INVERSION OF SPHERICALLY SYMMETRIC POTENTIALS FROM BOUNDARY DATA FOR THE WAVE EQUATION

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Abstract: Suppose $q(y)$ is a spherically symmetric potential which is zero outside B, the ball of radius R in R^3. Suppose

$$V_{tt} - \triangle_y V + q(y)V = \delta(y,t), \text{ for } y \in R^3, \; t \in R$$

$$V = 0 \text{ for } t < 0.$$

We show how $q(y)$ may be reconstructed from the values of $V(y,t)$ on $\partial B \times [R, 3R]$ by reducing it to a problem of inversion from transmission data for a one dimensional wave equation.

Introduction

For $q(y)$, a continuous function on R^3, consider the point source problem

$$\Box V + q(y)V = 4\pi\delta(y,t) \qquad \text{for } x \in R^3,\ t \in R \tag{1}$$

$$V = 0 \qquad for\ t < 0 \tag{2}$$

One may verify that (see [7] and [4])

$$V(y,t) = \frac{\delta(t-|y|)}{|y|} - W(y,t)$$

where $W(y,t)$ is supported in $t \geq |y|$ and is the solution of the Goursat problem

$$\begin{aligned} \Box W + q(y)W &= 0 \qquad \text{for } y \in R^3,\ t \geq |y| & (3)\\ W(y,|y|) &= \int_0^1 d\sigma\, q(\sigma y) \qquad \text{for } y \in R^3. & (4) \end{aligned}$$

It is shown in [7] that (3), (4) has a unique continuous solution.

Our goal is the solution of the inverse problem. Suppose B is the ball of radius R in R^3 and $q(y)$ is zero outside B. Our goal is to recover $q(y)$ given the values of $V(y,t)$ (hence $W(y,t)$) for $y \in \partial B$ and $t \in [R, 3R]$. From experience with similar one dimensional problems one knows that for successful inversion, $V(y,t)$ must be known for t in an interval at least twice as large as the radius of the support of $q(y)$.

In the above mentioned inverse problem the data depends on three parameters and the object to be reconstructed depends on three parameters. Only a few results are known for multi-dimensional inverse problems where the data and the unknown depend on the same number of parameters. For hyperbolic problems such results have been obtained by Romanov (see [8]), Bukhgeim and Klibanov (see [1]) and by Sacks and Symes (see [10]). However none of these deal with the problem under consideration in this article. Our goal is to solve the above problem in the case where q is spherically symmetric. We will show

Theorem 1 *Suppose $q(y)$ is a spherically symmetric, continuous function on R^3 which is supported in the ball B of radius R. Then q may be reconstructed from knowledge of $V(y,t)$ on $\partial B \times [R, 3R]$ provided we have an upper bound on $\|q\|_\infty$. (An upper bound is not needed if we are only interested in the uniqueness problem).*

If $q(y)$ is spherically symmetric then $W(y,t)$ is spherically symmetric in y. Let $r = |y|$ - we write $W(y,t)$ as $W(r,t)$ and $q(y)$ as $q(r)$ for convenience. Then rewriting (3), (4) in spherical coordinates we have

$$\begin{aligned} W_{tt} - W_{rr} - \frac{2}{r}W_r + q(r)W &= 0 \qquad \text{for } 0 \leq r \leq t & (5)\\ W(r,r) &= \int_0^1 q(\sigma r)\, d\sigma \qquad \text{for } r \geq 0 & (6) \end{aligned}$$

Let $w(r,t) = rW(r,t)$ then for $r \geq 0$

$$w(r,r) = r \int_0^1 q(\sigma r)\, d\sigma = \int_0^r q(\sigma)\, d\sigma$$

Further $w(0,t) = 0W(0,t) = 0$. Hence using (5) we find that $w(.,t)$ is the solution of a one dimensional characteristic boundary value problem

$$\begin{aligned} w_{tt} - w_{xx} + q(x)w &= 0 \qquad \text{for } 0 \leq x \leq t & (7)\\ w(x,x) &= \int_0^x q(\sigma)\, d\sigma \qquad \text{for } x \geq 0 & (8)\\ w(0,t) &= 0 \qquad \text{for } t \geq 0 & (9) \end{aligned}$$

So our inverse problems reduces to the following one dimensional inverse problem : Given that $q(x)$ is zero for $x \geq R$, determine $q(x)$ if $w(R,t)$ is known for $R \leq t \leq 3R$.

It turns out that $w(x,t)$ is related to the solution of a one dimensional initial boundary value problem with an impulsive source. Consider the problem

$$\begin{aligned} u_{tt} - u_{xx} + q(x)u &= 0 \qquad \text{in} \quad 0 \leq x,\ t \in R & (10)\\ u &= 0 \qquad \text{for } t < 0 & (11)\\ u(0,t) &= H(t) \qquad \text{for } t \in R & (12) \end{aligned}$$

The principal singularity in $u(x,t)$ is on $t = x$ and using the progressing wave expansion, as in Chapter 6, Section 4 of [3], one may write $u(x,t)$ as a sum of terms which capture the singularity in $u(x,t)$. We may show that, if q is continuous, then

$$u(x,t) = H(t-x) - \frac{1}{2}\left(\int_0^x q(\sigma)d\sigma\right)(t-x)H(t-x) + \text{ smoother terms} \qquad (13)$$

where H is the Heaviside function. Hence

$$u_t(x,t) = \delta(t-x) - \frac{1}{2}\int_0^x q(\sigma)d\sigma\, H(t-x) + \text{ smoother terms}$$

One may verify using the weak formulation that if $w(x,t)$ is defined to be zero for $t < x$ then

$$u_t(x,t) = \delta(t-x) - \frac{1}{2}w(x,t)$$

Hence Theorem 1 follows from

Theorem 2 *Suppose $q(x)$ is a continuous function on $[0,\infty)$ which is supported in $[0,X]$. Then q may be reconstructed from knowledge of $u_t(X,t)$ on $[X,3X]$ provided we have an upper bound on $\|q\|_\infty$. (An upper bound is not needed if we are only interested in the uniqueness problem).*

One dimensional inverse problems for the wave equation have received a lot of attention but most people have studied inversion from reflection data $u(0,t)$ (source and receiver at same location) or inversion from reflection and transmission data, instead of inversion from only transmission data $u(X,t)$ (source and receiver at different locations) - as in the problem under consideration. In [5] a problem similar to our one dimensional problem was studied using ideas from [6] and [2] - the only difference being the impulsive source (which was $u_x(0,t) = \delta(t)$) and which is $u_t(0,t) = \delta(t)$ in our problem.

Inversion from reflection data is usually accomplished using layer stripping where, given data at x=X, one recovers (for small h) $q(x)$ in the layer $X - h < x < X$, as well as the data at $x = X - h$ - and the process is repeated. This technique is not applicable to inversion from transmission data as explained in [6] and makes inversion from transmission data harder. We solve our problem by reducing it to a one dimensional inverse problem for the wave equation to which layer stripping techniques may be applied. In fact we show that if

$$m(t) \equiv 2(u_t(X,t) - \delta(t - X))$$

then

Proposition 3 *Suppose $q(x)$ is a continuous function on $[0,\infty)$ which is supported in $[0,X]$. If $K(x,t)$ is the solution of the following Goursat problem*

$$\begin{aligned} K_{tt} - K_{xx} + qK &= 0 \quad \textit{in} \quad |t| \leq x \leq X && (14) \\ K(x, \pm x) &= \pm\frac{1}{2}\int_0^x q(\sigma)\, d\sigma \qquad \textit{for} \qquad 0 \leq x \leq X && (15) \end{aligned}$$

then for $|t| \leq X$

$$K(X,t) = g(-t) - g(t), \qquad K_x(X,t) = g'(-t) - g'(t) \tag{16}$$

where $g(t)$ is the solution over $|t| \leq X$ of the Volterra equation

$$2g(t) + m(t + 2X) + \int_{-X}^{t} g(s)\, m(t + X - s)\, ds = 0 \tag{17}$$

Note that $m(t) = -w(X,t)$. Since q is continuous, one may show by standard arguments that the solution of (7)-(9) is C^1 on the region $0 \leq x \leq t$. Hence $m(t)$ is C^1 on $[X, 3X]$.

Hence the proof of Theorem 2 has been reduced to the recovery of $q(x)$ from the knowledge of $K(X,t)$ and $K_x(X,t)$ over $|t| \leq X$. This is a problem amenable to the layer stripping argument and was solved by Rundell and Sacks as Theorems 1 and 2 in [9].

Proposition 3 is proved by techniques similar to those in [5] but it is far from obvious that those techniques would work for this new problem. In fact we have not yet been able to resolve the inverse problem corresponding to the impulsive source $u_t(0,t) + \alpha u_x(0,t) = \delta(t)$ (instead of (12)). We have also improved upon the presentation given in [5].

One can also solve the corresponding problem for the impedance equation i.e. the problem discussed in [6] except with the impulsive source $u(0,t) = H(t)$. In fact the calculations for that problem would be a little simpler than the calculations for the potential problem in the present article.

Proof of Proposition 3

If we define $u_0(t) = u(0,t)$ and $u_1(t) = u_x(0,t)$ then may express $u(x,t)$ in terms of u_0 and u_1 and two "Green's Functions" F and G as

$$u(x,t) = \frac{1}{2}\{u_0(t-x) + u_0(t+x)\} + \int_{t-x}^{t+x} F(x,t-s)u_0(s)ds + \int_{t-x}^{t+x} G(x,t-s)u_1(s)ds \tag{18}$$

where

$$\begin{aligned} F_{tt} - F_{xx} + qF &= 0 \quad \text{in} \quad |t| \le x & \qquad G_{tt} - G_{xx} + qG &= 0 \quad \text{in} \quad |t| \le x \\ F(x,\pm x) &= \frac{1}{4}\int_0^x q(\sigma)d\sigma & G(x,\pm x) &= \frac{1}{2} \end{aligned} \tag{19}$$

Now $u_0(t) = H(t)$ and from (13), we have

$$u_x(x,t) = -\delta(t-x) + \frac{1}{2}\int_0^x q(\sigma)d\sigma \; H(t-x) + \text{ smoother terms}$$

So

$$u_1(t) = u_x(0,t) = -\delta(t) + 0 \cdot H(t) + \text{ smoother } \equiv -\delta(t) + v_1(t) \quad \text{(definition)}$$

Substituting these in (18) we obtain

$$u(x,t) = \frac{1}{2}\{H(t-x) + H(t+x)\} + \int_{t-x}^{t+x} F(x,t-s)H(s)ds - G(x,t) + \int_{t-x}^{t+x} G(x,t-s)v_1(s)ds.$$

Here G is defined to be zero outside $|t| \le x$. Therefore

$$\begin{aligned} u(x,t) - H(t-x) &= \frac{1}{2}\{H(t+x) - H(t-x)\} - G(x,t) \\ &\quad + \int_{t-x}^{t+x} \{F(x,t-s)H(s) + G(x,t-s)v_1(s)\} \end{aligned}$$

Let $v(x,t) \equiv u(x,t) - H(t-x)$. Noting that $G(x,\pm x) = \frac{1}{2}$ we have that

$$\frac{1}{2}\{H(t+x) - H(t-x)\} - G(x,t)$$

is continuous across $t = \pm\, x$. Therefore its derivatives will have a jump type singularity. Hence

$$\begin{aligned} v_x(x,t) &= -G_x(x,t) + F(x,-x)H(t+x) + G(x,-x)v_1(t+x) + F(x,x)H(t-x) \\ &\quad + G(x,x)v_1(t-x) + \int_{t-x}^{t+x} \{F_x(x,t-s)H(s) + G_x(x,t-s)v_1(s)\}\,ds \\ v_t(x,t) &= -G_t(x,t) + F(x,-x)H(t+x) + G(x,-x)v_1(t+x) - F(x,x)H(t-x) \\ &\quad - G(x,x)v_1(t-x) + \int_{t-x}^{t+x} \{F_t(x,t-s)H(s) + G_t(x,t-s)v_1(s)\}\,ds \end{aligned}$$

Hence

$$\begin{aligned}(v_t + v_x)(x,t) &= -\left(G_t + G_x\right)(x,t) + 2F(x,-x)H(t+x) + 2G(x,-x)v_1(t+x) \\ &\quad + \int_{t-x}^{t+x} \left\{(F_x + F_t)(x,t-s)H(s) + (G_x + G_t)(x,t-s)v_1(s)\right\} ds \qquad (20)\end{aligned}$$

and

$$\begin{aligned}(v_t - v_x)(x,t) &= -\left(G_t - G_x\right)(x,t) - 2F(x,x)H(t-x) - 2G(x,x)v_1(t-x) \\ &\quad + \int_{t-x}^{t+x} \left\{\left(F_t - F_x\right)(x,t-s)H(s) + \left(G_t - G_x\right)(x,t-s)v_1(s)\right\} ds \qquad (21)\end{aligned}$$

Here we define G_x, G_t to be zero outside $|t| \leq x$. Similarly we define $F,\ F_x,\ F_t$ to be zero outside $|t| \leq x$.

Since $q(x) = 0$ for $x > X$, and $u = 0$ for $t < 0$, there are no left moving waves in $t \geq x \geq X$. Hence $(u_t + u_x)(X,t) = 0$ implying $(v_t + v_x)(X,t) = 0$. Define

$$m(t) \equiv (v_t - v_x)(X,t) = (u_t - u_x)(X,t) - 2\delta(t-X) = 2(u_t(X,t) - \delta(t-X)). \qquad (22)$$

Note that from (13),

$$(u_t - u_x)(x,t) = 2\delta(t-x) - \int_0^x q(\sigma)d\sigma\ H(t-x) + \text{ smoother terms}$$

So $m(t)$ has only a jump type singularity at $t = X$. Also, we are given $u(X,t)$ for $X \leq t \leq 3X$, therefore $m(t)$ is known for $X \leq t \leq 3X$.

Then from (20) and (21), we have

$$\begin{aligned}0 &= H(t) * \left\{\left(F_t + F_x\right)(X,t) + 2F(X,-X)\delta(t+X)\right\} \\ &\quad + v_1(t) * \left\{\left(G_t + G_x\right)(X,t) + 2G(X,-X)\delta(t+X)\right\} - \left(G_t + G_x\right)(X,t) \\ m(t) &= H(t) * \left\{\left(F_t - F_x\right)(X,t) - 2F(X,X)\delta(t-X)\right\} \\ &\quad + v_1(t) * \left\{\left(G_t - G_x\right)(X,t) - 2G(X,X)\delta(t-X)\right\} - \left(G_t - G_x\right)(X,t)\end{aligned}$$

Here $*$ represents the convolution in the t variable. Eliminating $v_1(t)$ from the two equations by convolving the two equations with the appropriate quantities and then subtracting one from the other, we obtain

$$\begin{aligned}m(t) * \left\{\left(G_t + G_x\right)(X,t) + 2G(X,-X)\delta(t+X)\right\} = \\ &\left\{\left(G_t - G_x\right)(X,t) - 2G(X,-X)\delta(t-X)\right\} * \left(G_t + G_x\right)(X,t) \\ &- \left\{\left(G_t + G_x\right)(X,t) + 2G(X,-X)\delta(t+X)\right\} * \left(G_t - G_x\right)(X,t) \\ &+ H(t) * N(X,t) \qquad (23)\end{aligned}$$

where (noting that $F(x,-x) = F(x,x)$ and $G(x,-x) = G(x,x)$), we define

$$N(x,t) \equiv \left\{\left(G_t + G_x\right)(x,t) + 2G(x,x)\delta(t+x)\right\} * \left\{\left(F_t - F_x\right)(x,t) - 2F(x,x)\delta(t-x)\right\}$$

$$
\begin{aligned}
& -\left\{\left(G_t-G_x\right)(x,t)-2G(x,x)\delta(t-x)\right\} \; * \; \left\{\left(F_t+F_x\right)(x,t)+2F(x,x)\delta(t+x)\right\} \\
= \; & 2\left\{F_t(x,t)*G_x(x,t)-F_x(x,t)*G_t(x,t)\right\} \\
& -2F(x,x)\left\{\left(G_t+G_x\right)(x,t-x)+\left(G_t-G_x\right)(x,t+x)\right\} \\
& +2G(x,x)\left\{\left(F_t+F_x\right)(x,t-x)+\left(F_t-F_x\right)(x,t+x)\right\} \qquad (24)
\end{aligned}
$$

Hence

$$
\begin{aligned}
& m(t)*\left\{\left(G_t+G_x\right)(X,t)+2G(X,X)\delta(t+X)\right\}= \\
& \quad -2G(X,X)\left\{\left(G_t+G_x\right)(X,t-X)\;+\;\left(G_t-G_x\right)(X,t+X)\right\}+H(t)*N(X,t). \qquad (25)
\end{aligned}
$$

In the next section we will show that

Proposition 4

$$
N(x,t)*H(t)=(G_t-G_x)(x,t+x)-(G_t+G_x)(x,t-x)
$$

Noting that $G(x,x)=1/2$, we obtain, that for $0<t<2X$

$$
m(t)*\left\{\left(G_t+G_x\right)(X,t)+\delta(t+X)\right\}=-2\left(G_t+G_x\right)(X,t-X)
$$

which may be rewritten as

$$
2\left(G_t+G_x\right)(X,t-X)+m(t+X)+\int_{-\infty}^{\infty} m(t-s)\left(G_t+G_x\right)(X,s)\,ds=0
$$

Noting from (22) that $m(t)$ is supported in $t \geq X$, and replacing $t-X$ by t, we have for $|t| \leq X$

$$
2\left(G_t+G_x\right)(X,t)+m(t+2X)+\int_{-X}^{t} m(t+X-s)\left(G_t+G_x\right)(X,s)\,ds=0 \qquad (26)
$$

for $|t| \leq X$. So if we define $g(t)=\left(G_t+G_x\right)(X,t)$ then g satisfies (17). Further, since G is even in t, one may show that

$$
\begin{aligned}
2G_x(X,t) &= (G_x+G_t)(X,t)+(G_x+G_t)(X,-t)=g(t)+g(-t) \\
2G_t(X,t) &= (G_x+G_t)(X,t)-(G_x+G_t)(X,-t)=g(t)-g(-t)
\end{aligned}
$$

So if we define $K(x,t)=-2G_t(x,t)$ then K satisfies (14) and (16). In addition, using (19) and $G(x,\pm x)=1/2$ one may show that

$$
G_t(x,\pm x)=\mp\frac{1}{4}\int_0^x q(\sigma)\,d\sigma
$$

Hence (15) follows, proving Proposition 3.

Proof of Proposition 4

We will be using the following identity (for $\eta = 1$) which may be verified in a straightforward manner.

Remark 5 *Define*

$$\begin{aligned} f &= \eta(x)\left\{v_t(x,t-s)w_x(x,s) - v_x(x,t-s)w_t(x,s)\right\} \\ g &= \eta(x)\left\{v_t(x,t-s)w_t(x,s) - v_x(x,t-s)w_x(x,s)\right\} - q(x)v(x,t-s)w(x,s) \end{aligned}$$

Then

$$\frac{\partial f}{\partial x} - \frac{\partial g}{\partial s} = (Lv)(x,t-s)w_t(x,s) - v_t(x,t-s)(Lw)(x,s)$$

where

$$L = \eta(x)(\partial_t^2 - \partial_x^2) - \eta'(x)\partial_x + q(x).$$

Since F and G are supported in $|t| \leq x$ so using (24), $N(x,t)$ is supported in $|t| \leq 2x$. Also, since $F(x,t)$ and $G(x,t)$ are even in t, we have F_x, G_x are even in t and F_t, G_t are odd in t. This implies that $N(x,t)$ is odd in t, because

$$\begin{aligned} \frac{1}{2}N(x,-t) &= \int F_t(x,-t-s)G_x(x,s) - F_x(x,-t-s)G_t(x,s)ds \\ &\quad -F(x,x)\left\{(G_t+G_x)(x,-t-x) + (G_t-G_x)(x,-t+x)\right\} \\ &\quad +G(x,x)\left\{(F_t+F_x)(x,-t-x) + (F_t-F_x)(x,-t+x)\right\} \\ &= -\int F_t(x,t+s)G_x(x,s) + F_x(x,t+s)G_t(x,s)ds \\ &\quad +F(x,x)\left\{(G_t-G_x)(x,t+x) + (G_t+G_x)(x,t-x)\right\} \\ &\quad -G(x,x)\left\{(F_t-F_x)(x,t+x) + (F_t+F_x)(x,t-x)\right\} \\ &= -\int F_t(x,t-s)G_x(x,-s) + F_x(x,t-s)G_t(x,-s)ds + ('') + ('') \\ &= -\int F_t(x,t-s)G_x(x,s) - F_x(x,t-s)G_t(x,s)ds + ('') + ('') \\ &= -\frac{1}{2}N(x,t) \end{aligned}$$

Now for $0 \leq t \leq 2x$, using the suport of F and G,

$$\begin{aligned} \frac{1}{2}N(x,t) &= \int_{-x}^{x}(F_t(x,t-s)G_x(x,s) - F_x(x,t-s)G_t(x,s))ds \\ &\quad -F(x,x)(G_t+G_x)(x,t-x) + G(x,x)(F_t+F_x)(x,t-x). \end{aligned}$$

Now $F_t(x,t-s)$ and $F_x(x,t-s)$ are non-zero only if $-x \leq t-s \leq x$, that is if $t-x \leq s \leq t+x$. Further, since $0 \leq t \leq 2x$, we have $-x \leq t-x \leq x$, and $t+x \geq x$. So for $0 \leq t \leq 2x$,

$$\begin{aligned} \frac{1}{2}N(x,t) &= \int_{t-x}^{x}(F_t(x,t-s)G_x(x,s) - F_x(x,t-s)G_t(x,s))ds \\ &\quad -F(x,x)(G_t+G_x)(x,t-x) + G(x,x)(F_t+F_x)(x,t-x). \end{aligned}$$

Hence using (19)

$$N(x,2x) = -2F(x,x)(G_t+G_x)(x,x) + 2G(x,x)(F_t+F_x)(x,x) = p'(x) \tag{27}$$

where, for convenience, we define

$$p(x) \equiv F(x,\pm x) = \frac{1}{4}\int_0^x q(\sigma)d\sigma$$

So using Remark 5 and (19)

$$\begin{aligned}
\frac{1}{2}\frac{\partial N}{\partial x}(x,t) &= \int_{t-x}^{x} \frac{\partial}{\partial x}\left\{F_t(x,t-s)G_x(x,s) - F_x(x,t-s)G_t(x,s)\right\}ds \\
&\quad + \left\{F_t(x,t-x)G_x(x,x) - F_x(x,t-x)G_t(x,x)\right\} \\
&\quad + \left\{F_t(x,x)G_x(x,t-x) - F_x(x,x)G_t(x,t-x)\right\} \\
&\quad - p'(x)\left(G_t+G_x\right)(x,t-x) - p(x)\left\{G_{xt} - G_{tt} + G_{xx} - G_{xt}\right\}(x,t-x) \\
&\quad + \frac{1}{2}\left\{F_{xt} - F_{tt} + F_{xx} - F_{xt}\right\}(x,t-x) \\
&= \int_{t-x}^{x} \frac{\partial}{\partial s}\left\{F_t(x,t-s)G_t(x,s) - F_x(x,t-s)G_x(x,s)\right. \\
&\qquad \left. - q(x)F(x,t-s)G(x,s)\right\}ds \\
&\quad + \{''\} + \{''\} - p'(x)\left(G_t+G_x\right)(x,t-x) \\
&\quad - p(x)q(x)G(x,t-x) + \frac{1}{2}q(x)F(x,t-x) \\
&= F_t(x,t-x)G_t(x,x) - F_x(x,t-x)G_x(x,x) - q(x)F(x,t-x)G(x,x) \\
&\quad - F_t(x,x)G_t(x,t-x) + F_x(x,x)G_x(x,t-x) + q(x)F(x,x)G(x,t-x) \\
&\quad + \{''\} + \{''\} - p'(x)\left(G_t+G_x\right)(x,t-x) \\
&\quad - p(x)q(x)G(x,t-x) + \frac{1}{2}q(x)F(x,t-x). \\
&= F_t(x,t-x)\frac{d}{dx}(G(x,x)) - F_x(x,t-x)\frac{d}{dx}(G(x,x)) \\
&\quad - G_t(x,t-x)\frac{d}{dx}(F(x,x)) + G_x(x,t-x)\frac{d}{dx}(F(x,x)) \\
&\quad - p'(x)\left(G_t+G_x\right)(x,t-x) \\
&= p'(x)(G_x-G_t)(x,t-x) - p'(x)(G_t+G_x)(x,t-x) \\
&= -2p'(x)G_t(x,t-x) = -\frac{q(x)}{2}G_t(x,t-x).
\end{aligned}$$

Therefore for $0 \le t \le 2x$

$$\frac{\partial N}{\partial x}(x,t) = -q(x)G_t(x,t-x) \tag{28}$$

Now

$$N(x,t) * H(t) = \int_{-\infty}^{\infty} N(x,s)H(t-s)ds = \int_{-\infty}^{t} N(x,s)ds.$$

We observe $N(x,t) * H(t)$ is even in t because

$$(N * H)(x,-t) \quad = \quad \int_{-\infty}^{-t} N(x,s)ds = \int_{t}^{\infty} N(x,-s)ds$$

$$\begin{aligned} &= -\int_t^\infty N(x,s)ds \qquad \text{(because } N \text{ is odd in } t) \\ &= \int_{-\infty}^t N(x,s)ds \qquad \text{(because } \int_{-\infty}^\infty N(x,s)ds = 0) \end{aligned}$$

and since $N(x,t)$ is supported in $|t| \leq 2x$ and is odd in t, one may show that $N(x,t) * H(t)$ is supported in $|t| \leq 2x$. Further, for $|t| \leq 2x$,

$$(N * H)(x,t) = \int_{-\infty}^t N(x,s)ds = \int_{-2x}^t N(x,s)ds = -\int_t^{2x} N(x,s)ds.$$

So $(N * H)(x, \pm 2x) = 0$ and for $0 \leq t \leq 2x$, using (28) and (27)

$$\begin{aligned} \frac{\partial}{\partial x}(N * H(x,t)) &= -2N(x,2x) - \int_t^{2x} \frac{\partial N}{\partial x}(x,s)ds \\ &= -2p'(x) + q(x)\int_t^{2x} G_s(x,s-x)ds \\ &= -2p'(x) + q(x)\{G(x,x) - G(x,t-x)\} \\ &= -2p'(x) + \frac{1}{2}q(x) - q(x)G(x,t-x) = -q(x)G(x,t-x) \end{aligned}$$

Now from (19)

$$q(x)G(x,t-x) = (G_{xx} - G_{tt})(x,t-x) = \frac{\partial}{\partial x}\{(G_x + G_t)(x,t-x)\}$$

Hence

$$\frac{\partial}{\partial x}(N * H(x,t)) = -\frac{\partial}{\partial x}\{(G_t + G_x)(x,t-x)\}$$

Integrating with respect to x over the interval $[\frac{t}{2}, x]$,

$$(N * H)(x,t) - (N * H)(\frac{t}{2}, t) = (G_x + G_t)(\frac{t}{2}, \frac{t}{2}) - (G_x + G_t)(x, t-x)$$

and noting that $(N * H)(s, 2s) = 0$, $G(x,x) = 1/2$, we have

$$(N * H)(x,t) = -(G_x + G_t)(x, t-x).$$

if $0 \leq t \leq 2x$. Since $N * H$ is even in t and G is even in t, we get

$$N(x,t) * H(t) = -(G_x + G_t)(x,t-x) - (G_x - G_t)(x,t+x)$$

proving Proposition 4.

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