IMPEDANCE INVERSION FROM TRANSMISSION DATA FOR THE WAVE EQUATION

Rakesh
Rees Hall
University of Delaware
Newark, DE 19716
rakesh@math.udel.edu

Paul Sacks
Department of Mathematics
Iowa State University
Ames, IA 50011
psacks@iastate.edu

March 5, 1996

1 Introduction

Suppose $\eta(x)$ is a positive function in $W^1_\infty(0,\infty)$ (i.e. one bounded derivative), is constant for $x > X$ for some known positive number X. Consider the initial boundary value problem

$$\eta(x)u_{tt}(x,t) - (\eta(x)u_x(x,t))_x = 0, \qquad 0 \le x,\ t \in R \tag{1}$$
$$u(x,t) = 0, \qquad 0 \le x,\ t < 0 \tag{2}$$
$$u_x(0,t) = -\,\delta(t), \qquad t \in R \tag{3}$$

where δ is the usual Dirac delta function. From causality, u is zero for $t < x$, and using the progressing wave expansion, for $t \ge x$, u solves the characteristic boundary value problem (see e.g. [1])

$$\eta(x)u_{tt}(x,t) - (\eta(x)u_x(x,t))_x = 0, \qquad 0 \le x \le t \tag{4}$$
$$u(x,x) = \left(\frac{\eta(0)}{\eta(x)}\right)^{1/2}, \qquad 0 \le x \tag{5}$$
$$u_x(0,t) = 0, \qquad 0 \le t \tag{6}$$

Using energy estimates, it may be shown that $u(x,\cdot)$ is in $H^1_{loc}(x,\infty)$ and its H^1 norm over any finite time interval is uniformly bounded in x. So one may define the map

$$\begin{aligned} \mathcal{S} : W^1_{\infty,+}[0,X] &\longrightarrow H^1[X,3X] \\ \frac{\eta(\cdot)}{\eta(0)} &\longmapsto u(X,\cdot)|_{[X,3X]} \end{aligned}$$

Here $W^1_{\infty,+}[0,X]$ is the cone consisting of positive functions in $W^1_\infty[0,X]$. The inverse problem consists of analyzing the inverse of $\mathcal{S}$. In this article we prove the injectivity of $\mathcal{S}$, so $\mathcal{S}$ is invertible on its range, and we show how to invert $\mathcal{S}$. We do not characterize the range of $\mathcal{S}$ or make a detailed study of the map $\mathcal{S}^{-1}$.

Much work has been done on one dimensional inverse problems for the wave equation but almost all of it has been on inversion from reflection data (source and receiver are at the same location) i.e. from $u(0,t)$ and more. Inversion from reflection data generated by an impulsive source has been thoroughly analyzed - see [1], [2], [3], [4], and [5] and references mentioned there. Inversion from transmission plus reflection data is analyzed in [6]. Inversion from reflection data is in some sense a local problem since $u(0,t)$ for $0 < t < T$ is influenced by the value of $\eta(x)$ only if $0 < x < T/2$. This allows the use of so-called layer stripping techniques to resolve the inverse reflection problem. In contrast, for our problem - inversion from transmission data $u(X,t)$ - even the earliest signal sensed at $x = X$ has been influenced by the complete medium $0 \le x \le X$. This makes inversion from transmission data more difficult.

If the transmission data is given for all time i.e. $u(X,t)$ is known for all real t, then Carroll and Santosa showed in [7] that one could recover the reflection data $u(0,t)$ for all t, hence one could recover η. It is clearly of interest to study the situation when the transmission data is known for only a finite time interval. The finite time interval situation was analyzed by Maheswaran in [8] who showed that if $\log\eta$ has sufficiently small $H^1(0,X)$ norm then it is uniquely determined by the data $u(X,t)$ for $X \le t \le 3X$; he also studied a numerical reconstruction procedure there.

In the case of a piecewise constant medium (for which $\log\eta$ is obviously not in W^1_∞) a uniqueness result was actually proved much earlier by Claerbout in [9]. He assumed that the medium is Goupillaud layered, that is, one can subdivide the medium $[0,X]$ into subintervals of equal length, on each of which $\eta(x)$ is constant. He cleverly converted the inverse transmission problem into a local inversion problem which he then tackled by a discrete layer stripping method. It is quite instructive to read Chapter 8 of [9].

The purpose of this paper is to prove an analogue of Claerbout's result for continuously varying media. Aside from its intrinsic interest, we believe that this derivation in purely wave equation-theoretic terms will also provide a different and possibly useful conceptual understanding of the original method in [9]. Finally, the proof of the injectivity of $\mathcal{S}$ leads immediately to a reconstruction algorithm - results from its implementation are presented in Section 4.

The main result in this article is

Theorem 1 *Assume* $\log\eta \in W^1_\infty(0,2X)$, *and* η *is constant for* $X \le x \le 2X$. *Then* $\frac{\eta(x)}{\eta(0)}$ *is uniquely determined by* $u(X,t)$, $X \le t \le 3X$.

The main step in the proof is to relate the data of the transmission problem to that of a "reflection type" problem for which one can then apply known techniques. Here is the precise statement, assuming $\eta(0) = 1$ without loss of generality. Consider the Goursat problem

$$\eta M_{tt} - (\eta M_x)_x = 0, \qquad |t| \le x \le X \tag{7}$$
$$M(x, \pm x) = \pm\left(1 - \eta^{-1/2}(x)\right), \qquad 0 \le x \le X \tag{8}$$

If $\log\eta$ is in W^1_∞ and $\eta(0) = 1$, then using standard energy estimates one can prove that (7), (8) has a unique solution in H^1. Define $f(t) \equiv u(X, t + X)$.

Theorem 2 *Assume* $\log\eta \in W^1_\infty(0,2X)$, $\eta(0) = 1$, *and* η *is constant for* $X \le x \le 2X$. *Then for* $|t| \le X$,

$$M_t(X,t) = \eta(X)^{-1/2}(a(X+t) + a(X-t)) \tag{9}$$
$$M_x(X,t) = \eta(X)^{-1/2}(a(X+t) - a(X-t)) \tag{10}$$

where
$$f'(t) + \eta(X)^{-1/2}a(t) + \int_0^t a(r)f'(t-r)\,dr = 0, \qquad 0 \le t \le 2X \tag{11}$$

Given $u(X,t)$ on $[X,3X]$, (11) may be solved for a on $[0,2X]$ so that the Cauchy data $M_t(X,t)$, $M_x(X,t)$, are then known via (9), (10) for $|t| < X$. Then the proof of Theorem 1 is concluded by observing that well known results for the reflection inverse problem imply that η is uniquely determined by this boundary data for M, see the next section.

Along the way we will show that if $\sigma = \eta'/\eta$ and $A(x,t)$ solves

$$A_x(x,t) = \frac{\sigma(x)}{2}A(x,2x-t) \quad 0 \le \frac{t}{2} \le x \le X \tag{12}$$
$$A(x,2x) = \frac{\sigma(x)}{4} \quad 0 \le x \le X \tag{13}$$

then $A(X,t) = a(t)$ where $a(t)$, $0 \le t \le 2X$ is determined by $u(X,t)$, $X \le t \le 3X$ through (11). It is this set of equations which most closely parallels those in [9], and one could prove Theorem 1 by showing that σ may be recovered from $A(X,t)$, $0 \le t \le 2X$ by a contraction mapping argument. Instead, we have chosen to transform (12), (13) to (7), (8) - a problem which arises in doing inversion from reflection data, and appeal to known results for inversion from reflection data. We have done so to bring out the relationship between inversion from reflection and transmission data. We should emphasize, however, that this connection seems to be completely mathematical in nature, that is, we do not see any straightforward physical argument predicting this kind of relationship between the transmission and reflection inverse problems.

In (1) - (3) the boundary condition at $x = 0$ plays a significant role. In our problem it is $u_x(0,t) = 0$ for $t > 0$ and we expect the result to be true even if this boundary condition is replaced by $u(0,t) + hu_x(0,t) = 0$ (for any real h). However, if we replace it by $u_x(0,t) - u_t(0,t) = 0$, another natural boundary condition modeling complete absorption of the waves at $x = 0$, then one cannot recover η. For example, in the piecewise constant case, the three layer media $(\eta_1, \eta_2, \eta_3) = (1, 1/3, 1/6)$ has the same transmission data as $(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) = (1, 1/2, 1/6)$ See [9] for the construction of transmission data for a piecewise constant medium. Alternatively, if one formally linearizes the mapping $\eta \longmapsto u(X,\cdot)$ at $\eta \equiv 1$, one finds for this choice of boundary condition that the linearized mapping is zero (see [8] for a similar calculation) thus again indicating that uniqueness will fail. It may also be worth mentioning that in the case of the absorbing boundary condition, the inverse problem is formally equivalent to the standard one dimensional inverse scattering problem for the Schrödinger equation, (e.g. Chapter XVII of [10]) when the transmission coefficient $T(k)$ is the given data. This in turn is equivalent to the same inverse scattering problem when $|R_\pm(k)|$, the amplitude only of either the left or right hand reflection coefficient, is the prescribed data. So called phaseless inverse scattering problems have received much attention in recent years, see e.g. [11] for more discussion and references.

Since the speed of propagation is one, $u(X,t)$ is zero for $t < X$, and we require that $u(X,t)$ be known for $t \in [X, 3X]$, a time period twice the length of the medium. This would allow enough time for the right end receiver to sense a signal which originates at the left end, is reflected by the right end, and is reflected again by the left end. For a Goupillaud medium, the number of unknown constants (values of η) equals the number of distinct pieces of data when $u(X,.)$ is measured over $[X, 3X]$. So, given any proper subinterval $[X, Y]$ of $[X, 3X]$, one should be able to construct two distinct piecewise constant η which would have the same response at $x = X$ over the time interval $[X, Y]$, i.e. a smaller interval of measurement would not be adequate to fully reconstruct η.

2 Proofs of Theorems

Proof of Theorem 1

Since $f(t) = u(X, t+X)$, $0 \leq t \leq 2X$ is given and is in H^1, we have $f' \in L^2(0, X)$. Hence, we may determine the unique solution $a \in L^2(0, 2X)$ of the Volterra equation (11). So, from Theorem 2, we know $M_t(X,t), M_x(X,t)$, (which are in $L^2(-X, X)$), and to prove Theorem 1 it is enough to show that η is uniquely determined from $M_t(X,t), M_x(X,t)$, $|t| \leq X$, where $M(x,t)$ solves the Goursat problem (7), (8).

Introduce the change of variables $y = X - x$, $s = X + t$ which maps the region $|t| \leq x \leq X$ to $y \leq s \leq 2X - y$, $0 \leq y \leq X$. Define

$$w(y,s) = 1 - M(x,t), \qquad \rho(y) = \eta(x)$$

Then $w(y,s)$ satisfies

$$\rho w_{ss} - (\rho w_y)_y = 0 \qquad y \leq s \leq 2X - y, \; 0 \leq y \leq X \tag{14}$$

$$w(y,y) = \rho(y)^{-1/2} \qquad 0 \leq y \leq X \tag{15}$$

and $w_s(0,s), w_y(0,s)$ are known for $0 \leq s \leq 2X$. Appealing to Theorem 2 and Lemma 4 of [1], which deals with inversion from reflection data, we can recover $\rho(y)$, $0 \leq y \leq X$, and hence $\eta(x)$ on $0 \leq x \leq X$. Note that in [1], Theorem 2 is proved under the assumption $w_y(0, \cdot) = 0$, but the proof goes through without this assumption because Lemma 4 is proved without this assumption.

Proof of Theorem 2

We first convert our second order equation to a first order system for left and right going waves. Also, the factor $\eta^{1/2}$ appears in many places so we normalize expressions to remove that. Define

$$
\begin{aligned}
L(x,t) &= \frac{\eta^{1/2}(x)}{2}\left(u(x,t)+\int_{-\infty}^{t} u_x(x,s)\,ds\right)\\
R(x,t) &= \frac{\eta^{1/2}(x)}{2}\left(u(x,t)-\int_{-\infty}^{t} u_x(x,s)\,ds\right)
\end{aligned}
$$

Recall $\sigma=\eta'/\eta$. Then, using (1), we may verify that

$$
L_x = L_t + \frac{\sigma}{2}R, \quad R_x = -R_t + \frac{\sigma}{2}L, \qquad x\ge 0, \quad t\in\mathcal{R} \tag{16}
$$

Further, using $\eta(0)=1$ and (3) we have

$$
L(0,t) = \frac{1}{2}(u(0,t)-H(t)) \equiv v_0(t) \tag{17}
$$

$$
R(0,t) = \frac{1}{2}(u(0,t)+H(t)) = v_0(t)+H(t) \tag{18}
$$

in which H denotes the Heaviside function. Now η is constant for $X<x<2X$, and by causality we may as well assume that it is defined and constant for all $x>X$, so that u satisfies the constant coefficient wave equation in the region $x\ge X$, $t\in\mathcal{R}$. Also, $u=0$ for $t<0$, hence the region $x\ge X$ will have only right moving waves. So $u_x+u_t=0$ in $x\ge X$, in particular when $x=X$. Hence

$$
L(X,t) = 0 \tag{19}
$$

$$
\begin{aligned}
R(X,t) &= \frac{\eta^{1/2}(X)}{2}\left(u(X,t)-\int_{-\infty}^{t} u_x(X,s)\,ds\right)\\
&= \frac{\eta^{1/2}(X)}{2}\left(u(X,t)+\int_{-\infty}^{t} u_t(X,s)\,ds\right)\\
&= \eta^{1/2}(X)u(X,t) = u(X,t)/u(X,X)
\end{aligned} \tag{20}
$$

Note that $u(X,X)=\eta(X)^{-1/2}$ from (5). For the inverse problem, we are given $u(X,t)$, i.e. $L(X,t)$ and $R(X,t)$ are known, and also we know that $R(0,t)-L(0,t)=H(t)$ but we do not know $R(0,t)$ or $L(0,t)$, and we wish to determine η. So the relationship between the values of R and L at $x=0$ and $x=X$ and the value of η over $0\le x\le X$ will be crucial. This relationship will be established with the help of the Green's function for the time-like Cauchy problem for (16).

The time-like Cauchy problem for (16) is

$$
L_x = L_t + \frac{\sigma}{2}R, \qquad R_x = -R_t + \frac{\sigma}{2}L, \qquad 0\le x,\ t\in\mathcal{R} \tag{21}
$$

$$
L(0,t)=L_0(t), \qquad R(0,t)=R_0(t), \qquad t\in\mathcal{R} \tag{22}
$$

If $\log\eta\in W^1_\infty$ and $L_0,R_0\in L^2(-\infty,\infty)$ then (21), (22) has a unique solution for which $u(x,\cdot)\in L^2(-\infty,\infty)$ and is uniformly bounded in x in this norm. This follows from standard arguments for initial value problems with the roles of x and t reversed. To derive the relationship between $L(0,\cdot), R(0,\cdot), L(X,\cdot)$ and $R(X,\cdot)$ we will need a representation of the solution of (21), (22).

Consider, the Goursat problem

$$F_x(x,t) = F_t(x,t) + \frac{\sigma(x)}{2} G(x,t), \quad G_x(x,t) = -G_t(x,t) + \frac{\sigma(x)}{2} F(x,t), \quad |t| \le x \tag{23}$$

$$F(x,x) = 0, \quad G(x,-x) = \frac{\sigma(x)}{4} \tag{24}$$

If $\log \eta \in W^1_\infty$ then, using energy estimates, one may prove that (23), (24) has a unique solution $u(x,\cdot) \in L^2(-x,x)$, uniformly bounded in x in this norm. The next Lemma claims that this solution plays the role of the Green's function for (21), (22).

Lemma 1 *If $\log \eta \in W^1_\infty$, then the solution of (21), (22) has the representation*

$$\begin{aligned} L(x,t) &= L_0(t+x) + \int_{t-x}^{t+x} F(x,t-s) L_0(s)\, ds \ + \int_{t-x}^{t+x} G(x,s-t) R_0(s)\, ds & (25)\\ R(x,t) &= R_0(t-x) + \int_{t-x}^{t+x} G(x,t-s) L_0(s)\, ds \ + \int_{t-x}^{t+x} F(x,s-t) R_0(s)\, ds & (26) \end{aligned}$$

Using (25), (26) when $x = X$, and (17) - (20), we obtain

$$\begin{aligned} 0 &= v_0(t+X) + \int_{t-X}^{t+X} F(X,t-s) v_0(s)\, ds \ + \\ &\quad \int_{t-X}^{t+X} G(X,s-t)(v_0(s) + H(s))\, ds & (27)\\ u(X,t)/u(X,X) &= v_0(t-X) + H(t-X) + \int_{t-X}^{t+X} G(X,t-s) v_0(s)\, ds \ + \\ &\quad \int_{t-X}^{t+X} F(X,s-t)(v_0(s) + H(s))\, ds & (28) \end{aligned}$$

Since v_0 is unknown we wish to eliminate it between the two equations. We do so as follows. In (27) replace t by $t - X$ and in (28) replace t by $t + X$. We obtain

$$\begin{aligned} 0 &= v_0(t) + \int_{t-2X}^{t} F(X,t-X-s) v_0(s)\, ds \ + \\ &\quad \int_{t-2X}^{t} G(X,s-t+X)(v_0(s) + H(s))\, ds & (29)\\ u(X,t+X)/u(X,X) &= v_0(t) + H(t) + \int_{t}^{t+2X} G(X,t+X-s) v_0(s)\, ds \ + \\ &\quad \int_{t}^{t+2X} F(X,s-t-X)(v_0(s) + H(s))\, ds & (30) \end{aligned}$$

Extend F and G to be zero outside the region $|t| \le x$ and define

$$a(t) \equiv F(X, t-X) + G(X, X-t), \qquad b(t) \equiv G(X, t+X) + F(X, -X-t)$$

Then (29), (30) may be written as

$$\begin{aligned} 0 &= \{\delta(t) + a(t)\} * v_0(t) + G(X, X-t) * H(t) & (31) \\ u(X, t+X)/u(X,X) &= \{\delta(t) + b(t)\} * v_0(t) + \{\delta(t) + F(X, -X-t)\} * H(t) & (32) \end{aligned}$$

So to eliminate v_0 from the equations we convolve (31) with $\delta + b$ and subtract it from (32) convolved with $\delta + a$. We obtain

$$\begin{aligned} &\{\delta(t) + a(t)\} * u(X, t+X)/u(X,X) \\ &= \{\delta(t) + a(t)\} * \{\delta(t) + F(X, -X-t)\} * H(t) - \{\delta(t) + b(t)\} * G(X, X-t) * H(t) \\ &= H(t) + H(t) * \{ a(t) + F(X, -X-t) - G(X, X-t) \\ &\qquad + a(t) * F(X, -X-t) - b(t) * G(X, X-t) \} \\ &= H(t) + P(X,t) * H(t) & (33) \end{aligned}$$

where

$$\begin{aligned} P(x,t) \equiv\ & F(x, -x-t) + F(x, t-x) + F(x, t-x) * F(x, -x-t) \\ & - G(x, t+x) * G(x, x-t) & (34) \end{aligned}$$

the convolution being only in the t variable. We will show

Lemma 2 *$P(x,t) = 0$ for all $x \ge 0$ and all t.*

We will say more about this magical identity in Section 3. We now continue with the proof of Theorem 2.

From (33) and Lemma 2

$$u(X, t+X) + a(t) * u(X, t+X) = u(X,X)H(t) \tag{35}$$

Define

$$A(x,t) \equiv F(x, t-x) + G(x, x-t)$$

Then, A is zero outside $0 \le t \le 2x$, so we will think of A as defined only on $0 \le t \le 2x$. Using (23), (24) one may verify that (12) and (13) hold. Further, note that $a(t) = A(X,t)$, so if we define $f(t) \equiv u(X, X+t)$ then (35) may be written as

$$f(t) + \int_0^t a(r) f(t-r) = u(X,X) \qquad 0 \le t \le 2X$$

Differentiating this with respect to t and noting $f(0) = u(X,X) = \eta(X)^{-1/2}$ we obtain (11).

Now define

$$M(x,t) \equiv \eta(x)^{-1/2} \int_{x-t}^{x+t} A(x,s)\, ds \qquad 0 \le |t| \le x \tag{36}$$

Then, for $0 \le |t| \le x$, using (36) and (12)

$$\begin{aligned}
M_t(x,t) &= \eta(x)^{-1/2}(A(x,x+t) + A(x,x-t)) && (37)\\
M_x(x,t) &= \eta(x)^{-1/2}\left(A(x,x+t) - A(x,x-t) + \int_{x-t}^{x+t} A_x(x,s)\, ds\right)\\
&\quad - \frac{\eta'(x)}{2\eta(x)^{3/2}} \int_{x-t}^{x+t} A(x,s)\, ds\\
&= \eta(x)^{-1/2}\left(A(x,x+t) - A(x,x-t) + \frac{\sigma(x)}{2}\int_{x-t}^{x+t} A(x,2x-s)\, ds\right)\\
&\quad -\frac{\sigma(x)}{2\eta(x)^{1/2}} \int_{x-t}^{x+t} A(x,s)\, ds\\
&= \eta(x)^{-1/2}\left(A(x,x+t) - A(x,x-t)\right) && (38)
\end{aligned}$$

Noting that $a(t) = A(X,t)$ we see that (9), (10) are valid. Further, using calculations similar to the ones above one may may show that (7) is valid. Finally, to prove (8), using (13), for $0 \le x \le X$

$$\frac{d}{dx}M(x,x) = M_x(x,x) + M_t(x,x) = 2\eta(x)^{-1/2}A(x,2x) = \frac{\eta'(x)}{2\eta(x)^{3/2}}$$

Noting that $\eta(0) = 1$, and from (36) that $M(0,0) = 0$ and M is odd in t, one may show that M satisfies (8). This completes the proof of Theorem 2.

3 Proofs of Lemmas

Proof of Lemma 1

Proving this Lemma is just a matter of verification. It is clear from (25), (26) that (22) is true. Now we verify one of the equations in (21) - the proof of the other one is quite similar. From (25) we have

$$\begin{aligned}
L_x(x,t) &= L_0'(t+x) + F(x,-x)L_0(t+x) + F(x,x)L_0(t-x) + \int_{t-x}^{t+x} F_x(x,t-s)L_0(s)\, ds\\
&\quad + G(x,x)R_0(t+x) + G(x,-x)R_0(t-x) + \int_{t-x}^{t+x} G_x(x,s-t)R_0(s)\, ds\\
L_t(x,t) &= L_0'(t+x) + F(x,-x)L_0(t+x) - F(x,x)L_0(t-x) + \int_{t-x}^{t+x} F_t(x,t-s)L_0(s)\, ds
\end{aligned}$$

$$+ G(x,x)R_0(t+x) - G(x,-x)R_0(t-x) - \int_{t-x}^{t+x} G_t(x,s-t)R_0(s)\, ds$$

So, using (23), (24), and (26)

$$\begin{aligned} L_x(x,t) - L_t(x,t) &= 2F(x,x)L_0(t-x) + \int_{t-x}^{t+x} (F_x - F_t)(x,t-s)L_0(s)\, ds \\ &\quad + 2G(x,-x)R_0(t-x) + \int_{t-x}^{t+x} (G_x + G_t)(x,s-t)R_0(s)\, ds \\ &= \frac{\sigma(x)}{2}\int_{t-x}^{t+x} G(x,t-s)L_0(s)\, ds + \frac{\sigma(x)}{2}R_0(t-x) \\ &\quad + \frac{\sigma(x)}{2}\int_{t-x}^{t+x} F(x,s-t)R_0(s)\, ds \\ &= \frac{\sigma(x)}{2}R(x,t) \end{aligned}$$

Similarly the R equation may be established.

Above, we have been differentiating and using the Fundamental Theorem of Calculus for functions which are actually only in L^2. We have chosen the above presentation for clarity. These calculations could be rewritten without such operations, using integrals. To establish (21), (22) one would prove instead that L and R, given by (25), (26), satisfy

$$\begin{aligned} L(x,t) - L_0(t+x) &= \int_0^x \frac{\sigma(r)}{2}R(r,t+x-r)\, dr \\ R(x,t) - R_0(t-x) &= \int_0^x \frac{\sigma(r)}{2}L(r,t-x+r)\, dr \end{aligned}$$

using the following integral form of (23), (24) for $a \geq 0,\ s \geq 0$

$$\begin{aligned} F(a+s,a-s) &= \int_0^s \frac{\sigma(a+r)}{2}G(a+r,a-r)\, dr \\ G(a+s,s-a) &= \frac{\sigma(a)}{4} + \int_0^s \frac{\sigma(a+r)}{2}F(a+r,r-a)\, dr \end{aligned}$$

This completes the proof of Lemma 1.

Proof of Lemma 2

From (34) we have

$$\begin{aligned} P(x,t) = F(x,-x-t) + F(x,t-x) &+ \int_{-\infty}^{\infty} F(x,t-r-x)F(x,-x-r)\, dr \\ &- \int_{-\infty}^{\infty} G(x,t-r+x)G(x,x-r)\, dr \\ = F(x,-x-t) + F(x,t-x) &+ \int_{-\infty}^{\infty} \{F(x,t+s)F(x,s) - G(x,t+s)G(x,s)\}\, ds \quad (39) \end{aligned}$$

We first note that $P(x,t)$ is even in t, because

$$\begin{aligned}
P(x,-t) &= F(x,-x+t)+F(x,-t-x) \\
&+ \int_{-\infty}^{\infty}\{F(x,-t+s)F(x,s)-G(x,-t+s)G(x,s)\}\,ds \\
&= F(x,-x+t)+F(x,-t-x)+\int_{-\infty}^{\infty}\{F(x,p)F(x,p+t)-G(x,p)G(x,p+t)\}\,dp \\
&= P(x,t)
\end{aligned}$$

Since the support of F and G is $|x| \leq t$, one may observe that $P(x,t)$ is zero if $|t| > 2x$. Also, noting (24), one obtains $P(x,\pm 2x) = 0$. So, to prove that $P(x,t) = 0$ for $0 \leq t \leq 2x$ we shall establish that $P_x(x,t) = 0$ in $0 \leq t \leq 2x$.

For $0 < t < 2x$, noting the supports of F and G, from (39),

$$P(x,t) = F(x,t-x) + \int_{-x}^{x-t}\{F(x,t+s)F(x,s)-G(x,t+s)G(x,s)\}\,ds$$

(note one of the terms has dropped out because it is zero) so

$$\begin{aligned}
P_x(x,t) &= (F_x-F_t)(x,t-x)+F(x,x)F(x,x-t)-\ G(x,x)G(x,x-t) \\
&+ F(x,t-x)F(x,-x)-G(x,t-x)G(x,-x) \\
&+ \int_{-x}^{x-t}\{F_x(x,t+s)F(x,s)+F(x,t+s)F_x(x,s)\}\,ds \\
&- \int_{-x}^{x-t}\{G_x(x,t+s)G(x,s)+G(x,t+s)G_x(x,s)\}\,ds
\end{aligned} \tag{40}$$

Let us examine the integrand inside the two integrals. Using (23),

$$\begin{aligned}
\text{integrand} &= (F_t+\frac{\sigma}{2}G)(x,t+s)F(x,s)+F(x,t+s)(F_t+\frac{\sigma}{2}G)(x,s) \\
&- (-G_t+\frac{\sigma}{2}F)(x,t+s)G(x,s)-G(x,t+s)(-G_t+\frac{\sigma}{2}F)(x,s) \\
&= F_t(x,t+s)F(x,s)+F(x,t+s)F_t(x,s)+G(x,t+s)G_t(x,s)+G_t(x,t+s)G(x,s) \\
&= \frac{d}{ds}\{F(x,t+s)F(x,s)+G(x,t+s)G(x,s)\}
\end{aligned}$$

Using this in (40) and integrating and canceling terms we obtain

$$\begin{aligned}
P_x(x,t) &= (F_x-F_t)(x,t-x)+2F(x,x)F(x,x-t)-2G(x,t-x)G(x,-x) \\
&= \frac{\sigma(x)}{2}G(x,t-x)-2G(x,t-x)\frac{\sigma(x)}{4}\ = 0
\end{aligned}$$

where we have used (23) and (24) above.

Again, we have been differentiating and using the Fundamental Theorem of Calculus for functions which are actually only in L^2. These calculations could be rewritten without such operations. This completes the proof of Lemma 2.

Remark on Lemma 2

Lemma 2 seems almost magical. It deals with the Green's functions for (21) - the equation for the left and right going waves. We have not attempted to determine its counterpart for solutions of the original equation (1). However, the following seems to be the equivalent crucial observation. If $v(x,t)$ solves

$$\eta(x)v_{tt}(x,t) - (\eta(x)v_x(x,t))_x = 0, \qquad x,t \in R$$

define

$$E(x,t) = \eta(x)\int v_t(x,t-r)v_x(x,-r) + v_x(x,t-r)v_t(x,-r)\,dr$$

then one may show that

$$\frac{\partial E}{\partial x}(x,t) = -\int \frac{\partial}{\partial r}\{v_t(x,t-r)v_t(x,-r) + v_x(x,t-r)v_x(x,-r)\}\,dr$$

So if v had the "correct" boundary conditions to make the right hand side zero then $E(x,t)$ would be independent of x and would be zero if it was zero at $x=0$.

4 Numerical Results

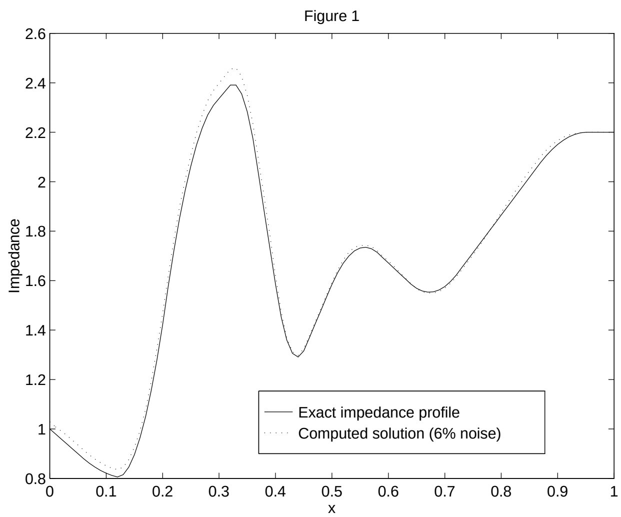
A numerical technique for the solution of the transmission inverse problem may now be presented. Given the data $u(X,t)$ for $X \le t \le 3X$ (equivalently $f(t) = u(X,t+X)$ for $0 \le t \le 2X$) we carry out the following four steps

1. Determine $\eta(X)$ - its value is $u(X,X)^{-2}$.
2. Solve the 2nd kind Volterra integral equation (11) for $a(t)$, $0 \le t \le 2X$.
3. Compute $M_t(X,t), M_x(X,t)$ for $-X \le t \le X$ from (9), (10).
4. Find $\eta(x)$, $0 < x < X$, using (7),(8) and the known $M_t(X,t), M_x(X,t)$ from step 3.

Only the fourth step calls for any further comment. As noted in the proof of Theorem 1, finding η from such data is equivalent to the following inverse problem: Find $\rho(y), 0 < y < X$ such that there exists $w = w(y,s)$ defined in the domain $D = \{(y,s) : y < s < 2X - s, 0 < y < X\}$ satisfying

$$\begin{aligned} \rho(y)w_{ss} - (\rho(y)w_y)_y &= 0 \qquad (y,s) \in D && (41)\\ w(y,y) &= \rho^{-1/2}(y) \qquad 0 < y < X && (42)\\ w_s(0,s) = w_1(s) \quad w_y(0,s) &= w_2(s) \qquad 0 < s < 2X && (43) \end{aligned}$$

where w_1, w_2 are given functions. This is a common version of the inverse reflection problem of geophysics and a number of fast and accurate computational algorithms are known, see e.g. [2], [4], and [12]. We refer the interested reader to these articles for more discussion and further references. In the example given below the iterative method of [12] was employed (and we make no claim that this is the optimal method). Of course it is not necessary to make the changes of variable just noted, instead the numerical method may be easily adapted to the case at hand.



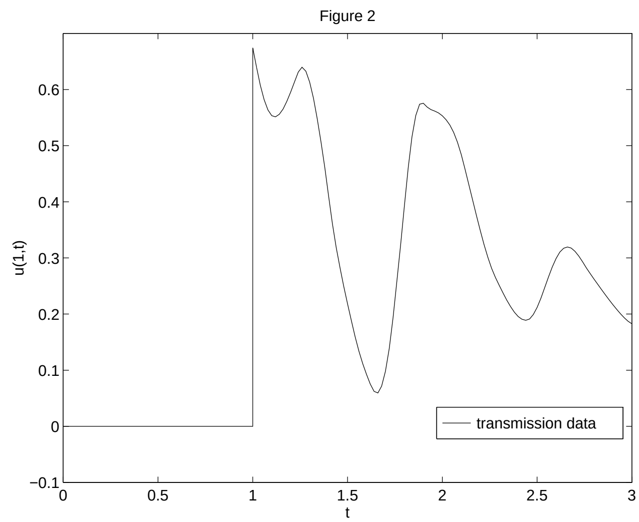


Figure 1 shows a target profile $\eta(x)$ and the corresponding transmission data $u(X,t), 0 < t < 3X$ is shown in figure 2. We have carried out the inversion procedure for this data, and again when the data has been contaminated by several kinds of numerically simulated random error. The interval length is $X = 1$, and the data is generated using a finite difference method for the Goursat problem (4) - (6). In Table 1 the $L^2(0,1)$ relative error in the reconstruction of both η and η' is displayed for the various choices of noise added to the data. In the first case no extra error is added, so that all the error in the reconstruction must be attributed to discretization error. In the second case the data used has the form $u_t(X,t)(1+\epsilon(t))$ where $||\epsilon||_{L^2(1,3)} \approx .06$. and the third is the same with relative error level $\approx .12$. Finally in the fourth case the relative error is again 6%, but now concentrated in the high frequency components of the data (i.e. $\hat{\epsilon}(k)$, the Fourier transform of ϵ is small for $k \approx 0$). Figure 1 also shows the reconstruction corresponding to the second line of the table.

It seems also worth mentioning here the counterpart of Claerbout's original method for carrying out the fourth step of the inversion procedure, which is to use the nonlocal ODE (12)-(13). More specifically it is clear from (37) and (38) that if $M_t(X,t), M_x(X,t)$ are known for $|t| < X$ then so are $A(X, X+t), A(X, X-t)$, i.e. one knows $A(X,t)$ for $0 < t < 2X$. Using this function as an initial condition the system (12)-(13) can be solved from right to left, updating σ in the process. If a step size Δx is chosen, then an obvious difference approximation to (12) yields an approximate solution for $A(X - \Delta x, t)$ for $0 < t < 2(x - \Delta x)$

% relative error in $u_t(X, \cdot)$	η	η'
0%	0.1%	1%
6%	1.7%	6%
12%	1.9%	8.4%
6% (high frequency)	0.3%	3.8%

Table 1: Relative $L^2(0,1)$ error in profile reconstruction

and then (13) provides a value for $\sigma(X - \Delta x)$. The procedure may then be repeated to step all the way across from $x = X$ to $x = 0$. To our knowledge no convergence analysis of this kind of scheme has been carried out.

5 Acknowledgments

This work was begun while the authors were visiting the Institute for Mathematics and its Applications. We wish to thank the IMA for its hospitality and financial support. The first author was on a sabbatical while the work was done and wishes to thank the University of Delaware. The second author was supported by the National Science Foundation under grant DMS-9504611. Finally, we wish to thank Bob Burridge for suggesting we examine [9] where we found the discrete version of our problem solved.

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