

NUMERICAL SOLUTION OF A TIMELIKE CAUCHY PROBLEM FOR THE WAVE  
EQUATION

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## Abstract

### NUMERICAL SOLUTION OF A TIMELIKE CAUCHY PROBLEM FOR THE WAVE EQUATION

Let  $D \subset R^n$  be a bounded domain with piecewise smooth boundary, and  $q(x, t)$  a smooth function on  $D \times [0, T]$ . Consider the time-like Cauchy Problem

$$u_{tt} - \Delta_x u + q(x, t)u = 0 \quad \text{in } D \times [0, T]$$

$$u = g, \quad u_n = h \quad \text{on } \partial D \times [0, T]$$

Given  $g, h$  for which the equation has a solution, we show how to approximate  $u(x, t)$  by solving a well posed fourth order elliptic pde. We use the method of quasi-reversibility to construct the approximating PDE. We derive error estimates and present numerical results.

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# 1 Introduction

Let  $D \subset R^n$  be a bounded domain with piecewise smooth boundary, and  $q(x, t)$  a function on  $D \times [0, T]$ . Consider the Cauchy problem

$$Lu \equiv u_{tt} - \Delta_x u + q(x, t)u = 0 \quad \text{in } D \times [0, T] \quad (1)$$

$$u = g, \quad u_n = h \quad \text{on } \partial D \times [0, T] \quad (2)$$

It was shown in [1] (also see [5]) that the above equation need not have a solution for arbitrary  $g$  and  $h$ . In particular, they showed that if  $q$  was zero, and a part of the boundary of  $D$  was planar, then the value of  $u$  and  $u_n$  on some part of this planar boundary in  $(x, t)$  space, determined the value of  $u$  on other parts of this planar boundary. So if the above problem is to have a solution then  $g$  and  $h$  can not be prescribed arbitrarily. Hence (1), (2) is not a well posed problem. We will say more about the uniqueness question later in this section.

Our goal is, given  $g$  and  $h$  for which (1), (2), does have a solution, to approximate  $u$  stably in the interior. This problem is of interest because of situations where only boundary measurements may be possible and the initial state of the system may be unknown. Further, Klibanov and Malinsky in [7] have proposed a numerical scheme to solve an inverse problem related to (1), in which the most difficult step is the construction of solutions of (1), (2).

If  $D$  were unbounded, or if Cauchy data was prescribed only on "one side" of  $\partial D$ , then we have no hope of constructing a stable algorithm. In this case, the problem is not stable because of exponentially growing solutions. e.g. if  $n = 2$  and  $D$  is the half plane  $y > 0$ , then consider the equation

$$\square u \equiv u_{tt} - u_{xx} - u_{yy} = 0 \quad \text{on } x \in R, \quad y > 0, \quad 0 < t < T$$

$$u(x, 0, t) = \frac{1}{k^2} e^{(ix\sqrt{1+k^2}+it)}, \quad u_y(x, 0, t) = \frac{1}{k} e^{(ix\sqrt{1+k^2}+it)}, \quad \text{if } 0 < t < T, \quad x \in R$$

Its solution is

$$u_k(x, y, t) = \frac{1}{k^2} e^{(ix\sqrt{1+k^2}+it+ky)}$$

In any neighbourhood of a point  $(a, b, c)$  with  $b > 0$ ,

$$\|u_k\|_{0,loc} \sim \frac{e^{kb}}{k^2}$$

which approaches infinity as  $k$  approaches infinity. On the other hand, the cauchy data

$$\|u_k(x, 0, t)\|_{1,loc} + \left\| \frac{\partial u_k}{\partial y}(x, 0, t) \right\|_{0,loc} \sim \frac{1}{k}$$

approaches zero as  $k$  approaches infinity. Also, see [6] for some other interesting examples.

For piecewise, smooth, bounded domains the situation is much better. If  $D$  is contained in a ball of diameter  $T$ , then it was shown, by Lop Fat Ho in [3] for  $q = 0$ , by Komornik and Zuazua in [9] for  $q \geq 0$ , and by Klibanov and Malinsky in [7] for  $q \in L^\infty(D \times [0, T])$ , that the solution of (1), (2) depends stably on  $g$  and  $h$ . i.e.

$$\|u\|_{1, D \times [0, T]} \leq C(\|g\|_1 + \|h\|_0)$$

For general  $D$ , bounded and piecewise smooth, one can prove stability over parts of  $D \times [0, T]$ , using Carleman estimates, as done in [11] and [5]. This region will be described more carefully in Proposition 2 - also see Section 2 for some examples. This region may be empty for certain domains  $D \times [0, T]$ .

Stability implies uniqueness, so  $u$  is uniquely determined over the subsets (possibly empty) of  $D \times [0, T]$  mentioned in the previous paragraph. Cohen constructed examples, see [4], which show lack of local uniqueness for (1), (2) for certain choices of  $D$ . However, the question of uniqueness for bounded domains has not been resolved completely, especially for real valued  $q$ .

We approximate the solution of (1), (2), by solving a well posed fourth order elliptic equation. This equation is constructed using the method of quasi-reversibility introduced by Lattes and Lions in [10]. This method was used by Klibanov and Santosa in [8] to construct numerical solutions of the Cauchy problem for Laplace's equation. Danilaev and Klibanov in [2], using quasi-reversibility, implemented a scheme to solve an inverse problem for a one dimensional parabolic equation. Quasi-reversibility has also been used by Strauss in [13] to prove the existence of solutions of certain non-linear evolution equations. See [12] for references for other applications of the method of quasi-reversibility.

Klibanov and Malinsky in [7] introduced the method of quasi-reversibility for (1), (2), and proved the convergence estimate. But they were only interested in the case where  $D$  was contained in a ball of diameter  $T$  and did not present any numerical results. In this article, using a modification of their arguments, we extend the method to apply to any smooth, bounded domain  $D$  and we implement the proposed algorithm. If  $D$  is contained in a ball of diameter  $T$  then theoretically the convergence is of order  $1/2$  (as  $\epsilon$  approaches zero), see [7]. In the general case the convergence is slower.

Let us remark again that we work only with bounded domains  $D$ , and our method will approximate  $u$  over all of  $D \times [0, T]$  provided  $D$  is contained in a ball of diameter  $T$ . In the general case,  $u$  is approximated only over a subset (possibly empty) of  $D \times [0, T]$ . Solving fourth order elliptic equations, in 3 dimensions, is rather time consuming. However, to the best of our knowledge, no other method is known for approximating the solution to (1), (2).

## 2 The Algorithm and its Convergence

First, we reduce (1), (2) to an equation with zero boundary data, but a non-zero right hand side. This may always be done by subtracting from  $u$  a function which has the same boundary data as  $u$ . So for the rest of the article we will be concentrating on

$$Lu \equiv u_{tt} - \Delta_x u + q(x, t)u = f \quad \text{in } D \times [0, T] \quad (3)$$

$$u = 0, \quad u_n = 0 \quad \text{on } \partial D \times [0, T] \quad (4)$$

So the goal is - Given  $f \in L^2(D \times [0, T])$ , for which (3), (4) has a solution, obtain a numerical approximation of  $u$ .

Define the differential operator

$$L^\epsilon v \equiv \frac{1}{\epsilon} L^* L v + v_{ttt} + v_{iiii} + v$$

where the  $i$  term is to be summed over the indices 1 to  $n$ . Then using an integration by parts one may show that for smooth  $v, w$

$$\int_{D \times [0, T]} dx dt w L^\epsilon v = \langle v, w \rangle_H + \text{bdryterm1} + \text{bdryterm2} \quad (5)$$

where

$$\begin{aligned} \langle v, w \rangle_H &= \int_{D \times [0, T]} \left\{ \frac{1}{\epsilon} L v L w + v_{tt} w_{tt} + v_{ii} w_{ii} + v w \right\} \\ \text{bdryterm1} &= \int_{\partial D \times [0, T]} \{ v_{iii} n_i w - v_{ii} w_i n_i \} + \int_{\partial D \times [0, T]} \frac{1}{\epsilon} \{ w_n L v - (L v)_n w \} \\ \text{bdryterm2} &= \int_D dx w \left\{ v_{ttt} + \frac{1}{\epsilon} (L v)_t \right\} - w_t \left\{ v_{tt} + \frac{1}{\epsilon} L v \right\} \Big|_0^T \end{aligned}$$

Now  $\langle v, w \rangle$  is an inner product on  $C^2(D \times [0, T])$ , and its completion is  $H^2(D \times [0, T])$ . Let  $H$  be the completion, under the above norm, of the space of functions  $v$  in  $C^2(D \times [0, T])$  such that  $v$  and  $v_n$  vanish on  $\partial D \times [0, T]$ . Then  $H$  consists of functions  $v$  which are in  $H^2(D \times [0, T])$  such that  $v$  and  $v_n$  are zero on  $\partial D \times [0, T]$ .

**Proposition 1** *Given  $f \in L^2(D \times [0, T])$ , there exists a unique  $u^\epsilon \in H$  which solves*

$$L^\epsilon u^\epsilon = \frac{1}{\epsilon} L^* f \quad (6)$$

$$u^\epsilon = 0, \quad u_n^\epsilon = 0 \quad \text{if } (x, t) \in \partial D \times [0, T] \quad (7)$$

$$u_{ttt}^\epsilon + \frac{1}{\epsilon} (L u^\epsilon)_t = f_t \quad \text{on } t = 0, T \quad (8)$$

$$u_{tt}^\epsilon + \frac{1}{\epsilon}Lu^\epsilon = f \quad \text{on } t = 0, T \quad (9)$$

in the weak sense. In addition

$$\|u^\epsilon\|_H^2 \leq \frac{1}{\epsilon}\|f\|_0^2$$

Remark: Let us determine the weak form of (6) - (9). Suppose  $u^\epsilon$  and  $w$  are  $C^4$  functions with  $w$  and  $w_n$  zero on  $\partial D \times [0, T]$ . Then using (5) and (6)

$$\frac{1}{\epsilon} \int_{D \times [0, T]} wL^*f = \int_{D \times [0, T]} wL^\epsilon u^\epsilon = \langle u^\epsilon, w \rangle_H + \text{boundary terms}$$

But the boundary terms which are integrals over  $\partial D \times [0, T]$  are zero because  $w$  and its normal derivative vanishes on  $\partial D \times [0, T]$ . Again, noting that  $w$  and  $w_n$  are zero on  $\partial D \times [0, T]$ , we have

$$\begin{aligned} \int_{D \times [0, T]} wL^*f &= \int_{D \times [0, T]} fLw + \int_D \{f_t w - fw_t\} \Big|_0^T + \int_{\partial D \times [0, T]} \{wf_n - fw_n\} \\ &= \int_{D \times [0, T]} fLw + \int_D \{f_t w - fw_t\} \Big|_0^T \end{aligned}$$

Using the above relations and noting the boundary conditions (8), (9) satisfied by  $u^\epsilon$ , one concludes that

$$\langle u^\epsilon, w \rangle_H = \frac{1}{\epsilon} \int_{D \times [0, T]} fLw \quad \forall w \in H \quad (10)$$

This is the weak formulation of (6) - (9).

Proof of Proposition 1 Consider the following linear functional on  $H$

$$w \longmapsto \frac{1}{\epsilon} \int_{D \times [0, T]} fLw, \quad w \in H$$

Then

$$\begin{aligned} \frac{1}{\epsilon} \left| \int fLw \right| &\leq \frac{1}{\epsilon} \|f\|_0 \|Lw\|_0 \\ &\leq \frac{1}{\sqrt{\epsilon}} \|f\|_0 \|w\|_H \end{aligned}$$

So from the Riesz Representation Theorem there exists a unique  $u^\epsilon \in H$  such that

$$\langle u^\epsilon, w \rangle_H = \frac{1}{\epsilon} \int_{D \times [0, T]} fLw \quad \forall w \in H$$

Further  $\|u^\epsilon\|_H^2 \leq \frac{1}{\epsilon} \|f\|_0^2$ .

**Lemma 1** Suppose  $f \in L^2(D \times [0, T])$  and  $u \in H^2(D \times [0, T])$  solves (3) - (4). Suppose  $u^\epsilon$  is the solution to (6) - (9) guaranteed by Proposition 1. Then

$$\|u - u^\epsilon\|_2^2 \leq 4\|u\|_2^2 \quad (11)$$

$$\|L(u - u^\epsilon)\|_0^2 \leq 4\epsilon\|u\|_2^2 \quad (12)$$

Proof: From the definition of  $u^\epsilon$  we have

$$\langle u^\epsilon, w \rangle_H = \frac{1}{\epsilon} \int fLw \quad \forall w \in H$$

Also, noting that  $u$  satisfies (3), we have

$$\langle u, w \rangle_H = \frac{1}{\epsilon} \int fLw + (u, w)_2 \quad \forall w \in H$$

Here, we have used the notation

$$(v, w)_2 = \int_{D \times [0, T]} \{v_{tt}w_{tt} + v_{ii}w_{ii} + vw\}$$

So subtracting the previous two relations we obtain

$$\langle u - u^\epsilon, w \rangle_H = (u, w)_2 \quad \forall w \in H$$

Given that  $u \in H^2(D \times [0, T])$  and  $u$  satisfies (4), we have  $u \in H$ . Also  $u^\epsilon \in H$ . So taking  $w = u - u^\epsilon$  in the previous equation we obtain

$$\begin{aligned} \|u^\epsilon - u\|_H^2 &= (u, u - u^\epsilon)_2 \\ &\leq \|u\|_2 \|u - u^\epsilon\|_2 \\ &\leq 2\|u\|_2^2 + \frac{1}{2}\|u - u^\epsilon\|_2^2 \\ &\leq 2\|u\|_2^2 + \frac{1}{2}\|u - u^\epsilon\|_H^2 \end{aligned}$$

Therefore

$$\|u^\epsilon - u\|_H^2 \leq 4\|u\|_2^2$$

and noting the definition of  $\|\cdot\|_H$  we obtain the Lemma.

**Proposition 2** Suppose  $u \in H^2(D \times [0, T])$  and  $f \in L^2(D \times [0, T])$  satisfy (6), (7). Suppose  $u^\epsilon$  is the solution to (6) - (9) guaranteed by Proposition 1. Let  $\phi$  be a function in  $C^2(R^{n+1})$  with

- $\nabla\phi \neq 0$  in  $D \times [0, T]$

- $\phi$  pseudoconvex w.r.t  $L$  in  $D \times [0, T]$ . See [4] for the definition.
- $a < b$  implies  $\Omega_b \subset \Omega_a$  where

$$\Omega_c = \{ (x, t) \in D \times [0, T] : \phi(x, t) \geq c \}$$

- $\Omega_0$  does not intersect  $t = 0$  or  $t = T$  (for convenience)

Then for any  $\delta > 0$  and  $\epsilon$  small enough

$$\|u - u^\epsilon\|_{1, \Omega_{2\delta}}^2 \leq C \epsilon^{\frac{\delta}{m-2\delta}} \|u\|_2^2 \quad (13)$$

where  $m$  is the largest value of  $\phi$  on  $\Omega_0$ .

Remark So Proposition 2 proves a Hölder rate of convergence. If  $D$  is contained in a ball of diameter  $T$ , then the convergence is faster and over the whole region. It was shown by Klibanov and Malinsky in [7] that

$$\|u - u^\epsilon\|_{1, D \times [0, T]}^2 \leq C \epsilon \|u\|_{2, D \times [0, T]}^2$$

Proof: Construct a function  $\theta \in C^\infty(R^{n+1})$  such that  $\theta$  is 1 on  $\Omega_\delta$  and

$$\text{supp } \theta \cap (D \times [0, T]) \subset \Omega_0$$

Let  $v = u - u^\epsilon$  and  $w = \theta v$ . Since  $v$  and  $v_n$  are zero on  $\partial D \times [0, T]$  and  $\theta$  vanishes near  $t = 0, T$ , one concludes that  $w \in H_0^2(D \times [0, T])$ . So from the Carleman estimate in [4], for large  $\lambda$ ,

$$\lambda \int_{D \times [0, T]} |\nabla w|^2 e^{2\lambda\phi} + \lambda^3 \int_{D \times [0, T]} |w|^2 e^{2\lambda\phi} \leq C \int_{D \times [0, T]} |Lw|^2 e^{2\lambda\phi} \quad (14)$$

Now

$$Lw = L(\theta v) = \theta Lv + [L, \theta]v$$

Therefore, using the definition of  $\theta$

$$|Lw|^2 \leq \chi(\Omega_0) |Lv|^2 + C \chi(\Omega_0 \setminus \Omega_\delta) \{ |\nabla v|^2 + |v|^2 \}$$

where  $\chi(A)$  represents the characteristic function of the set  $A$ . Also

$$\begin{aligned} |\nabla w|^2 &= |\nabla(\theta v)|^2 \\ &\geq \frac{\theta^2}{2} |\nabla v|^2 - |[\nabla, \theta]v|^2 \\ &\geq \frac{1}{2} \chi(\Omega_\delta) |\nabla v|^2 - \chi(\Omega_0 \setminus \Omega_\delta) |v|^2 \\ |w|^2 &\geq \chi(\Omega_\delta) |v|^2 \end{aligned}$$

So (14) implies that

$$\int_{\Omega_\delta} e^{2\lambda\phi} \left\{ \lambda |\nabla v|^2 + \lambda^3 |v|^2 \right\} \leq C \left\{ \int_{\Omega_0} |Lv|^2 e^{2\lambda\phi} + \lambda \int_{\Omega_0 \setminus \Omega_\delta} (|\nabla v|^2 + |v|^2) e^{2\lambda\phi} \right\}$$

Let  $m$  be the largest value of  $\phi$  on  $\Omega_0$ . Noting that  $\phi \leq \delta$  on  $\Omega_0 \setminus \Omega_\delta$ , and that  $\phi \geq 2\delta$  on  $\Omega_{2\delta}$ , and using (11), (12), one obtains

$$e^{2\lambda(2\delta)} \left\{ \lambda \int_{\Omega_{2\delta}} |\nabla v|^2 + \lambda^3 \int_{\Omega_{2\delta}} |v|^2 \right\} \leq C \left\{ \epsilon e^{2\lambda m} \|u\|_2^2 + \lambda e^{2\lambda(\delta)} \|u\|_2^2 \right\}$$

Therefore for large  $\lambda$

$$\lambda \int_{\Omega_{2\delta}} |\nabla v|^2 + \lambda^3 \int_{\Omega_{2\delta}} |v|^2 \leq C \|u\|_2^2 \left\{ \epsilon e^{2\lambda(m-2\delta)} + \lambda e^{-2\lambda\delta} \right\} \quad (15)$$

Choose  $\lambda$  such that

$$\epsilon e^{2\lambda(m-2\delta)} = e^{-2\lambda\delta}$$

that is

$$\lambda = \frac{1}{2(m-\delta)} \ln\left(\frac{1}{\epsilon}\right)$$

If  $\epsilon$  is small enough then the  $\lambda$  chosen will be large enough so that (15) is valid. Then (15) implies

$$\int_{\Omega_{2\delta}} \left\{ |\nabla v|^2 + |v|^2 \right\} \leq C \epsilon^{\frac{\delta}{m-\delta}} \|u\|_2^2$$

which proves the proposition.

### 3 Numerical Results

We solve

$$\begin{aligned} \square u &\equiv u_{tt} - u_{xx} - u_{yy} = f(x, y, t) \quad \text{in } D \times [0, T] \\ u &= 0, \quad u_n = 0 \quad \text{on } \partial D \times [0, T] \end{aligned}$$

Here  $T$  varies in size and

$$D = \{ (x, y) : 0 < x < 1, \quad 0 < y < 1 \}$$

We start by choosing a  $u$  such that  $u$  and  $u_n$  vanish on  $\partial D \times [0, T]$ , and obtain an expression for  $f$  i.e.  $\square u$ . We add a certain percentage of uniformly distributed random error to  $f$  and then construct the approximation  $u^\epsilon$  using our algorithm. Then we compute the  $L^2$  norm of  $u - u^\epsilon$  (and the relative error) over subsets of  $D \times [0, T]$ .

These subsets are determined by the function

$$\begin{aligned}\phi(x, y, t) &= \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 - \sigma\left(t - \frac{T}{2}\right)^2 + \sigma\frac{T^2}{4} - \frac{1}{2} \\ &= x^2 + y^2 - x - y - \sigma t^2 + \sigma T t\end{aligned}$$

Here  $\sigma$  is a number in  $(0, 1)$  to be chosen later. If

$$\Omega_c = \{ (x, y, t) \in D \times [0, T] : \phi(x, y, t) \geq c \}$$

We study the error in the approximation over the regions  $\Omega_c$ , as suggested by Proposition 2.

Clearly  $\Omega_c$  decreases as  $c$  increases, but also note that  $\Omega_c$  increases as  $\sigma$  increases. Note that  $\sigma$  influences  $m$  the largest value of  $\phi$  on  $D \times [0, T]$  which influences the rate of convergence. Further  $\phi(x, y, t)$  is pseudoconvex with respect to the wave operator (see [4] for the definition) provided  $0 < \sigma < 1$ . Also note that the largest region  $\Omega_c$  which does not intersect  $t = 0, T$  is  $\Omega_0$ . So  $\phi$  has been arranged so that Proposition 2 is applicable. Note that  $\nabla\phi$  is zero at  $(1/2, 1/2, T/2)$  but we can take care of this problem by using another  $\phi$  based at a point slightly perturbed from  $(1/2, 1/2, T/2)$ .

The approximate solution  $u^\epsilon$  is the solution of

$$\int_{D \times [0, T]} \left\{ \frac{1}{\epsilon} \square u^\epsilon Lw + u_{tt}^\epsilon w_{tt} + u_{ii}^\epsilon w_{ii} + u^\epsilon w \right\} = \frac{1}{\epsilon} \int_{D \times [0, T]} f Lw$$

for all  $w$  in  $H^2(D \times [0, T])$  with  $w$  and  $w_n$  vanishing on the  $\partial D \times [0, T]$ . We solve this using the finite element method (in  $x, y, t$  space). We use rectangular grids and the *Bogner – Schmit* conforming elements in 3 dimensions. The resulting system of linear equations was solved by the Conjugate Gradient method until the relative error was less than  $10^{-5}$ . The integrals were computed using a Gaussian Quadrature scheme.

The random error in  $f$  is generated as follows. Following the FEM we first generate the linear equation  $Ax = b$  with exact  $f$ . Then we generate a random vector  $e$  of the same dimension as  $b$ . Then we choose as our new  $b$  the vector

$$b + \frac{\text{percent\_error}}{100} \frac{e}{\|e\|}$$

For all the calculations we assume that  $\sigma = 0.8$ , so

$$\phi(x, y, t) = x^2 + y^2 - x - y - 0.8t^2 + 0.8Tt$$

For this  $\phi$ ,  $\Omega_0$  and its subsets are the regions for which Proposition 2 is valid,  $\Omega_c$  with  $c = 0.2T^2$  is the empty set, and  $\Omega_c$  with  $c = -0.5$  is  $D \times [0, T]$ .

### Example 1

$$u(x, y, t) = (1 - \cos(2\pi x))(1 - \cos(2\pi y))(t + 1)^2, \quad T = 1.0, \quad \text{grid} = 8 \times 8 \times 8$$

(a) input error = 0%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 9.4% | 0.7%   |
| 0.0          | 9.4% | 0.6%   |
| 0.1          | 7.8% | 0.6%   |

(b) input error = 5%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1   | 0.0001 |
|--------------|-------|--------|
| -0.5         | 10.0% | 1.0%   |
| 0.0          | 10.5% | 0.9%   |
| 0.1          | 9.6%  | 1.1%   |

(c) input error = 10%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 9.5% | 1.9%   |
| 0.0          | 9.5% | 1.7%   |
| 0.1          | 7.7% | 1.8%   |

Example 2 (same as Example 1 except  $T = 2.0$ )

$$u(x, y, t) = (1 - \cos(2\pi x))(1 - \cos(2\pi y))(t + 1)^2, \quad T = 2.0, \quad \text{grid} = 8 \times 8 \times 16$$

(a) input error = 0%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 8.7% | 0.4%   |
| 0.0          | 8.6% | 0.5%   |
| 0.7          | 6.5% | 0.7%   |

(b) input error = 5%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 9.0% | 1.0%   |
| 0.0          | 9.0% | 0.9%   |
| 0.7          | 6.6% | 1.0%   |

(c) input error = 10%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 8.6% | 1.6%   |
| 0.0          | 7.6% | 1.3%   |
| 0.7          | 6.5% | 1.0%   |

Example 3

$$u(x, y, t) = 100x^2(1-x)^2y^2(1-y)^2, \quad T = 1.0, \quad \text{grid} = 8 \times 8 \times 8$$

(a) input error = 0%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 8.9% | 0.2%   |
| 0.0          | 8.9% | 0.1%   |
| 0.1          | 9.3% | 0.1%   |

(b) input error = 5%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 9.0% | 1.0%   |
| 0.0          | 9.6% | 1.0%   |
| 0.1          | 9.7% | 1.0%   |

(c) input error = 10%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 8.3% | 2.2%   |
| 0.0          | 8.9% | 1.8%   |
| 0.1          | 9.3% | 1.3%   |

Example 4

$$u(x, y, t) = 100x^2(1-x)^2y^2(1-y)^2e^{(x+y+t)}, \quad T = 1.0, \quad \text{grid} = 8 \times 8 \times 8$$

(a) input error = 0%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 9.0% | 0.8%   |
| 0.0          | 9.3% | 0.7%   |
| 0.1          | 8.5% | 0.6%   |

(b) input error = 5%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 9.0% | 1.0%   |
| 0.0          | 9.6% | 1.0%   |
| 0.1          | 9.0% | 0.9%   |

(c) input error = 10%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1   | 0.0001 |
|--------------|-------|--------|
| -0.5         | 9.6%  | 2.4%   |
| 0.0          | 10.1% | 2.3%   |
| 0.1          | 9.7%  | 2.4%   |

### Example 5

$$u(x, y, t) = 100x^2(1-x)^2y^2(1-y)^2e^{(x+y+t)}, \quad T = 0.7, \quad \text{grid} = 8 \times 8 \times 8$$

input error = 0%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1   | 0.0001 |
|--------------|-------|--------|
| -0.5         | 11.1% | 5.9%   |
| 0.0          | 11.1% | 5.9%   |
| 0.05         | 11.9% | 5.5%   |
| 0.09         | 12.3% | 4.6%   |

### Example 6

$$u(x, y, t) = 100x^2(1-x)^2y^2(1-y)^2e^{(x+y+t)}, \quad T = 2.0, \quad \text{grid} = 8 \times 8 \times 10$$

input error = 0%, relative  $L^2$  error in output is

| $c/\epsilon$ | 0.1  | 0.0001 |
|--------------|------|--------|
| -0.5         | 8.7% | 0.5%   |
| 0.3          | 8.4% | 0.5%   |
| 0.8          | 6.8% | 0.53%  |

## CONCLUSIONS

From our examples it seems the algorithm works very well for the examples we have chosen. We observe from our calculations that the numerical solution approximates the exact solution with greater accuracy as  $\epsilon$  gets smaller. However, the accuracy does not seem to improve much when  $c$  increases (i.e. when  $\Omega_c$ , the region of comparison decreases), though the theory predicts an increase in accuracy. Of course, the accuracy is quite good for large  $\Omega_c$ , not leaving much room for improvement as  $\Omega_c$  decreases in size. Also, studying examples 5 and 6, it becomes clear that the accuracy over regions not covered by the theory can be

poor, and if  $D$  is contained in a ball of diameter  $T$  then the accuracy over the whole region is quite good, as predicted by Klibanov and Malinsky in [7].

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