Mathematical modeling and analysis of a continuum model for three-zone swarming behavior

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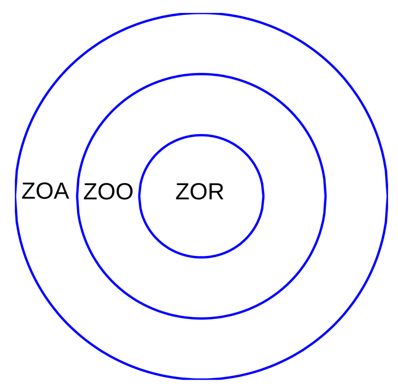
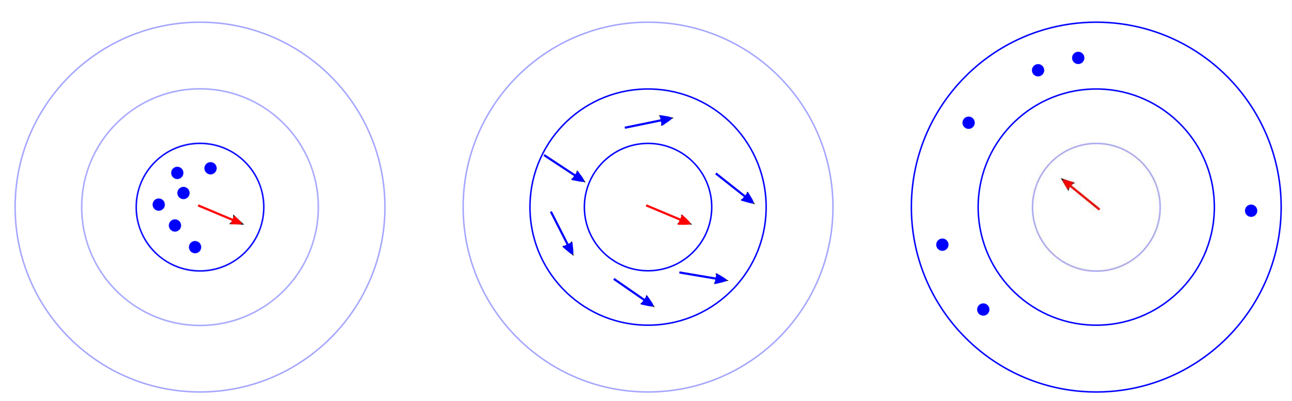
Basics for modeling swarms

A swarm is an aggregation of individuals which may be moving together as a group or a stationary group in which the individuals are moving. Depending upon individual behavioral tendencies, swarms can exhibit a variety of dynamics. The central goal of this project is to mathematically connect individual behavior and emergent swarm structures. One biologically inspired set of behaviors is the the three-zone model [1, 2, 3]:

- Avoid others who are very close,
- Orient with individuals nearby, and
- Stay with the group.



These three zones are called the zone of repulsion (ZOR), zone of orientation (ZOO), and zone of attraction (ZOA). Depending on the relative sizes of these zones, dramatically different swarm behavior has been observed [1].



Continuum modeling for swarms and a first-order model

In nature, a swarm is a discrete system rather than a continuum. When modeling with individuals in an agent-based model, the number of interactions and calculations increases rapidly as the size of the swarm increases. Instead of an agent-based model, we consider two continuum models, which are valid for swarms consisting of large numbers of individuals.

In a continuum model, we replace individuals by a continuous density $\rho(\vec{x}, t)$ and individual velocities by a velocity field $\vec{v}(\vec{x}, t)$, and we use the continuity equation $\rho_t + \nabla \cdot (\rho\vec{v}) = 0$ to conserve the mass of the swarm [4]. The velocity is the sum of responses determined by the convolution of ρ or $\rho\vec{v}$ with social interaction kernels. For example, we propose a first-order model of the form

$$\vec{v} = \underbrace{H_{\sigma_1} \star \rho}_{\text{repulsion}} + f_1(\rho)\left(\underbrace{\frac{G_{\sigma_2} \star (\rho\vec{v})}{G_{\sigma_2} \star \rho}}_{\text{orientation}} + \omega_a \underbrace{K_{\sigma_3} \star \rho}_{\text{attraction}}\right).$$

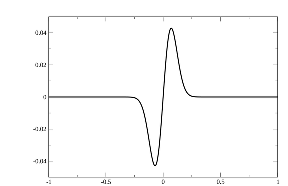
For this model, the speed varies depending on the strength of attraction and repulsion. The orientation response has been normalized with respect to the density.

The function $f_1(\rho)$ has value one when ρ is small in the zone of repulsion and zero when the ρ is large there. This gives priority to repulsion by turning off the orientation and attraction terms. The parameter ω_a allows us to weight attraction in comparison to repulsion.

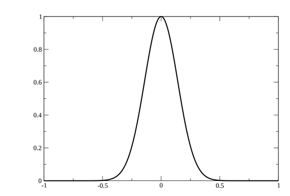
The social interaction kernels

The kernels in our model are:

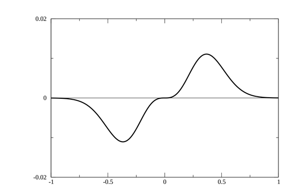
repulsion: $H_{\sigma_1} = \dfrac{\vec{x}}{8\pi\sigma_1^4} e^{-|\vec{x}|^2/4\sigma_1^2}$



orientation: $G_{\sigma_2} = \dfrac{1}{4\pi\sigma_2^2} e^{-|\vec{x}|/4\sigma_2^2}$



attraction: $K_{\sigma_3} = \dfrac{\vec{x}|\vec{x}|^2}{64\pi\sigma_3^6} e^{-|\vec{x}|/4\sigma_3^2}$



We have normalized the orientation kernel so that a uniform swarm moving at a constant velocity continues to move with the same velocity, and the repulsion and attraction kernels have been normalized to make convolution with a linear density $\rho = ax + b$ result in a response of size a in the appropriate direction. In this case, the zones of interaction have some overlap.

A second order, constant speed model

Our second-order model uses the same continuity equation, but now we consider the change in the angle $\theta(\vec{x}, t)$ of the velocity vector $\vec{v} = \langle \cos\theta, \sin\theta \rangle$. In this case, the speed is constant.

$$\theta_t + (\vec{v} \cdot \nabla)\theta = \langle -\sin\theta, \cos\theta \rangle \cdot \left[H_{\sigma_1} \star \rho + f_1(\rho)\left(\frac{G_{\sigma_2} \star (\rho\vec{v})}{G_{\sigma_2} \star \rho} + \omega_a K_{\sigma_3} \star \rho\right)\right]$$

$$\rho_t + \nabla \cdot (\rho\vec{v}) = 0$$

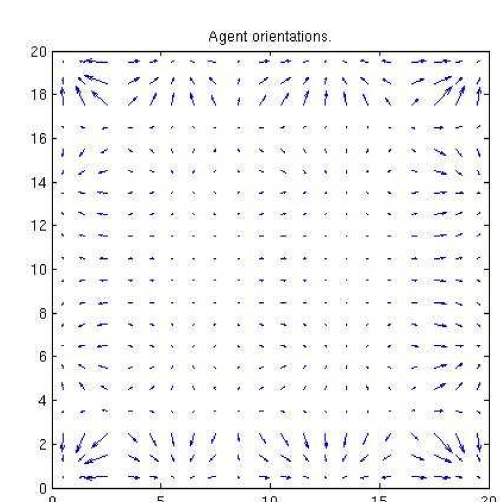
The blue terms highlight the differences from the first-order model. The left hand side of the interaction equation is the material derivative of θ. On the right hand side, we have a vector perpendicular to the velocity dotted with the interaction response from the first-order model, which indicates how much θ needs to change to reach the desired velocity determined by the interactions.

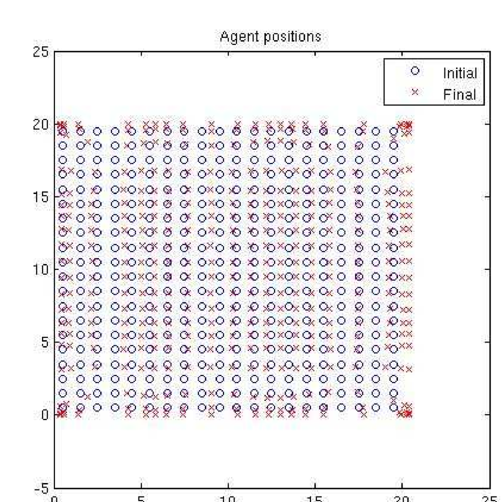
Linear stability analysis

We use linear stability analysis to determine the robustness of an infinite uniform swarm. We perturb the constant density ρ_0 by a plane wave $\epsilon\rho^{(1)} = \epsilon A e^{i(\vec{\xi}\cdot\vec{x} - \omega t)}$ and assume that this results in a perturbation $\epsilon\vec{v}^{(1)}$ of the constant velocity $\vec{v}_0$. Matching terms that are $O(\epsilon)$ yields the dispersion relation

$$\omega = \vec{v}_0 \cdot \vec{\xi} + i\rho_0|\xi|^2 \frac{\left(\exp(-\sigma_1^2|\vec{\xi}|^2) + \frac{\omega_a}{2}(-2 + \sigma_3^2|\vec{\xi}|^2)\exp(-\sigma_3^2|\vec{\xi}|^2)\right)}{1 - \exp(-\sigma_2^2|\vec{\xi}|^2)}.$$

When the imaginary part of ω is positive, the disturbances from the plane wave will grow. If repulsion is weighted more heavily than attraction ($\omega_a < 1$), the imaginary part of ω is non-negative for all ξ. If attraction is weighted more heavily than repulsion ($\omega_a \geq 1$), the imaginary part of ω will be negative for small wavenumbers, i.e. small $|\xi|$, and positive for larger wavenumbers. The figure below shows the perturbed positions and velocities in the discretized model for $\omega_a = 0.1$ after 100 Euler time steps.





Conclusions and future work

We have first- and second-order models for swarms based on the rules of repulsion, orientation, and attraction. Linear stability analysis of the first-order model shows that if attraction is weighted less than repulsion, all wavenumbers are unstable and the system is ill-posed. If attraction is weighted more heavily than repulsion, there is an interval of stable wavenumbers. We would expect uniform swarms to be stable.

Future work will include connecting our continuum models with agent-based simulations. The continuum model analysis considers infinite uniform swarms and the discretized version of this model considers infinite periodic swarms. We want to be able to compare the dynamics of continuum swarms to an agent-based simulation with finite swarms and eventually use the continuum model to predict the dynamics. We are also currently seeking nontrivial coherent structures other than uniform swarms.

References

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[3] C. Topaz and A. Bertozzi. Swarming patterns in a two-dimensional kinematic model for biological groups. *SIAM Journal of Applied Mathematics*, 2004.

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