

A remark on unique continuation along and across lower dimensional planes for the wave equation

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Abstract

We prove unique continuation of solutions of the wave equation along and across lower dimensional planes containing the t axis. This is a sharpening and a generalization of a result of Cheng, Ding and Yamamoto as well as a simplification of the proof.

Keywords Unique continuation; wave equation.

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1 Introduction

Unique continuation of solutions of hyperbolic PDEs in domains in R^{l+1} , with coefficients independent of t , across l dimensional surfaces have been studied by John, Romanov, Hormander, Robbiano, Tataru, Zuily and others - see [5], [7], [9], [13], [11]. Unique continuation along or across lower dimensional analytic manifolds for operators with analytic coefficients has been studied in [1] and [2] and the results may be summarized loosely as saying that there is unique continuation along and across all lower dimensional non-characteristic manifolds. In the non-analytic category, Lebeau in [10] showed that if $u(x, t)$ is a solution of

$$\partial_t^2 u - \sum_{i,j} \partial_j (a_{ij} \partial_i u) = 0$$

in a neighborhood of $\{p\} \times [0, T]$ with (a_{ij}) positive definite and dependent only on x , and u and all its x, t derivatives vanishing on $\{p\} \times [0, T]$ then u is zero in a neighborhood of $\{p\} \times [0, T]$. The proof required the symmetry of the differential operator and there seems no obvious extension of the proof to the asymmetric (first order perturbation) case. In [3] and [4], Cheng et al showed that unique continuation holds, for solutions of the wave equation, in (along) two dimensional planes containing the t axis.

We prove unique continuation results along and across lower dimensional subspaces for special operators. These follow from some simple observations and an application of unique continuation results across hypersurfaces.

Suppose l, m, n are non-negative integers with $l = m + n$ and $n > 0$. Points $x \in R^l$ will also be written as $x = (y, z)$ with $y \in R^m$ and $z \in R^n$. Define the operator

$$P := \sum_{ij=1}^m a_{ij}(y) \partial_{y_i} \partial_{y_j} + \sum_{i=1}^m b_i(y) \partial_{y_i} + c(y)$$

where $(a_{ij}(y))$ is uniformly positive definite on compact subsets. Define a distance d between points in R^{m+n} associated with the Riemannian metric associated with the elliptic operator $P + \Delta_z$. Since the coefficients of P are independent of z one can show that the shortest curve joining (y, z) to (y', z') has components consisting of the shortest curves joining y to y' and z to z' . For any positive integer k , $B_k(r)$ will denote the origin centered open ball of radius r in R^k , $S_k(r)$ will denote its boundary, and $|p|$ will denote the distance of p from the origin - all this corresponding to a distance function made clear from the context. We also define $B_0(r)$ to consist of just a point.

We prove the following results - the first is a unique continuation result *along* a subspace and the second a unique continuation result *across* a subspace.

Theorem 1 *Suppose l, m, n are positive integers with $l = m + n$, ρ, T and ϵ (small) are positive numbers, $a_{ij}(y), b_i(y), c(y)$ are C^∞ and $u(y, z, t)$ is a C^∞ solution of*

$$u_{tt} - Pu - \Delta_z u = 0, \quad (y, z, t) \in B_l(\rho) \times (-T, T) \tag{1}$$

so that $u(y, 0, t) = 0$ on $B_m(\epsilon) \times (-T, T)$. Then $u(y, 0, t)$ is zero on

$$\{(y, t) \in R^m \times R : |y| + |t| < T, |y| < \rho\}.$$

One cannot expect unique continuation across $z = 0$ in Theorem 1 because we have assumed only $u = 0$ on $z = 0$ and not assumed anything about the value of z derivatives of u on

$z = 0$. This lack of unique continuation across $z = 0$ in Theorem 1 may be clearly seen in solutions of the wave equation which are odd in z . If we also assume that the z derivatives of u are zero on $z = 0$ then we obtain unique continuation across $z = 0$ as shown in Theorem 2. Note that if u is a solution of (1), then knowing u and the first order z derivatives of u on $z = 0$ is not enough to determine the higher order z derivatives of u on $z = 0$ if $n > 1$.

Theorem 2 *Suppose l, m, n are non-negative integers with $n > 0$ and $l = m + n$, ρ, T and ϵ (small) are positive numbers, $a_{ij}(y), b_i(y), c(y)$ are C^∞ and $u(y, z, t)$ is a C^∞ solution of*

$$u_{tt} - Pu - \Delta_z u = 0, \quad (y, z, t) \in B_l(\rho) \times (-T, T) \quad (2)$$

so that $(\partial_z^\alpha u)(y, 0, t) = 0$ on $B_m(\epsilon) \times (-T, T)$ for all multi-indices α . Then $u(y, z, t)$ is zero on

$$\{(y, z, t) \in R^{m+n} \times R : |(y, z)| + |t| < T, |(y, z)| < \rho\}.$$

Theorem 1 is a generalization and a sharpening of the results in [3], [4] with a simplification of the proof. If the operator P is symmetric then the hypothesis can be weakened - instead of requiring that $u(y, 0, t)$ be zero on $B_m(\epsilon) \times (-T, T)$, one needs only that all the y derivatives of u are zero on $\{0\} \times [0, T]$ (see the proof of Theorem 1). We conjecture that Theorem 1 generalizes to the case when Δ_z is replaced by a second order elliptic operator in z with coefficients depending only on z . If P is symmetric then Theorem 2 is just a special case of the much stronger result of Lebeau in [10]. However, Lebeau's result does not cover the case when P is not symmetric so Theorem 2 is a new result, but we conjecture that Lebeau's result generalizes to the asymmetric case.

Theorem 1 is valid if $u(y, z, t)$ is continuous, instead of being C^∞ , as shown below. Suppose $\chi(t)$ is a smooth function supported in a neighborhood of the origin. Define $u'(x, t) = (\chi * u)(x, t)$; then u' also satisfies the hypothesis of Theorem 1 except with a slightly smaller T . Now $WF(u')$ is contained in $\tau^2 = |\xi|^2$ and on the region $|\tau|/2 \leq |\xi| \leq 2\tau$, $\hat{u}'(\xi, \tau) = \hat{\chi}(\tau) \hat{u}(\xi, \tau)$, decays faster than any polynomial in ξ, τ (localization in (x, t) is also needed); so u' is C^∞ . If the theorem is valid for smooth solutions then u' is zero on the appropriate set, which implies u is zero on the appropriate set because χ was arbitrary.

2 Proof of Theorem 1

Since we can add a single real variable to z with u independent of this real variable, there is no loss of generality in assuming that n is an odd integer (to shorten the proof).

For any real number r with $(y, r) \in B_{m+1}(\rho)$, define the mean value function

$$v(y, r, t) = \frac{1}{|S_n(1)|} \int_{S_n(1)} u(y, r\theta, t) d\theta;$$

$v(y, r, t)$ is a smooth function which is even in r . Then from the fact that the mean values satisfy Darboux's equation (see [5]), we have that $v(y, r, t)$ satisfies

$$v_{tt} - Pv - v_{rr} - \frac{n-1}{r}v_r = 0, \quad (y, r) \in B_{m+1}(\rho), \quad |t| < T$$

and $v(y, 0, t) = 0$ on $B_m(\epsilon) \times (-T, T)$. Also, since v is even in r , we have $v_r(y, 0, t) = 0$ on $B_m(\rho) \times (-T, T)$. Theorem 1 will follow from the following proposition because $v(y, 0, t) = u(y, 0, t)$.

Proposition 1 *Suppose m, n are non-negative integers, $n > 0$ and odd, ρ, T and ϵ (small) are positive numbers, $a(y)$ and $b(y)$ are smooth and $v(y, r, t)$ is a smooth, even in r , solution of*

$$v_{tt} - Pv - v_{rr} - \frac{n-1}{r}v_r = 0, \quad (y, r, t) \in B_{m+1}(\rho) \times (-T, T) \quad (3)$$

so that $v(y, 0, t)$ is zero on $B_m(\epsilon) \times (-T, T)$. Then $v(y, r, t)$ is zero on

$$\Omega_{\rho, T} = \{(y, r, t) \in R^{m+1} \times R : |(y, r)| + |t| < T, |(y, r)| < \rho\}.$$

The proposition is true for even n too but requires a transform slightly different from the one used for the odd n case.

Proof of Proposition 1

When $n = 1$ the proposition follows quickly from the unique continuation results for the wave operator in [13], as is shown below. When $n > 1$ the proposition would also follow from [13] except for the singularity $(n-1)/r$ in the coefficient of ∂_r . However, there is a well known transform, in the study of spherical mean values of functions, for reducing the $n > 1$ equation to the $n = 1$ case.

($n = 1$ case)

This case is handled by a well known, but rarely written, short and simple argument which we have included for the reader's convenience. This also follows from a more general result in [8] - see Theorem 3.16 there. (We have not addressed the possible lack of smoothness of the function $|x|$ - see [8] for how this is handled). If $m = 0$ then (3) is the one dimensional wave equation and the result is standard. So assume $m > 0$. The proposition will follow from a unique continuation result in [13] which asserts the following : suppose v is an H^1 solution

of a homogeneous hyperbolic pde of second order with coefficients which are independent of t , the second order coefficients are C^1 and the lower order coefficients are bounded; then v has unique continuation across every non-characteristic surface. If $\rho < T$ then noting that

$$\Omega_{\rho,T} = \cup_{\tau \in (-T+\rho, T-\rho)} \{(y, r, t) \in R^{m+1} \times R : |(y, r)| + |t - \tau| < \rho\}$$

and that Proposition 1 is valid under a translation in t , there is no loss of generality in assuming that $\rho \geq T$ in which case $\Omega_{\rho,T}$ is independent of ρ and ρ plays no role.

In the region $B_{m+1}(\epsilon) \times (-T, T)$, v satisfies (3), we are given $v(y, 0, t) = 0$ and $v_r(y, 0, t) = 0$ because $v(y, r, t)$ is even in r . Hence we can redefine v to be zero on the region $r < 0$ in $B_{m+1}(\epsilon) \times (-T, T)$ and v will still be a solution of (3) in $B_{m+1}(\epsilon) \times (-T, T)$. Then using the non-characteristic surface $|y|^2 + |r + \epsilon/3|^2 = \epsilon^2/9$ in (r, y, t) space, from the unique continuation result there is neighborhood of the t axis in which v is zero. This can be done on both sides of $r = 0$ so $v = 0$ in a neighborhood of the t axis (with $|t| < T$) for the original v and not just the redefined v .

For any $\delta \geq 0$ define the region

$$G_{\delta,T} = \{(y, r, t) \in R^{m+1} \times R : |(y, r)| + \sqrt{t^2 + \delta} < T\}$$

and we have to show that $v = 0$ on $G_{\delta,T}$. Note that the set $G_{\delta,T}$ increases in size as δ decreases and $G_{0,T}$ is the union of the $G_{\delta,T}$ as δ varies over positive reals. Also, for $\delta > 0$, the boundary of $G_{\delta,T}$ is non-characteristic; the boundary is singular at $(y, r) = 0$ but that will not be a problem since $v = 0$ is zero in a neighborhood of the t axis. Let δ_0 be the infimum of all the δ 's > 0 for which $v = 0$ on $G_{\delta,T}$; the existence of a $\delta_0 < T$ is assured because $v = 0$ in a neighborhood of the t axis. If $\delta_0 > 0$ then $v = 0$ on $G_{\delta_0,T}$; since the boundary of $G_{\delta_0,T}$ is non-characteristic, from unique continuation, we have $v = 0$ in a neighborhood of the closure of $G_{\delta_0,T}$. Since this closure is compact, there is a $0 < \delta_1 < \delta_0$ such that $v = 0$ in $G_{\delta_1,T}$ which contradicts the definition of δ_0 . Hence $\delta_0 = 0$.

($n > 1$ case)

We show how this may be reduced to the $n = 1$ case. We have shown that it is enough to assume n odd; see pages 700-704 of [5] for the appropriate modifications for the even n case.

For even functions $\phi(r)$ and for any integer k , define the operators D, I, L_k by

$$(D\phi)(r) = \frac{1}{2r} \frac{d\phi}{dr}, \quad (I\phi)(r) = 2 \int_0^r s\phi(s) ds, \quad L_k\phi = \phi_{rr} + \frac{k-1}{r}\phi_r.$$

Note that D, I, L_k preserve even functions, and we have $D \circ I = \text{identity}$, and $L_k D = D L_{k-2}$. Further, using induction and a reversal of the order of integration, we can show that for any

positive integer p ,

$$(I^p \phi)(r) = \frac{2}{(p-1)!} \int_0^r s(r^2 - s^2)^{p-1} \phi(s) ds = \frac{2}{(p-1)!} r^{2p} \int_0^1 \sigma(1 - \sigma^2)^{p-1} \phi(\sigma r) d\sigma. \quad (4)$$

If $n > 1$ then $n = 2p + 1$ for some positive integer p . Define

$$w(y, r, t) = (I^p v)(y, r, t)$$

and note that $v = D^p w$. Further, since $n - 2p = 1$, we have

$$\begin{aligned} 0 &= v_{tt} - \Delta_y v - L_n v + a(y) \cdot \nabla_y v + b(y)v \\ &= \partial_t^2 D^p w - \Delta_y D^p w - L_n D^p w + a(y) \cdot \nabla_y D^p w + b(y) D^p w \\ &= D^p w_{tt} - D^p \Delta_y w - D^p L_1 w + D^p (a(y) \cdot \nabla_y w) + D^p (b(y)w) \\ &= D^p (w_{tt} - \Delta_y w - w_{rr} + a(y) \cdot \nabla_y w + b(y)w). \end{aligned}$$

Since w is even in r and D corresponds to differentiation with respect to r^2 , we have

$$(w_{tt} - \Delta_y w - w_{rr} + a \cdot \nabla_y w + bw)(y, r, t) = \sum_{i=0}^{p-1} \alpha_i(y, t) r^{2i}, \quad (y, r) \in B_{m+1}(\rho) \times (-T, T). \quad (5)$$

Let $W = \partial_r^{2p} w$; differentiating (5) respect to r to order $2p$ and noting that a, b are independent of r , we have

$$W_{tt} - \Delta_y W - W_{rr} + a \cdot \nabla_y W + bW = 0, \quad (y, r) \in B_{m+1}(\rho) \times (-T, T). \quad (6)$$

Now W is even in r and from (4) we have

$$W(y, 0, t) = (\partial_r^{2p} I^p v)_{r=0} = cv(y, 0, t)$$

for some constant c , implying $W(y, 0, t) = 0$ on $B_m(\epsilon) \times (-T, T)$. Hence from the proposition in the $n = 1$ case we conclude that $W(y, r, t)$ is zero on $\Omega_{\rho, T}$.

Since $w = I^p v$, from (4) we have $(\partial_r^i w)(y, 0, t) = 0$ on $B_m(\rho) \times (-T, T)$ for $i = 1, \dots, 2p-1$; further $\partial_r^{2p} w(y, r, t) = W(y, r, t) = 0$ on $\Omega_{\rho, T}$. Hence $w(y, r, t) = 0$ on Ω_T . Since $w(y, r, t) = (I^p v)(y, r, t)$ and from (4) the kernel of I^p is non-negative, we may conclude that $v(y, r, t) = 0$ on $\Omega_{\rho, T}$.

3 Proof of Theorem 2

Again we have assumed that n is odd. There is an orthonormal basis of $L^2(S_n(1))$ consisting of homogeneous harmonic polynomials - see [12]. Let $\phi(\theta)$ be one element of this basis with

k being the degree of homogeneity of ϕ ; define $v(y, r, t) = r^{-k} f(y, r, t)$ where

$$f(y, r, t) = \int_{S_n(1)} u(y, r\theta, t) \phi(\theta) d\theta.$$

Since $\phi(\theta)$ is orthogonal to all polynomials of degree less than k , one may observe that $f(y, r, t)$ and all its r derivatives of order less than k are zero at $r = 0$. Hence $v(y, r, t)$ is smooth and even in r . Further, for $(y, t) \in B_m(\epsilon) \times (-T, T)$, we have

$$v(y, 0, t) = \frac{1}{k!} \partial_r^k f(y, r, t)|_{r=0} = \frac{1}{k!} \int_{S_n(1)} ((\theta \cdot \nabla_z)^k u)(y, 0, t) \phi(\theta) d\theta = 0$$

because of the hypothesis of Theorem 2; hence v satisfies one part of the hypothesis of Proposition 1.

If Δ_S is the Laplace-Beltrami operator on $S_n(1)$ then

$$\Delta_z = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_S$$

and because $\phi(\cdot)$ is harmonic and homogeneous of degree k , one may show that $\Delta_S \phi = -k(n+k-2)\phi$ on $S_n(1)$. Hence (see page 1235 of [6])

$$\begin{aligned} r^k(v_{rr} + \frac{n+2k-1}{r}v_r) &= (r^k v)_{rr} + \frac{n-1}{r}(r^k v)_r - \frac{k(n+k-2)}{r^2}(r^k v) \\ &= f_{rr} + \frac{n-1}{r}f_r - \frac{k(n+k-2)}{r^2}f \\ &= \int_{S_n(1)} (u_{rr} + \frac{n-1}{r}u_r)(y, r\theta, t)\phi(\theta) d\theta - \frac{k(k+n-2)}{r^2} \int_{S_n(1)} u(y, r\theta, t)\phi(\theta) d\theta \\ &= \int_{S_n(1)} (u_{rr} + \frac{n-1}{r}u_r)(y, r\theta, t)\phi(\theta) d\theta + \frac{1}{r^2} \int_{S_n(1)} u(y, r\theta, t)\Delta_S \phi(\theta) d\theta \\ &= \int_{S_n(1)} (u_{rr} + \frac{n-1}{r}u_r)(y, r\theta, t)\phi(\theta) d\theta + \frac{1}{r^2} \int_{S_n(1)} (\Delta_S u)(y, r\theta, t)\phi(\theta) d\theta \\ &= \int_{S_n(1)} (\Delta_z u)(y, r\theta, t)\phi(\theta) d\theta. \end{aligned}$$

Hence, from (2), $v(y, r, t)$ satisfies (3) in Proposition 1, except with n replaced by $n+2k$.

So from Proposition 1, we have $v(y, r, t) = 0$ on $\Omega_{\rho, T}$, that is

$$\int_{S_n(1)} u(y, r\theta, t)\phi(\theta) d\theta = 0$$

for all $(y, r, t) \in \Omega_{\rho, T}$ for all basis elements $\phi(\cdot)$. This implies $u(y, z, t)$ is zero on

$$\{(y, z, t) \in R^{m+n} \times R : |(y, z)| + |t| < T, |(y, z)| < \rho\}.$$

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