_m - T_{\infty})^3. \quad (21)$$

Using approximations for each of the terms, we are able determine that convection is about 100 times more important than radiation. For this reason we will neglect radiation in our mathematical model.

Solving the Model:

We reduce Equation (15) to ignore radiation, which allows us to arrive at an exact solution. The model is shown below.

$$U_t = Pf(U)g(t) - BU \quad (22)$$

As mentioned before, we have assumed for simplicity that resistance, and therefore conductivity, varies linearly versus temperature, so

$$\sigma(T) = aT + b \quad (23)$$

where a and b are constants that depend on the material. Substituting U in for T and by noting that $\sigma(T_{\infty}) = \sigma_o = aT_{\infty} + b$, we get

$$\sigma(T) = \sigma_o \frac{a[(T_m - T_{\infty})U + T_{\infty}] + b}{\sigma_o} = \sigma_o \left[\frac{a(T_m - T_{\infty})}{aT_{\infty} + b} U + 1 \right]. \quad (24)$$

Hence

$$f(U) = CU + 1 \quad (25)$$

where

$$C = \frac{a(T_m - T_\infty)}{aT_\infty + b}. \quad (26)$$

The linearity of $f(U)$ allows an exact solution to Equation (22).

$$U = \frac{P \int_0^t g(z) \cdot e^{\int^{B-PCg(t)} dz}}{e^{\int^{B-PCg(t)} dt}}. \quad (27)$$

The strength of this model is that it can provide us with an exact solution. Its weaknesses are that it lumps the effects of conduction and radiation into the convection term and assumes that conductivity is linear. If we directly include the conduction and radiation terms in our model or use a different equation for conductivity, we will have to solve it numerically because of its non-linearity.

Using Equation (27), our goal is to optimize the voltage function $g(t)$. To begin we will choose an optimum temperature profile based on the information that Altea has provided. Possible choices for a temperature profile include a plateau at 700° F, a peak at 700° F, and a profile that minimizes rise time. Once we choose our optimal temperature profile $T(t)$, we can compare it to the temperature profile $U(t)$ given from Equation (27) when different voltage pulses are used. Our goal is to minimize the difference between these two temperature profiles. We can quantify the difference with the following equation:

$$error = \int_0^t (U(t) - T(t))^2 dt. \quad (28)$$

By taking the derivative of Equation (28) with respect to P and again with respect to B , we can set them equal to zero. If we can solve for B and P , we can find the maximum voltage and the time pulse that minimizes the error.

Solutions with Specific Voltage Profiles:

Now that we have a general solution to our simplified model, we can prescribe a specific voltage profile to get a specific solution. Then we can compare this theoretical solution with experimental data to test the validity of our model.

The simplest voltage profile is a constant voltage. Since the voltage is scaled, we get the Heaviside function

$$g(t) = H(1-t) \quad (29)$$

where $g(t) = 1$ for $0 \leq t \leq 1$ and $g(t) = 0$ for $t > 1$.

We substitute Equation (29) into the general solution and get

$$U(t) = \frac{P}{B - PC} (1 - e^{-(B-PC)t}) \quad (30)$$

$$U(t) = \left(\frac{\frac{P}{B - PC} (1 - e^{-(B-PC)t})}{e^{-B}} \right) e^{-Bt} . \quad (31)$$

Note that Equation (30) holds for $0 \leq t \leq 1$ and Equation (31) holds for $t > 1$.

The graph of the solution depends on the relationship between B and PC. If $B > PC$, then the solution for $0 \leq t \leq 1$ has an exponential decay term. If $B < PC$, then the solution for $0 \leq t \leq 1$ had an exponential growth term. A general graph of the solution can be seen below for both cases. Note these graphs illustrate the heating during constant voltage and the cooling when the voltage is removed.

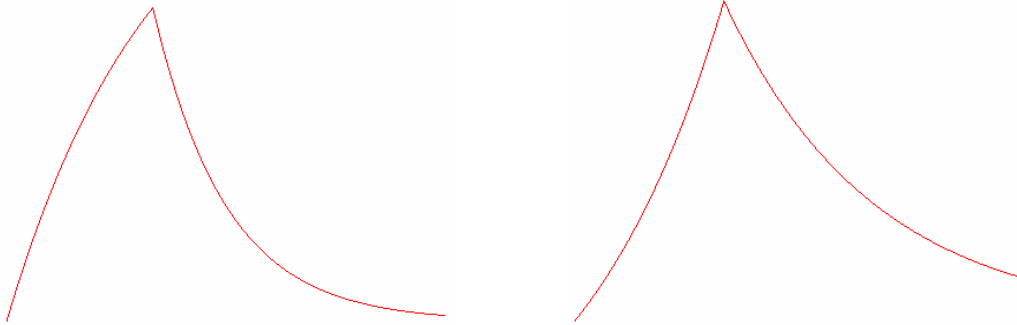


Figure 3a: $B > PC$

Figure 3b: $B < PC$

Figure 3: Temperature Profile Based on B/ PC Relationship

Figure 3a shows the temperature profile when B is greater than PC. In this case, the B term, which corresponds to the cooling effect, dominates the heating parameter, PC. As time approaches infinity, with a constant voltage, the heating and cooling of the system reach equilibrium resulting in a constant temperature. This relationship between B and PC forces the solution to have a downward concavity during its heating phase. Figure 3b shows the temperature profile when B is less than PC. Here the heating of the system dominates the cooling, resulting in the thermal runaway phenomenon. During this phenomenon, as time approaches infinity, with constant voltage, the temperature of the system exponentially increases. This causes an upward concavity. Physically, this phenomenon will result in the melting of the material.

We must also examine when $B = PC$. Again by examining Equation (27), we can see that the solution is

$$U(t) = Pt. \quad (32)$$

In this case, the slope of the temperature profile will be constant as seen by the linear solution.

Since we have a specific solution for constant voltage, we can minimize Equation (28). Unfortunately, we cannot solve for P and B, and, thus, we cannot find the optimal values for maximum voltage and time pulse. A contour plot and a surface plot of P versus B can be seen in the appendix. The contour plot shows the values of P and B that correspond to an error of .5. With these values for P and B, we can back out the values for maximum voltage and time pulse to have an error of .5.

Now that we are able to arrive at a solution to our mathematical model we would like to validate this solution with experimental data. To do this we performed an experiment (Experiment 4) for comparison. We used the values from the experiment (temperature, voltage, etc.) to plug into our analytical solution for comparison. The following graph shows the data collected in the experiment and the solution that our mathematical model provided. The procedure that was used to make these graphs can be seen in the appendix.

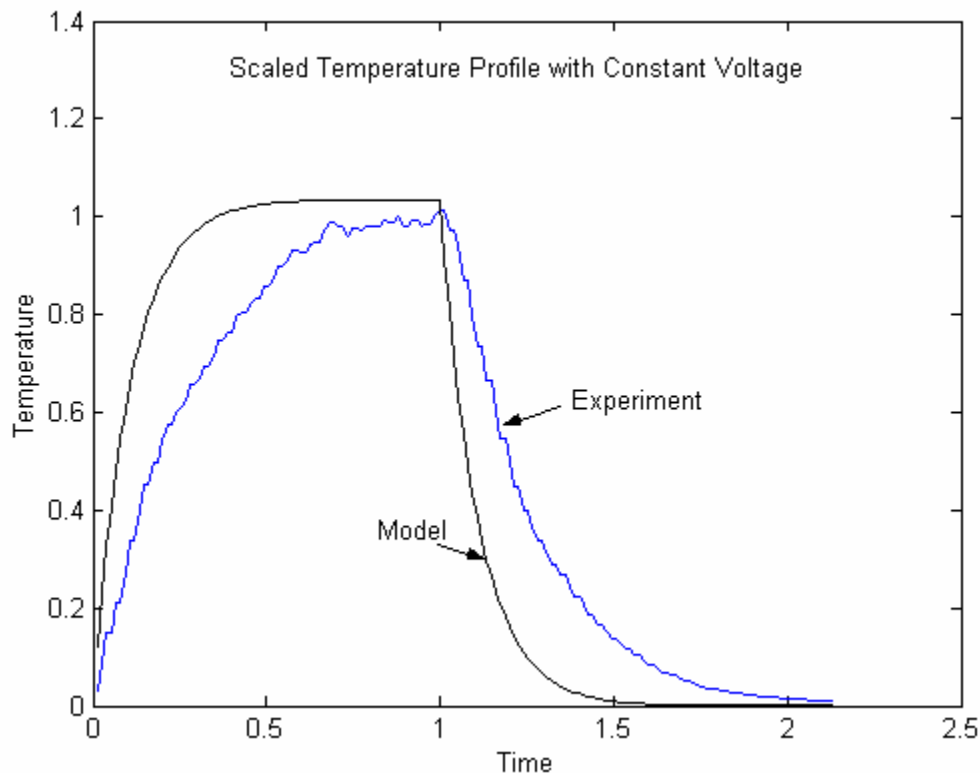


Figure 4: Experimental Data and Analytical Solution

Quantitatively, our mathematical model does not exactly predict values that are obtained experimentally. However, qualitatively the two graphs are quite similar. They both appear to exhibit exponential behavior while the temperature is increasing. Also, both graphs exhibit exponential decay after the voltage is removed and the wire is allowed to cool to room temperature.

Experiments:

This section focuses on the experiments we did in the lab. Experiments 1-3 are primarily concerned with finding the coefficient of convection used in our mathematical model. The results obtained from Experiment 4 will be used to compare with the solution given analytically by our mathematical model. Comparing our experimental data to our analytical solution will be the basis for determining the value of our mathematical model.

Experiment 1:

The goal of the first experiment is to obtain an approximate value for the coefficient of convection that is used in Equation (8). To validate the value of h that we obtain from the experiment, we will compare it to a value of h calculated theoretically. To simplify our experiment, we will disregard radiation and conduction along the wire. The Joule heating term can also be eliminated from the equation because in the experiment we will not begin measuring the temperature change of the wire until after the voltage is stopped. Using these assumptions, our model is simplified to the following:

$$mc_p \frac{dT}{dt} = -hA_s(T - T_\infty) \quad (33)$$

$$T(0) = T_{\max} . \quad (34)$$

Using the temperature scale from Equation (12), and the time scale

$$t = \frac{t' h A_s}{mc_p} \quad (35)$$

we get the following scaled solution:

$$U = e^{-t} . \quad (36)$$

Then we take the log of Equation (36) to get

$$\ln(U) = -t = -\frac{t' h A_s}{mc_p} . \quad (37)$$

We use this equation with experimental data to find h .

To setup the first experiment we first attach the ends of the wire to two aluminum blocks that serve as the anode and cathode of the electrical circuit. The wire is made of nichrome, is .042 meters in length, and has a radius of 4.56 E^{-4} meters. We have approximated the conductivity of nichrome to be $\sigma(T) = -112.956T + 958,467$ where conductivity is in $\frac{1}{\Omega m}$ and temperature is in Kelvin. (For details of the approximation, see the appendix). Voltage is applied to the wire elevating its temperature to a set value. The voltage is then removed and the wire is allowed to cool. Measurements are taken of the temperature of the wire in the middle of the wire versus time after the voltage is removed. We scale the temperature measurements and take the log of them. Then we find the slope of the resulting line using least squares approximation, so we can solve for h. An example of this graph can be found in the appendix. This experiment is performed twice with different values of the maximum temperature of the wire. When the maximum temperature reaches 307.5 K, the resulting h is 95.5 watts per meter squared Kelvin. When the maximum temperature reaches 321.3 K, the resulting h is 78.75.

This experiment is believed to have several weaknesses that could affect the reliability of the data. The anode and cathode in this experiment (the aluminum blocks) have an extremely large heat capacity relative to the wire. For this reason they act as heat sinks and draw heat from the wire. Our simplified model does not take this into account explicitly, thereby giving us inaccurately large values of h. This is because the heat lost by conduction to the electrodes is lumped into the value of h because there is not a conduction term to account for it.

Experiment 2:

The goal of this experiment is to determine a value for the coefficient of convection that is more accurate than that obtained in the first experiment. We accomplish this by reducing the relative heat capacity of the electrodes such that conduction does not play such an integral role in the experiment. In this experiment, we use a piece of nichrome wire of length .082 meters. The procedure is the same as in the first experiment except that the aluminum blocks (electrodes) are substituted with small clips that send the current through the wire. These clips absorb much less heat than the larger aluminum blocks. When the maximum temperature of the wire is 406.6 K, the resulting h is 50 watts per meter squared Kelvin, and when the maximum temperature reaches 414.1 K, the resulting h is 63. Note that the lower values for h are predictable since there is less heat loss in general.

It is clear that conduction was reduced in this experiment but we cannot be certain that it is now negligible.

Experiment 3:

The goal of this experiment is to calculate h and be certain that conduction is playing a negligible role. To minimize the conduction term, and isolate convective heat loss, we minimize the size of the electrodes and maximize the length of the wire, thus reducing the

relative heat capacity of the electrodes. Replacing the aluminum blocks with metal clips in experiment 2 reduces the size of the electrodes. With the smaller electrodes there exists a length of wire such that conduction to the electrodes is relatively small and can be neglected. To be sure that we have effectively isolated convection, we heat a wire up to a designated temperature and perform the same calculations as in the previous experiments to determine h . We then increase the length of the wire (reducing the relative heat capacity of the electrodes) and repeat the experiment. The value of h is compared to the previous trial. If the values from each trial are different, it can be concluded that conduction is still a significant factor. We repeat this experiment, each time increasing the length of the wire, until the coefficient of convection for trial (call it trial n) is negligibly different from the preceding trial ($n-1$).

$$h_n - h_{n-1} \approx 0 \quad (38)$$

Once this condition is satisfied we can say that conduction is negligible and that the only significant heat lost is due to convection. Using the same procedure as above, we use the data gathered from each trial to calculate the value of the coefficient of convection for use in our mathematical model. The results of this experiment are as follows:

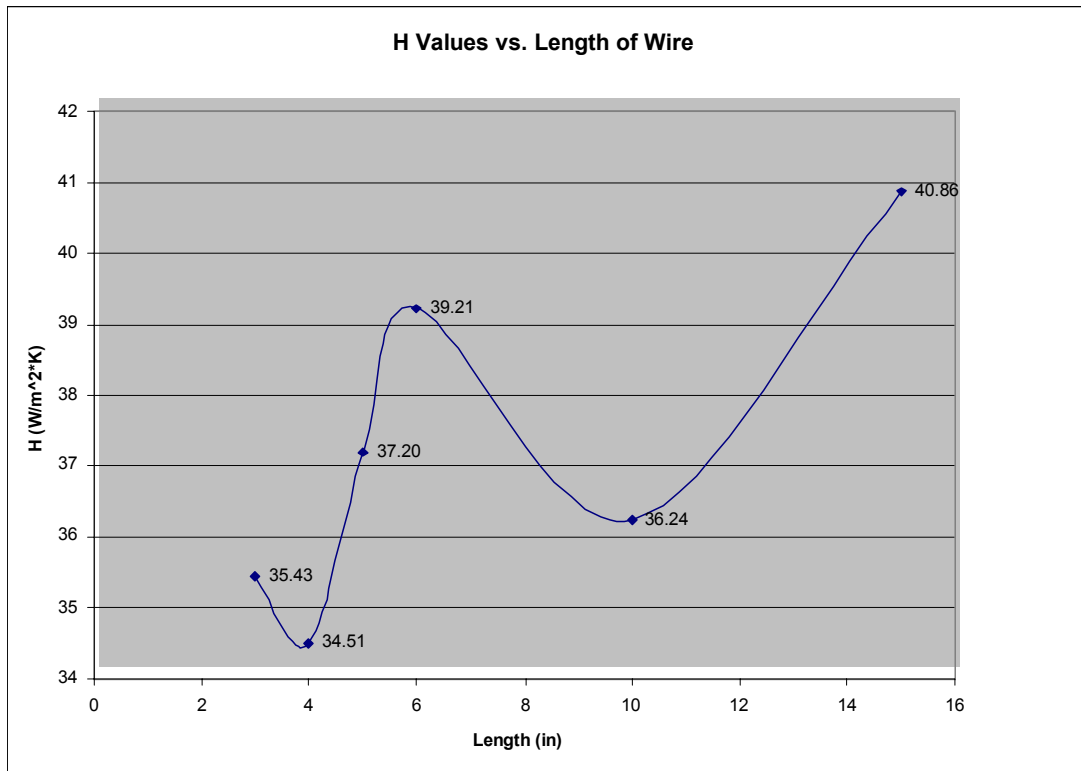


Figure 5: H Values vs. Length of Wire

From this graph of our data, we see no obvious correlation between the length of the wire and the value of h ; however, we did a linear regression of the points and got that

$h = .38 * L + 34.49$ with a correlation coefficient of .72. The correlation coefficient is a measure of how well the points fit into a line with 1 being a perfect fit and 0 being no relationship. Thus, there is a pretty good relationship between h and length. Also, the h values in experiment 3 are lower than those obtained in experiment 2 in all cases including similar length wires in both experiments. The values of h for experiment 2 and 3 with the same length wire were expected to be close. One possible source of error in this experiment is the thermocouple used to measure the temperature of the wire. The data recorded from this thermocouple oscillated by plus or minus 10 degrees even when the temperature of the wire was constant. This might be because one side of the thermocouple is exposed to the ambient air while the other side is in contact with the wire. To correct this, we will coat the thermocouple with thermally conductive paste to obtain more accurate readings.

Experiment 4:

The goal of this experiment is to obtain temperature data during constant voltage heating and no voltage cooling to compare to the analytical solution of our model. This experiment was run two times with a different voltage each time. To obtain better values, thermal paste was used at the contact area between the thermocouple and the wire. In this experiment, a four inch wire is heated as the voltage and temperature are recorded. The first iteration is performed with a voltage of 1.2 V and the second iteration with a voltage of .61 V. Then the voltage is turned off and temperature readings are continued to be taken until the wire has cooled to room temperature plus or minus one degree. The data is then scaled and graphed. The second iteration can be seen in Figure 4.

Conclusion:

The goal of this project was to design a voltage profile that will be delivered to the array of stainless steel filaments such that the filaments achieve an optimal temperature profile. To do this we attempted to develop a mathematical model that can accurately predict the temperature profile given certain conditions; however, our model does not directly take into account spatial variation, radiation, or the dependence of material properties on temperature. The constant voltage solution to our model qualitatively resembles the data acquired experimentally. Further analysis must be done to find the optimal maximum voltage and time pulse. We recommend that in the future Altea completes this analysis and also refines our model to include spatial variance, radiation, and material properties that depend on temperature.

Annotated Bibliography:

The following books and journals provide relevant information for our project. Their summaries can be found below.

- 1) John A. Pelesko and David H. Bernstein, *Modeling MEMS and NEMS* (2003), Chapman & Hall/CRC, Boca Raton.

This book derives a system of equations governing the electrostatic and thermal problems encountered during the joule heating of a cylinder. First, it uses Maxwell's equations and Ohm's law to derive an equation for the electrostatic potential. This is then combined with boundary conditions to form a system of equations representing the electrostatic problem. Next, it derives the heat equation as it applies to isotropic, homogeneous materials, which is what we will be working with. After a source term is added to the heat equation, it is also combined with boundary conditions to develop a system of equations representing the thermal problem. Since both systems depend on electric conductivity, they are combined into one system to be solved simultaneously. In addition, an in depth analysis of this system using scaled variables is done to obtain fewer non-dimensional parameters.

- 2) Yasunori Asano, Takaaki Nishi, and Jun Yanagimoto, *Continuous Heating System Using Electric Resistance Heating for the Hot Rolling of Stainless Steel*. *ISIJ International*, 42 (2002), pp.1112-1118.

This article addresses that the electrical resistance of metals increases according to the elevation in temperature. Furthermore, it confirms our assumption that joule heating provides uniformly distributed cross-sectional temperature in the filament. This validates our supposition that the thermal model is only a function of time and distance along the filament.

- 3) Vincent P. Coletta, *College Physics* (1995), Mosby-Year Book, Inc., St. Louis.

This introductory college physics textbook provides us with some common equations to govern electrical and thermal conductivity. Specifically, it covered Ohm's Law and the equation relating electrical resistivity to electrical resistance. It also discusses the linear relationship between resistivity and temperature over certain ranges of temperature.

- 4) Frank P. Incropera and David P. DeWitt, *Introduction to Heat Transfer* 4th Edition (2002), John Wiley and Sons, Inc., New York.

This book provides us with information relevant to all of the modes of heat transfer occurring in our problem. Conduction occurs throughout the steel filament as well as from the ends of the filaments to the copper fingers.

Convection occurs from the surface of the filaments to the surrounding air. Heat is also radiated from the surface of the filaments when they are heated. This book supplies equations that govern each of these modes of heat transfer as well as examples that closely resemble particular aspects of our problem.

5) A.A. Lacey, Thermal Runaway in a Non-Local Problem Modeling Ohmic Heating. I. Model Derivation and Some Special Cases, *European J. Appl. Math.*, 6 (1995), pp. 127-144.

A.A. Lacey, Thermal Runaway in a Non-Local Problem Modeling Ohmic Heating. II. General Proof of Blow-Up and Asymptotics of Runaway, *European J. Appl. Math.*, 6 (1995), pp. 201-204.

The above two articles describe the non-local mathematical model of the joule heating of a cylinder. This model governs the temperature of a material, with temperature dependent electrical conductivity, when an electrical current is passed through it with a fixed voltage. These articles further discuss the situations in which a solution can be found for the model.

Appendix

Minimizing the Error Function in Maple

A sample definition for $T(t)$

```
> restart:
```

```
> T1:=1:
```

```
> T2:=exp(1-t):
```

We use the values of C and total time from experiment 4 to plot contour plots of P vs. B

```
> C:=-.0089:
```

```
> totaltime:=160:
```

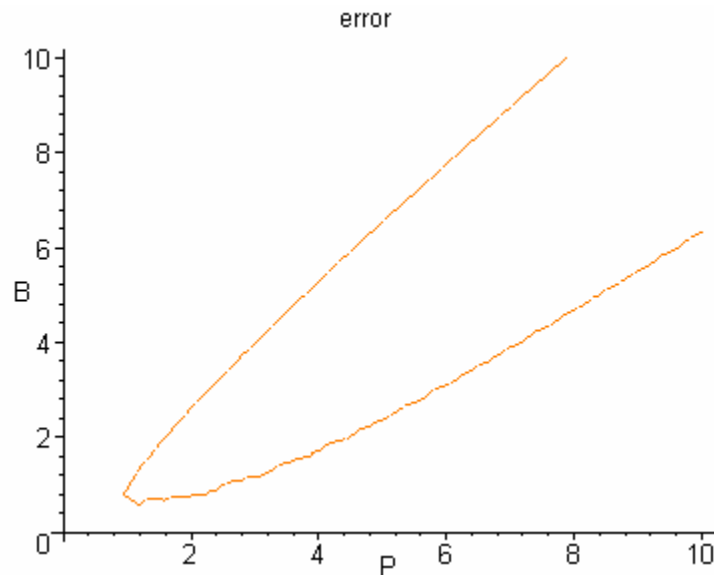
```
> E1:=((P/(B-P*C))*(1-exp(-(B-P*C)*t))-T1)^2:
```

```
> E2:=((P/(B-P*C))*(1-exp(-(B-P*C)*t)))/exp(-B)*exp(-B*t)-T2)^2:
```

```
> E:=int(E1,t=0..1)+int(E2,t=1..totaltime):
```

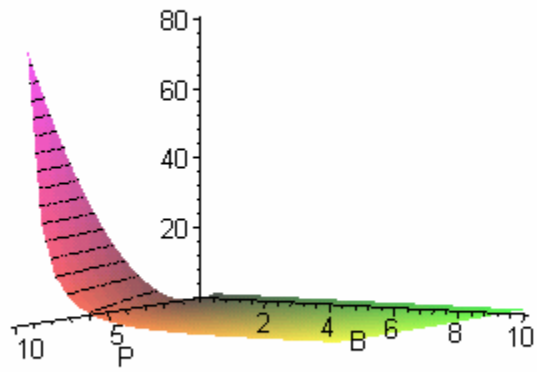
```
> with(plots):
```

```
> contourplot(E,P=0..10, B=0..10,title='error',contours=[0,1/2]);
```



The contour where error equals 0 is not appearing in the graph, so we could not find values of P and B that eliminates error completely. So we tried to find the smallest value for the error that would have a graph. The contour where the error equals $1/2$ shows the values for P and B where the error is $.5$. Using these values for P and B , we can back out values for maximum voltage and time pulse that results in an error of $.5$.

```
> plot3d(E,P=0..10, B=0..10,title='error',axes=normal);  
error
```



Comparison of an Experiment with Constant Voltage to the Solution of our Model using Matlab

```

Uexp=[299.4,303.7,308.1,308.1,312.7,312.7,317.2,322,322,326.4,330.1,330.1,333.6,333.
6,336.7,339.1,339.1,341,341.6,342.6,345,345,345.9,347.7,347.7,349.3,351.5,351.5,352.9,
352.9,355.1,355.9,355.9,356.9,357.8,357.8,359.5,359.5,360.8,362.6,362.6,363.6,365.1,36
5.1,364.6,364.6,366,366.1,366.1,368,369.2,369.2,368.6,368.6,367.1,368.2,368.2,368.1,36
8.7,368.7,368.6,368.6,369.6,369.1,369.1,370.2,368.7,368.7,369.4,369.4,368.5,368.8,368.
8,370.5,370.9,370.9,368.1,368.1,365.5,360.4,360.4,355.2,350.8,350.8,345.8,345.8,341.2,
337,337,333.7,329.9,329.9,326.4,326.4,323.7,321.8,321.8,320,318.3,318.3,316.7,316.7,3
14.9,313.6,313.6,312.4,310.7,310.7,309.4,309.4,308.4,307.4,307.4,306.7,305.7,305.7,304
.9,304.9,304.1,303.5,303.5,302.8,302.3,302.3,301.8,301.8,301.4,301,301,300.6,300.2,300
.2,299.9,299.9,299.7,299.5,299.5,299.3,299.2,299.2,299,299,298.8,298.7,298.7,298.6,298
.5,298.5,298.4,298.4,298.3,298.3,298.3,298.2,298.1,298.1,298.1,298.1,298,298];
Texp=[];
for i =1:160
Texp=[Texp,i];
end
Uexpscaled=(Uexp-297.3)/(370-297.3);
Texpscaled=Texp./75;
plot(Texpscaled,Uexpscaled)
Tmodel1=[];
for i = 1:75
Tmodel1=[Tmodel1,i];
end
Tmodel1scaled=Tmodel1./75;
Umodel= 1.035428824-1.035428824*exp(-9.410498120.*Tmodel1scaled);
%The values from above were found by finding the values for P, B, and C and
%using the solution of the model for when the voltage is turned on.
%We also set h, the coefficient of convection, to 100.
hold on
plot(Tmodel1scaled,Umodel,'black')
Tmodel2=[];
for i = 75:160
Tmodel2=[Tmodel2,i];
end
Tmodel2scaled=Tmodel2./75;
Umodel2= 11598.58981*exp(-9.323905001.*Tmodel2scaled);
%The values from above were found by finding the values for P, B, and C
%using the solution for when the voltage is turned off
plot(Tmodel2scaled,Umodel2,'black')
%The graph can be seen in the Solutions with Specific Voltage Profiles section.

```

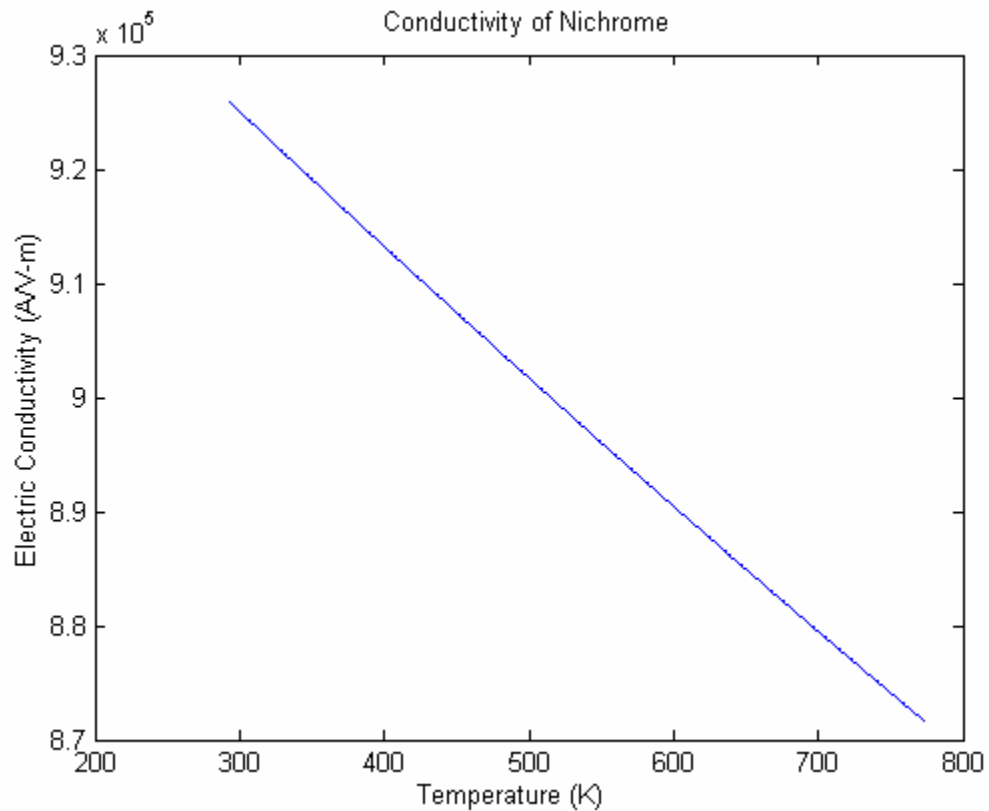
Approximation of the Conductivity of Nichrome using Matlab

```
t=[];  
for i = 1:481  
t=[t;292+i];  
end  
r = 1.400000E-10*t + 1.038980E-06;  
c=1./r;  
polyfit(t,c,1)
```

ans =

1.0e+005 *

-0.00112955632262 9.58467645316957



Least Squares Approximation of Experimental Data

