

RECONSTRUCTION FOR AN INVERSE PROBLEM FOR THE WAVE
EQUATION WITH CONSTANT VELOCITY

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SHORT TITLE - RECONSTRUCTION FOR AN INVERSE PROBLEM

classification - 0260

Abstract

Let $\Omega \subset R^n$, $n > 1$ be a bounded domain with smooth boundary. Consider

$$u_{tt} - \Delta_x u + q(x)u = 0 \quad \text{in} \quad \bar{\Omega} \times [0, T]$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0 \quad \text{if} \quad x \in \Omega$$

$$u(x, t) = f(x, t) \quad \text{on} \quad \partial\Omega \times [0, T]$$

Define the Dirichlet to Neumann map

$$\begin{aligned} \Lambda_q : H^1(\partial\Omega \times [0, T]) &\longrightarrow L^2(\partial\Omega \times [0, T]) \\ f(x, t) &\longmapsto \frac{\partial u}{\partial n} \end{aligned}$$

where \vec{n} is the unit outward normal to $\partial\Omega$. If $T > \text{diam}(\Omega)$, we express $q(x)$ in terms of Λ_q , and show that if $q(x)$ is piecewise constant then $q(x)$ is uniquely determined by $\Lambda_q(f_i)$, $i = 1 \dots N$ for some integer N .

Our problem is motivated by a desire to obtain properties e.g. density of an inhomogeneous medium by probing it with disturbances generated on the boundary. The data is the response of the medium to these disturbances, measured on the boundary, and the goal is to recover the function which measures the property of the medium.

We consider a very simple model. Let $\Omega \subset R^n$, $n > 1$ be a bounded domain with smooth boundary, which is the region occupied by our medium. Suppose $q(x)$ is a function on Ω which represents some property of the medium. If $u(x, t)$ is a measure of the magnitude of the disturbance at the point x at time t , then the interaction of the medium with the disturbance is assumed to be modelled by

$$\begin{aligned} \square u + q(x)u &\equiv u_{tt} - \Delta_x u + q(x)u = 0 && \text{in } \bar{\Omega} \times [0, T] \\ (1) \quad u(x, 0) &= 0, \quad u_t(x, 0) = 0 && \text{if } x \in \Omega \\ u(x, t) &= f(x, t) && \text{on } \partial\Omega \times [0, T] \end{aligned}$$

Here we have assumed that the medium is quiet initially, and $f(x, t)$ is the disturbance which is used to probe the medium. Roughly speaking, the data is $\frac{\partial u}{\partial n}$ measured on $\partial\Omega \times [0, T]$, for different choices of $f(x, t)$, and the goal is to recover $q(x)$. Here \vec{n} is the outward pointing normal to $\partial\Omega$.

For a fixed $q(x)$, (1) is a well posed initial boundary value problem, hence one may define the Dirichlet to Neumann map

$$\begin{aligned} \Lambda_q : H^1(\partial\Omega \times [0, T]) &\longrightarrow L^2(\partial\Omega \times [0, T]) \\ f(x, t) &\longmapsto \frac{\partial u}{\partial n} \end{aligned}$$

So knowing Λ_q is equivalent to knowing the response of the medium, on the boundary, to all possible input disturbances.

In [3] Rakesh and Symes showed that the map

$$\begin{aligned} L^\infty(\Omega) &\longrightarrow B(H^1(\partial\Omega \times [0, T]), L^2(\partial\Omega \times [0, T])) \\ q &\longmapsto \Lambda_q \end{aligned}$$

is injective provided $T > \text{diameter}(\Omega)$. In [7] Sun showed that that the above map restricted to $H^s(\Omega)$, $s > n/2$, is an open map for $T > \text{diameter}(\Omega)$.

In this paper we prove two results. First we show that if $q(x)$ is piecewise constant on a known grid on Ω , then $q(x)$ is uniquely determined given $\Lambda_q(f)$ for a finite number of f (only a Uniqueness Theorem). Note that the uniqueness result in the general case needed a knowledge of $\Lambda_q(f)$ for an infinite

number of f . Next we give a representation of $q(x)$ in terms of Λ_q , valid for all $q \in L^\infty(\Omega)$. So it may be thought of as an inversion formula, except that it is not a very practical one.

REMARKS Romanov in [5] proved a uniqueness result for a related problem in the case when the region is a half plane and the (known) wave speed is of a special type. (It must not be constant.)

The derivations of the two results start from an identity which was motivated by one in [4] for elliptic differential equations. Many people have obtained a similar identity independently e.g. Sylvester and Uhlmann, Alessandrini, Nachman, Sun - to mention a few. The inversion formula is motivated by ideas used by Nachman in [1] to construct an inversion formula for an inverse problem for an elliptic differential equation. We also draw on ideas of Sylvester and Uhlmann in [6] and on our work with Bill Symes in [3]. It turns out that the hyperbolic case is far easier to handle because of the ease with which one can construct oscillatory solutions to hyperbolic equations.

Our result dealing with piecewise constant coefficients actually implies a continuous dependence result which is a consequence of the uniqueness result and the finite dimensionality of the space of piecewise constant $q(x)$. Similar results may be obtained in case q is built with splines rather than constant functions. This result is important because of its implications for numerical schemes to recover q from Λ_q .

The main idea is to probe the medium by highly oscillatory geometric optics solutions of the wave equation, concentrated along a line, starting on one side of the boundary, and measure the response of the medium on the other side of the boundary. The main part of the response is a term independent of $q(x)$ and another term based on the line integral of $q(x)$. If we use enough of these probing waves we will be able to recover $q(x)$.

Also note that the ideas do not carry over to unbounded domains, in particular half planes, because their boundaries do not have another side on which to measure the response i.e. the data for unbounded domains consists mostly of reflected energy unlike the bounded domain case where the data consists of reflected as well as transmitted energy (though we use only the transmitted part). That is why Romanov in [5], when dealing with the half-plane, had to use special kinds of wave velocity, and why he could not use the constant wave velocity.

DERIVATION OF IDENTITIES

Suppose

$$(2) \quad \begin{aligned} \square u + q(x)u &= 0 & \text{in } & \bar{\Omega} \times [0, T], \\ u(x, 0), \partial_t u(x, 0) &= 0 & \text{if } & x \in \Omega \\ u(x, t) &= f(x, t) & \text{on } & \partial\Omega \times [0, T] \end{aligned}$$

and

$$(3) \quad \begin{aligned} \square v &= 0 & \text{in } & \bar{\Omega} \times [0, T] \\ v(x, T), \partial_t v(x, T) &= 0 & \text{if } & x \in \Omega \\ v(x, t) &= g(x, t) & \text{on } & \partial\Omega \times [0, T] \end{aligned}$$

Then

$$(4) \quad \begin{aligned} \int_{\Omega \times [0, T]} dx dt q(x) u v &= - \int_{\Omega \times [0, T]} dx dt \square u v \\ &= - \int_{\Omega \times [0, T]} dx dt u \square v + \int_{\partial\Omega \times [0, T]} dS \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \\ &= \int_{\partial\Omega \times [0, T]} dS \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \\ &= \int_{\partial\Omega \times [0, T]} dS \left(v \Lambda_q(f) - f \frac{\partial v}{\partial n} \right) \end{aligned}$$

Given $\omega \in R^n$, $|\omega| = 1$, $\sigma > 0$, $\chi \in C_0^\infty(R^n)$, taking

$$\begin{aligned} f &= \chi(x + t\omega) e^{+i\sigma(x \cdot \omega + t)} \\ g &= \chi(x + t\omega) e^{-i\sigma(x \cdot \omega + t)} \end{aligned}$$

we can construct geometric optics solutions of (2) and (3), (concentrated near a line in (x, t) space if the support of χ is small), of the form

$$\begin{aligned} u &= \chi(x + t\omega) e^{+i\sigma(x \cdot \omega + t)} + R_1 \\ v &= \chi(x + t\omega) e^{-i\sigma(x \cdot \omega + t)} + R_2 \end{aligned}$$

with

$$(5) \quad \|R_i(x, t)\|_{L^2(\Omega \times [0, T])} \leq \frac{C}{\sigma} \quad i = 1, 2$$

and C depending only on Ω , T , $\|q\|_\infty$, and $\|\chi\|_{C^3(\bar{\Omega})}$, provided

$$(6) \quad \text{supp } \chi \cap \bar{\Omega} = \phi$$

$$(7) \quad \text{supp } \chi - T\omega \cap \bar{\Omega} = \phi$$

See [3] for details.

With this u and v we have

$$\begin{aligned}
 & \int_{\Omega \times [0, T]} dx dt q(x) u v = \\
 & \int_{\Omega \times [0, T]} \chi^2(x + t\omega) q(x) + \int_{\Omega \times [0, T]} \chi(x + t\omega) e^{-i\sigma(x \cdot \omega + t)} q(x) R_1 \\
 (8) \quad & + \int_{\Omega \times [0, T]} q(x) R_1 R_2 + \int_{\Omega \times [0, T]} \chi(x + t\omega) e^{+i\sigma(x \cdot \omega + t)} q(x) R_2
 \end{aligned}$$

UNIQUENESS FOR PIECEWISE CONSTANT COEFFICIENTS

THEOREM Suppose Ω is a bounded region in R^n with smooth boundary, and R^n has been subdivided into a rectangular grid. Given $\rho > 0$, we can find an integer k and functions f_1, \dots, f_k in $C_0^\infty(\partial\Omega \times [0, T])$ such that $\Lambda_q(f_i)$, $i = 1, \dots, k$ uniquely determines q , provided

- (i) q is constant on the grid rectangles
- (ii) q is zero on any grid rectangle not completely contained in Ω
- (iii) $\|q\|_\infty \leq \rho$

and T is greater than the length of the diagonal of the smallest rectangle (made up of grid rectangles) containing Ω . The f_i, k depend only on Ω, T, ρ, n , and the grid.

COROLLARY Given $\rho > 0$ and T as above, we can find an integer k and functions f_1, \dots, f_k in $C_0^\infty(\partial\Omega \times [0, T])$ such that for any q_1, q_2 satisfying (i), (ii), and (iii)

$$\|q_1 - q_2\|_\infty \leq C \max_{i=1, \dots, k} \|\Lambda_{q_1}(f_i) - \Lambda_{q_2}(f_i)\|_{L^2(\partial\Omega \times [0, T])}$$

with C depending only on Ω, T, ρ , and n .

We can weaken condition (ii) considerably - in fact we feel that condition (ii) may be dropped, but the geometry involved seems very complicated and we have decided to go with the result which we can prove easily. Also, the lower bound on the size of T can be sharpened much more without too much effort. Also, for the proof we need to measure the impulse response i.e. $\Lambda_q(f)$ only on certain parts of $\partial\Omega$.

PROOF

As explained in the remarks made earlier, the corollary follows very easily from the theorem, so we shall only prove the theorem.

We shall choose χ to be one of χ_1, \dots, χ_k , and ω to be one of $\omega_1, \dots, \omega_k$. Let u_j represent the solution of (2) for $j = 1, \dots, k$, when

$$f = f_j \equiv \chi_j(x + \omega_j t) e^{i\sigma(x \cdot \omega_j + t)}$$

Also let v_j represent the solution of (3) when

$$g = g_j \equiv \chi_j(x + \omega_j t) e^{-i\sigma(x \cdot \omega_j + t)}$$

Define maps A_j, B_j for $j = 1, \dots, k$ via

$$(9) \quad A_j(q) = \int_{\Omega \times [0, T]} dx dt \chi_j^2(x + t\omega_j) q(x)$$

$$(10) \quad \begin{aligned} B_j(q) &= \int_{\Omega \times [0, T]} \chi_j(x + t\omega_j) e^{-i\sigma(x \cdot \omega_j + t)} q(x) R_1^j \\ &+ \int_{\Omega \times [0, T]} \chi_j(x + t\omega) e^{+i\sigma(x \cdot \omega_j + t)} q(x) R_2^j \\ &+ \int_{\Omega \times [0, T]} q(x) R_1^j R_2^j \end{aligned}$$

Then for $j = 1, \dots, k$, from (4) and (8) we have

$$(11) \quad A_j(q) + B_j(q) = \int_{\partial\Omega \times [0, T]} dS (v_j \Lambda_q(f_j) - f_j \frac{\partial v_j}{\partial n})$$

Define maps

$$\begin{aligned} A, B : L^\infty(\Omega) &\longrightarrow R^k \\ A &= (A_1, \dots, A_k), \quad B = (B_1, \dots, B_k) \end{aligned}$$

then (11) implies

$$(12) \quad (A + B)(q) = \int_{\partial\Omega \times [0, T]} dS (\vec{v} \cdot \Lambda_q(\vec{f}) - \vec{f} \cdot \frac{\partial \vec{v}}{\partial n})$$

where the product is componentwise and

$$\begin{aligned} \vec{v} &= (v_1, \dots, v_k) \\ \vec{f} &= (f_1, \dots, f_k) \end{aligned}$$

We are going to restrict the domain of A and B to finite dimensional subspaces. Assume for the moment that A has a left inverse on this smaller, finite dimensional domain, then (12) will imply that

$$(13) \quad (I + A^{-1}B)(q) = A^{-1} \int_{\partial\Omega \times [0, T]} dS (\vec{v} \cdot \Lambda_q(\vec{f}) - \vec{f} \cdot \frac{\partial \vec{v}}{\partial n})$$

Since A is independent of σ so will be A^{-1} . From (5) and (10)

$$\|B\|_{L(L^\infty(\Omega), R^k)} \leq \frac{C}{\sigma}$$

So if we take σ large enough we will be assured that $I + A^{-1}B$ is invertible hence also injective. This would prove our theorem because v is known explicitly since v solves (3).

It remains to prove that A has a left inverse on a certain subspace of $L^\infty(\Omega)$. Denote by $L_{pwc}^\infty(\Omega)$ the finite dimensional subspace of $L^\infty(\Omega)$ consisting of those $q(x)$ which satisfy conditions (i) and (ii) of the Theorem. We shall show that we can find $k, \chi_j, \omega_j, j = 1, \dots, k$, such that the map

$$A : L_{pwc}^\infty \longrightarrow R^k$$

is injective, which would imply that A has a left inverse because the spaces involved are finite dimensional.

Since q is zero outside Ω , from (9)

$$\begin{aligned} A_j(q) &= \int_{R^n \times [0, T]} dx dt \chi_j^2(x + t\omega_j) q(x) \\ (14) \quad &= \int_{R^n} dx \chi_j^2(x) \int_0^T dt q(x - t\omega_j) \end{aligned}$$

Let $Rect(\Omega)$ be the smallest rectangle, made up of grid rectangles, which contains Ω . Then for proving the invertibility of A we may as well work with $Rect(\Omega)$ instead of Ω , provided we tighten the restrictions (6) and (7) on χ_j to

$$\begin{aligned} \text{supp } \chi_j \cap Rect(\Omega) &= \phi \\ \text{supp } \chi_j - T\omega_j \cap Rect(\Omega) &= \phi \end{aligned}$$

If $T - \text{diam}(Rect(\Omega)) > 4\epsilon > 0$ then the above conditions are satisfied if

$$(15) \quad \text{supp } \chi \subseteq Rect(\Omega)^c \cap \{\epsilon \text{ neighborhood of boundary of } Rect(\Omega)\}$$

From here onwards we cover only the case where Ω is 2 dimensional, because the ideas carry over to the higher dimensional case - only the presentation is cumbersome.

In (14) the second integral represents a line integral over the line starting at x , parallel to $-\omega_j$, of length T . So A_j represents a weighted integral of q along a tube of lines, the width of the tube determined by the support of χ_j . We choose χ_j, ω_j in the following fashion:

- All the χ 's chosen will satisfy (15)

- Referring to the Figure, we choose $w_{1,1}$ and $\chi_{1,1}$ so that $A_{1,1}$ is an integral along a tube of lines going from left to right but going through only the (1,1) rectangle. This can be done if the support of $\chi_{1,1}$ is chosen small enough. Hence $A_{1,1}q$ uniquely determines the value of q in the (1,1) rectangle.
- Next choose $\chi_{1,2}$, $\omega_{1,2}$ so that $A_{1,2}$ is an integral along a tube of lines crossing only the rectangles (1,1) and (1,2). Hence $A_{1,2}q$ determines the value of q on the (1,2) rectangle.
- Proceeding in this fashion we can construct χ 's and ω 's so that the value of q is determined on all the rectangles in the first row.
- Starting on the second row, we choose $\chi_{2,1}$, $\omega_{2,1}$ such that $A_{2,1}q$ is an integral along a tube of lines going through the rectangles (1,1) and (2,1). Hence $A_{2,1}q$ will determine the value of q on the (2,1) rectangle. Proceeding in this fashion the A_jq will determine the value of q on the second row of rectangles, and in fact all the rows.

So by choosing χ_j and ω_j appropriately (determined purely by $Rect(\Omega)$)

we can construct an A so that

$$A : L_{pwc}^\infty(\Omega) \longrightarrow R^k$$

is injective.

THE INVERSION FORMULA

We start again with the identities (4) and (8). If $q \in L^\infty(\Omega)$ then because of (5) the L^∞ norms of the last three terms on the right hand side of (8) approach zero as σ tends to infinity. So from (4) and (8)

$$\begin{aligned} \lim_{\sigma \rightarrow \infty} \int_{\partial\Omega \times [0, T]} dS \left(f \frac{\partial v}{\partial n} - v \Lambda_q(f) \right) &= \lim_{\sigma \rightarrow \infty} \int_{\Omega \times [0, T]} dx dt q(x) u v \\ &= \int_{\Omega \times [0, T]} dx dt \chi^2(x + t\omega) q(x) \\ &= \int_{R^n \times [0, T]} dx dt \chi^2(x + t\omega) q(x) \\ &= \int_{R^n} dx \chi^2(x) \int_0^T dt q(x - t\omega) \end{aligned}$$

Since v solves (3) we can construct v without knowledge of q , so given Λ_q we can recover

$$\int_{R^n} dx \chi^2(x) \int_0^T dt q(x - t\omega)$$

for every $\omega \in R^n$ with $|\omega| = 1$, and for every $\chi \in C_0^\infty(R^n)$ satisfying (6) and (7).

If $T - \text{diameter}(\Omega) > 2\epsilon$ then χ satisfies (6) and (7) provided

$$(16) \quad \text{supp}\chi \subseteq \Omega_\epsilon \equiv \Omega^c \cap \{\epsilon \text{ neighborhood of } \partial\Omega\}$$

Given any point a in Ω_ϵ we can find a sequence of χ' 's, with support in Ω_ϵ , which approach $\delta(x - a)$ as distributions. So if $q \in C(\overline{\Omega})$ then we can recover

$$\int_{R^n} dx \delta(x - a) \int_0^T dt q(x - t\omega) = \int_0^T dt q(a - t\omega)$$

for every $\omega \in R^n$ with $|\omega| = 1$, and for every $a \in \Omega_\epsilon$. Since q is supported in Ω and $T - 2\epsilon > \text{diameter}(\Omega)$ we can recover the integral of q on every line. This is enough to determine $q(x)$ because we can relate the line integrals to the fourier transform of q .

ACKNOWLEDGEMENTS

We would like to thank Professor William Symes of Rice University for his useful comments on this work.

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