AN INVERSE PROBLEM FOR THE WAVE EQUATION WITH A TIME DEPENDENT COEFFICIENT

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Abstract

Let $\Omega \subset R^n$, $n \geq 2$, be a bounded domain with smooth boundary. Consider

$$u_{tt} - \triangle_x u + q(x,t)u = 0 \qquad in \qquad \overline{\Omega} \times [0,T]$$

$$u(x,0) = 0, \;\; u_t(x,0) = 0 \qquad if \qquad x \in \Omega$$
$$u(x,t) = f(x,t) \qquad on \qquad \partial\Omega \times [0,T]$$

We show that if u and f are known on $\partial\Omega \times [0,T]$, for all $f \in C_0^\infty(\partial\Omega \times [0,T])$, then $q(x,t)$ may be reconstructed on

$$\mathcal{C} = \{ \; (x,t) \; : \; x{\in}\Omega, \; 0 < t < T, \;\; x - t\omega \;\; \& \;\; x + (T-t)\omega \notin \overline{\Omega} \;\; \forall \;\; \omega \in R^n, \; |\omega| = 1 \; \}$$

provided q is known at all points not in $\mathcal{C}$.

If u, f are known on $\partial\Omega \times [0,T]$, and u is known on $t = T$, for all $f \in C_0^\infty(\partial\Omega \times [0,T])$ then $q(x,t)$ may be reconstructed on

$$\mathcal{D} = \{ \; (x,t) \; : \; x{\in}\Omega, \; 0 < t < T, \;\; x - t\omega \notin \overline{\Omega} \;\; \forall \;\; \omega \in R^n, \; |\omega| = 1\}$$

provided q is known at all points not in $\mathcal{D}$.

Let $\Omega \subset R^n$ be a bounded domain with smooth boundary, and $q(x,t)$ a function on Ω. Suppose $u(x,t)$ satisfies

$$
\begin{aligned}
\Box u + q(x,t)u \equiv u_{tt} - \triangle_x u + q(x,t)u = 0 \qquad in \qquad \overline{\Omega} \times [0,T] \\
u(x,0) = 0, \;\; u_t(x,0) = 0 \qquad if \qquad x \in \Omega \\
u(x,t) = f(x,t) \qquad on \qquad \partial\Omega \times [0,T]
\end{aligned}
\tag{1}
$$

For a fixed $q(x,t)$, (1) is a well posed initial boundary value problem, hence one may define the Dirichlet to Neumann map

$$
\begin{aligned}
\Lambda_q : H^1(\partial\Omega \times [0,T]) &\longrightarrow L^2(\partial\Omega \times [0,T]) \\
f(x,t) &\longmapsto \frac{\partial u}{\partial n}
\end{aligned}
$$

and the boundary to final data map

$$
\begin{aligned}
\Gamma_q : H^1(\partial\Omega \times [0,T]) &\longrightarrow H^1(\Omega) \times L^2(\Omega) \\
f(x,t) &\longmapsto (\; u(x,T) \; , \; u_t(x,T) \;)
\end{aligned}
$$

We prove the following two results -

THEOREM A
If $n > 1$, and $q(x,t)$ is in $C^1(\overline{\Omega} \times [0,T])$, then knowing Λ_q we can reconstruct $q(x,t)$ on the region

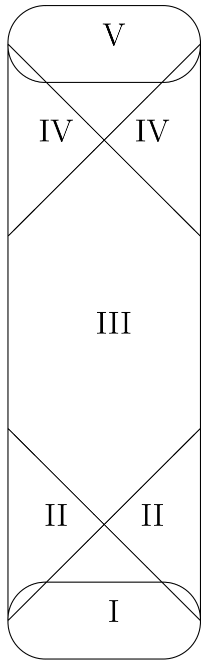
$$
\mathcal{C} = \{ \; (x,t) \; : \; x{\in}\Omega, \; 0 < t < T, \;\; x - t\omega \;\; \& \;\; x + (T-t)\omega \notin \overline{\Omega} \;\; \forall \;\; \omega \in R^n, \; |\omega| = 1 \; \}
$$

provided $q(x,t)$ is known at all points not in $\mathcal{C}$. $\mathcal{C}$ is region III in the figure, and it is a subset of $\overline{\Omega} \times [0,T]$ consisting of those points through which every line making an angle of 45^o with the vertical meets the planes $t = 0$ and $t = T$ outside $\overline{\Omega}$.

THEOREM B
If $n > 1$, and $q(x,t)$ is in $C^1(\overline{\Omega} \times [0,T])$, then knowing Λ_q and Γ_q we can reconstruct $q(x,t)$ on the region

$$
\mathcal{D} = \{ \; (x,t) \; : \; x{\in}\Omega, \; 0 < t < T, \;\; x - t\omega \notin \overline{\Omega} \;\; \forall \;\; \omega \in R^n, \; |\omega| = 1 \; \}
$$

provided $q(x,t)$ is known at all points not in $\mathcal{D}$. $\mathcal{D}$ is region $III \cup IV \cup V$ in the figure, and is a subset of $\overline{\Omega} \times [0,T]$ consisting of those points through which every line making an angle of 45^o with the vertical meets the planes $t = 0$ outside $\overline{\Omega}$.



REMARKS

In [5] Rakesh and Symes showed that Λ_q uniquely determines q provided q is independent of t. Romanov in [7] also proved a uniqueness result for a related problem in the half plane when the (known) wave speed is of special form, again when q is independent of t. [4] contains a reconstruction formula for recovering q from Λ_q, again, provided q is independent of t. We refer the reader to [5] for other articles related to the time independent case.

In [3] Isakov proved a theorem related to Theorem B. He proved uniqueness for $q(x,t)$ over the region $\Omega \times [0,T]$, except his data was the response of the medium not just for zero initial data (and all possible Dirichlet data), as in Theorem B, but all possible initial data (and all possible Dirichlet data). So he proved uniqueness over a larger region but he needed much more information. The proof of Theorem B may be easily modified to prove Isakov's result.

Given Λ_q and Γ_q, from domain of dependence arguments it is clear that with zero initial data one cannot hope to recover $q(x,t)$ over the region I given by

$$\{ \ (x,t) \ : \ x \in \Omega, \ 0 < t < T, \ \ dist(x, \partial\Omega) < t \ \}$$

But our inability to determine q on region II , we feel, is just a shortcoming of our technique.

If we are given only Λ_q, the response measured on the boundary, then from domain of dependence arguments one can see that we cannot hope to recover $q(x,t)$ on the subregion $I \cup V$ given by

$$\{ \ (x,t) \ : \ x \in \Omega, \ 0 < t < T, \ \ dist(x, \partial\Omega) < t, \ or \ dist(x, \partial\Omega) < T - t \ \}$$

However, we feel that one should be able to recover $q(x,t)$ on regions II and IV , but we have not succeeded in doing so.

The derivations of the two results start from an identity which was motivated by one in [6] for elliptic differential equations. Alessandrini, Sylvester and Uhlmann (for elliptic problems), and Ziqi Sun (for hyperbolic problems) amongst others, have used similar identities.

We also draw upon ideas of Sylvester and Uhlmann in [10] and upon the work of the first author with Bill Symes in [5].

NOTE For the rest of this article we will assume that $q(x,t)$ is in $C^1(\overline{\Omega} \times [0,T])$, and $q(x,t)$ is defined to be zero outside $\overline{\Omega} \times [0,T]$.

DERIVATION OF IDENTITY

Suppose

$$\begin{aligned} \Box u + q(x,t) u &= 0 \quad in \quad \overline{\Omega} \times [0,T] \\ u(x,0), \ \ u_t(x,0) &= 0 \quad if \quad x \in \Omega \end{aligned} \tag{2}$$

and

$$\Box v = 0 \quad in \quad \overline{\Omega} \times [0,T] \tag{3}$$

Then using an integration by parts

$$\begin{aligned} &\textstyle\int_{\Omega \times [0,T]} dx \ dt \ q(x) \ u \ v \\ &= \ - \textstyle\int_{\Omega \times [0,T]} dx \ dt \ \Box u \ v \\ &= \ - \textstyle\int_{\Omega \times [0,T]} dxdt \ u \ \Box v + \int_{\partial\Omega \times [0,T]} dS \ (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) \end{aligned}$$

$$+\int_{\Omega\times\{t=T\}} dx\ (uv_t - vu_t)$$

$$(4) \qquad = \int_{\partial\Omega\times[0,T]} dS\ (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})\ +\ \int_{\Omega\times\{t=T\}} dx\ (uv_t - vu_t)$$

We construct special solutions of (2) and (3) as described in the following Lemma which is proved later.

LEMMA 1 Given a unit vector $\omega\in R^n$, $\sigma > 0$, and $\chi\in C_0^\infty(R^n)$ with

$$(5) \qquad supp\ \chi \bigcap \overline{\Omega} = \phi$$

we can construct special solutions of (2) and (3) of the form

$$u = \chi(x + t\omega)e^{+i\sigma(x\cdot\omega+t)} + R_1$$
$$v = \chi(x + t\omega)e^{-i\sigma(x\cdot\omega+t)} + R_2$$

with $R_1,\ R_2$ zero on $\partial\Omega \times [0, T]$, $(R_1,\ R_{1t})$ zero on $t = 0$, $(R_2,\ R_{2t})$ zero on $t = T$, and

$$(6) \qquad ||R_i(x, t)||_{L^2(\Omega\times[0,T])} \leq \frac{C}{\sigma} \qquad i = 1, 2$$

where C depends only on Ω, T, $||q||_{C^1(\Omega\times[0,T])}$, and $||\chi||_{C^3(\overline{\Omega})}$.

With such u and v we have

$$\int_{\Omega\times[0,T]} dx\ dt\ q(x, t)\ u\ v =$$
$$\int_{\Omega\times[0,T]} \chi^2(x + t\omega)q(x, t) \ + \ \int_{\Omega\times[0,T]} \chi(x + t\omega)e^{-i\sigma(x\cdot\omega+t)}q(x, t)R_1$$
$$(7) \qquad + \int_{\Omega\times[0,T]} q(x, t)R_1R_2 \ + \ \int_{\Omega\times[0,T]} \chi(x + t\omega)e^{+i\sigma(x\cdot\omega+t)}q(x, t)R_2$$

So for $q(x, t)$ bounded on $\overline{\Omega} \times [0, T]$ we have from (4), (6), and (7)

$$\int_{\Omega\times[0,T]} \chi^2(x + t\omega)q(x, t)$$
$$(8) \qquad = \ lim_{\sigma\to\infty} \left\{\int_{\partial\Omega\times[0,T]} dS\ (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})\ +\ \int_{\Omega\times\{t=T\}} dx\ (uv_t - vu_t)\right\}$$

for any $\omega \in R^n$, $|\omega| = 1$, and any $\chi \in C_0^\infty(R^n)$ satisfying (5).

Given any point $a \notin \overline{\Omega}$, we can find a sequence of $\chi_\epsilon \in C_0^\infty(R^n)$ satisfying (5) and so that, as ϵ approches zero, χ_ϵ^2 approaches $\delta(x - a)$ as distributions on R^n. So if $q(x, t)$ is defined to be zero outside $\overline{\Omega} \times [0, T]$ then (8) implies

$$\int_0^T dt\ q(a - t\omega, t) \ = \ lim_{\epsilon\to 0}lim_{\sigma\to\infty} \left\{\int_{\partial\Omega\times[0,T]} dS\ (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) \right.$$
$$\left. + \int_{\Omega\times\{t=T\}} dx\ (uv_t - vu_t)\right\}$$

But $q(x,t)$ is zero for $t < 0$ or $t > T$. So if we define

$$f(x,t) \equiv u = \chi_\epsilon(x + t\omega)e^{i\sigma(x\cdot\omega+t)} \qquad (x,t) \in \partial\Omega \times [0,T]$$

then

$$\int_{-\infty}^{\infty} dt\ q(a - t\omega, t) \;=\; lim_{\epsilon\to 0} lim_{\sigma\to\infty} \Big\{ \textstyle\int_{\partial\Omega\times[0,T]} dS\ (v\Lambda_q(f) - f\frac{\partial v}{\partial n})$$

$$+ \textstyle\int_{\Omega\times\{t=T\}} dx\ (uv_t - vu_t) \Big\} \tag{9}$$

for any $\omega \in R^n$, $|\omega| = 1$, and any $a \notin \overline{\Omega}$.

We shall need the following lemma proved later in this article.

LEMMA 2 If $q(x,t)$ is a bounded, measurable function on $R^n \times R$, with compact support, then knowing the light ray transform

$$\mathcal{R}q(\omega, a) \;=\; \int_{-\infty}^{\infty} dt\ q(a - t\omega, t) \qquad \forall\omega \in R^n,\ \ |\omega| = 1,\ \forall a \in R^n$$

we can reconstruct q.

PROOF OF THEOREM A

Let $\{\ (a - t\omega, t)\ :\ -\infty < t < \infty\ \}$ be a 45^o line through a point in $\mathcal{C}$. Then from the definition of $\mathcal{C}$ we have that this line meets $t = 0$ and $t = T$ outside $\overline{\Omega}$ i.e. $a \notin \overline{\Omega}$ and $a - T\omega \notin \overline{\Omega}$. So we can construct a sequence of functions χ_ϵ in $C_0^\infty(R^n)$ such that

$$supp\chi_\epsilon \cap \overline{\Omega} = \phi, \qquad supp\chi_\epsilon - T\omega \cap \overline{\Omega} = \phi$$

and χ_ϵ^2 approaches $\delta(x - a)$ as distributions in R^n. So the v constructed by Lemma 1 and used in (9) may be chosen so that v, v_t are zero on $t = T$. So if Λ_q is known then (9) implies that $\mathcal{R}q(\omega, a)$ is known for all 45^o lines which go through a point of $\mathcal{C}$. For lines not going through any point of $\mathcal{C}$, $\mathcal{R}q(\omega, a)$ is known anyway because q is known at all points outside $\mathcal{C}$ (from the hypothesis of Theorem A). Hence $\mathcal{R}q(\omega, a)$ is known for all $a \in R^n$ and all unit vectors $\omega \in R^n$. Hence q is determined on $\mathcal{C}$ by Lemma 2.

PROOF OF THEOREM B

Since v in (3) is independent of q, if Λ_q and Γ_q are known then from (9), $\mathcal{R}q(\omega, a)$ is known $\forall a \notin \overline{\Omega}$ and $\forall\omega \in R^n$, $|\omega| = 1$. For $a \in \overline{\Omega}$ the line

$$\{\ (a - t\omega, t)\ : -\infty < t < \infty\}$$

lies in the complement of $\mathcal{D}$. From the hypothesis of Theorem B, q is known at all points not in $\mathcal{D}$. So in fact $\mathcal{R}q(\omega, a)$ is known is known $\forall \omega \in R^n$, $|\omega| = 1$, and $\forall a \in R^n$. So Lemma 2 allows us to recover q on $\mathcal{D}$.

PROOF OF LEMMA 1

Our proof of Lemma 1 is a modification of the proof of a similar result in [5] which dealt with the situation where q did not depend on t.

To prove our Lemma it would be enough to show that if

$$\Box R_1 + q(x,t) R_1 = -(\Box + q)\left(\chi(x + t\omega)\, e^{+i\sigma(x\cdot\omega + t)}\right) \quad in \quad \overline{\Omega} \times [0,T]$$

$$R_1(x,0) = 0, \ \ \partial_t R_1(x,0) = 0 \quad if \quad x \in \overline{\Omega} \tag{10}$$

$$R_1 = 0 \quad on \quad \partial\Omega \times [0,T]$$

and

$$\Box R_2 = -\Box\left(\chi(x + t\omega)\, e^{-i\sigma(x\cdot\omega + t)}\right) \quad in \quad \overline{\Omega} \times [0,T]$$

$$R_2(x,T) = 0, \ \ \partial_t R_2(x,T) = 0 \quad if \quad x \in \overline{\Omega}$$

$$R_2 = 0 \quad on \quad \partial\Omega \times [0,T]$$

then the estimates (6) hold. We shall prove the estimate for R_1, and the R_2 case may be handled in a similar fashion.

Let us define

$$\phi(x,t) = -(\Box + q)\left(\chi(x + t\omega)\, e^{+i\sigma(x\cdot\omega + t)}\right) = -e^{+i\sigma(x\cdot\omega + t)}\, (\Box + q)\chi(x + t\omega) \tag{11}$$

From estimates for hyperbolic initial boundary value problems as in [2], for any $w \in H^1(\overline{\Omega} \times [0,T])$ with $(w,\ w_t)$ zero on $t = 0$, w zero on $\partial\Omega \times [0,T]$, and large τ

$$\tau \textstyle\int_{\Omega\times[0,T]} dx\ dt\ e^{-2\tau t} |w_t(x,t)|^2 \ \le\ C \textstyle\int_{\Omega\times[0,T]} dx\ dt\ e^{-2\tau t} |\Box w(x,t)|^2$$

with C depending only on $\overline{\Omega}$ and T. So taking $w(x,t) = \int_0^t ds\ R_1(x,s)$ in the previous equation and integrating equation (10) with respect to t one obtains

$$\begin{aligned}
&\tau \textstyle\int_{\Omega\times[0,T]} dx\ dt\ e^{-2\tau t} |R_1(x,t)|^2 \\
&\le\ C \textstyle\int_{\Omega\times[0,T]} dx\ dt\ e^{-2\tau t} \left\{ \left| \displaystyle\int_0^t ds\ \phi(x,s) \right|^2 + \left| \displaystyle\int_0^t ds\ q(x,s)\ R_1(x,s) \right|^2 \right\} \\
&\le\ C \textstyle\int_{\Omega\times[0,T]} dx\ dt\ \left\{ e^{-2\tau t} \left| \displaystyle\int_0^t ds\ \phi(x,s) \right|^2 + \left| \displaystyle\int_0^t ds\ e^{-\tau s} q(x,s)\ R_1(x,s) \right|^2 \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; C_1 \int_{\Omega\times[0,T]} dx\, dt \; \left\{ e^{-2\tau t} \Big| \int_0^t ds\; \phi(x,s)\Big|^2 + \int_0^T ds\; e^{-2\tau s} |q(x,s)|^2 \; |R_1(x,s)|^2 \right\} \\
&\leq \; C_2 \int_{\Omega\times[0,T]} dx\, dt \; \left\{ e^{-2\tau t} \Big| \int_0^t ds\; \phi(x,s)\Big|^2 + \; e^{-2\tau t} \; |R_1(x,t)|^2 \right\}
\end{aligned}
$$

using the boundedness of q. So by taking τ large enough one obtains

$$
\int_{\Omega\times[0,T]} dx\, dt\, |R_1(x,t)|^2 \leq C \int_{\Omega\times[0,T]} dx\, dt\, \Big| \int_0^t ds\; \phi(x,s)\Big|^2 \tag{12}
$$

Looking at the form of $\phi(x,t)$ in (11), we define $\chi_1(x,t)$

$$
\chi_1(x,t) = -(\Box + q(x,t))\chi(x,t)
$$

Then

$$
\begin{aligned}
\int_0^t ds \phi(x,s) \;&=\; \int_0^t ds\; \chi_1(x,s)\; e^{+i\sigma(x\cdot\omega+s)} \\
&=\; \frac{1}{i\sigma}\int_0^t ds\; \chi_1(x,s) \frac{d}{ds}(e^{+i\sigma(x\cdot\omega+s)} \\
&=\; -\frac{1}{i\sigma}\int_0^t ds\; \frac{\partial \chi_1}{\partial s}(x,s) e^{+i\sigma(x\cdot\omega+s)} \\
&\quad +\; \frac{1}{i\sigma}\chi_1(x,t)e^{+i\sigma(x\cdot\omega+t)} \;-\; \frac{1}{i\sigma}\chi_1(x,0)e^{i\sigma x\cdot\omega}
\end{aligned}
$$

Using this in (12) we have

$$
\int_{\Omega\times[0,T]} dx\, dt\, |R_1(x,t)|^2 \leq \frac{C}{\sigma}
$$

with C depending only on Ω, T, $||\chi||_{C^3(R^n)}$, and $||q||_{C^1(\Omega\times[0,T])}$.

PROOF OF LEMMA 2

Since q has compact support and is bounded, from Fubini's theorem

$$
\begin{aligned}
\int_{R^n} da\; \mathcal{R}q(\omega,a)e^{ia\cdot\xi} \;&=\; \int_{R^n} da \int_{-\infty}^{\infty} dt\; q(a-t\omega,t)\; e^{ia\cdot\xi} \\
&=\; \int_{-\infty}^{\infty} dt \int_{R^n} da\; q(a-t\omega,t)\; e^{ia\cdot\xi} \\
&=\; \int_{-\infty}^{\infty} dt \int_{R^n} da\; q(a,t)\; e^{ia\cdot\xi}e^{it\omega\cdot\xi} \\
&=\; \hat{q}(\xi,\omega\cdot\xi)
\end{aligned}
$$

for all $\xi \in R^n$ and all unit vectors $\omega \in R^n$. Here $\hat{q}$ is the fourier transform of q. So $\hat{q}(\xi,\tau)$ is known in $\{\ (\xi,\tau) : \xi \in R^n,\ \tau \in R,\ |\tau| < |\xi|\}$, which contains an open set in $R^n \times R$. Since $\hat{q}(\xi, \tau)$ is analytic in the variables ξ and τ, the above information is enough to recover $\hat{q}$ everywhere, hence q may be recovered.

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