

POTENTIAL INVERSION FROM TRANSMISSION DATA FOR THE ONE DIMENSIONAL
WAVE EQUATION

Rakesh
Rees Hall
University of Delaware
Newark, DE 19716
rakesh@math.udel.edu

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Abstract: Consider an inhomogeneous medium on the half line with nonconstant potential which is zero beyond depth X . This medium is probed by an impulsive source generated on the boundary and the transmitted signal is measured at depth X over the time period $X \leq t \leq 3X$. The interaction between the medium and the signal is governed by the one dimensional wave equation with the potential being the coefficient of the zeroth order term. We prove that the potential may be reconstructed from the transmitted data provided we have an upper bound on the potential.

1 Introduction

Suppose $q(x) \in C[0, \infty)$ and is zero for $x > X$ for some known positive number X . Consider the initial boundary value problem

$$u_{tt} - u_{xx} + q(x)u = 0 \quad x \geq 0, \quad t \in R \quad (1)$$

$$u = 0 \quad t < 0 \quad (2)$$

$$u_x(0, t) = -\delta(t) \quad (3)$$

Proposition 1 (1) - (3) has a unique solution which is locally bounded. Further, $u(x, t) - H(t - x)$ is a continuous function which is differentiable in the region $t > x$, is zero in $t < x$, and its derivatives have a jump type singularity across $t = x$.

Hence one may define the map

$$\begin{aligned} S : C[0, X] &\longrightarrow C^1[X, 3X] \\ q &\longmapsto (u(X, t) - H(t - X))|_{[X, 3X]} \end{aligned}$$

We are interested in analyzing the inverse of S . In this article we prove the injectivity of S , so S is invertible on its range, and we show how to invert S . At the moment we are unable to characterize the range of S . The main result in this article is

Theorem 1 Assume $q \in C[0, \infty)$, and $q(x)$ is zero for $x \geq X$. Then q may be reconstructed from $u(X, t)$, $X \leq t \leq 3X$ provided we are given an upper bound on $\|q\|_\infty$.

In the above theorem we require a knowledge of an upper bound on $\|q\|_\infty$ but it still proves the injectivity of S . The upper bound is needed in the reconstruction step - it provides a positive lower bound on the step size for the downward continuation algorithm used in the reconstruction. Also, we believe, a careful analysis of our proof, with some modifications will establish Theorem 1 even if $q \in L^\infty$ (i.e. continuity may be replaced by boundedness).

There has been considerable work on one dimensional inverse problems for the wave equation but almost all the work has been on inversion from reflection data (source and receiver are at the same location) i.e. from $u(0, t)$ and more. Inversion from reflection data generated by an impulsive source has been thoroughly analyzed - see [1], [8] and [9] and references mentioned there. Inversion from transmission plus reflection data is analyzed in [4]. Inversion from transmission data is a more difficult problem and this point is discussed in [6].

Rakesh and Sacks in [6] showed how to recover coefficients from transmission data for the Webster's Horn equation, based on an idea of Claerbout in [3] for solving the discrete version of that problem. In this paper we prove an analogue of the result in [6] for the one dimensional Plasma equation.

The main idea in [6] was to relate inversion from transmission data to another problem involving inversion from reflection data. The link between the two problems was established by transforming Webster's Horn equation to a first order system of PDEs in the left and right going waves and using a magical identity for this system of first order equations. We establish a similar identity for the Plasma equation but note that while Webster's Horn equation is energy preserving the Plasma equation is not and the Plasma equation cannot be reduced to a first order system of PDEs. So it is not obvious that the ideas used in [6] would work here too.

One important clue in the solution of the problem in [6] was the solution of the discrete problem in [3]. For the Plasma equation, the discrete version of our problem i.e assuming q is piecewise constant, may be solved quite easily, because at an interface, the reflected wave is one order smoother and the transmitted wave is the same as the incoming wave - so the discrete transmitted inverse problem may be solved by linear inversion. Unfortunately this is of no help in the solution of the continuous problem.

The change of dependent variables $u = v\sqrt{\eta}$ transforms Webster's Horn Equation (with $\eta > 0$)

$$\eta(x)v_{tt} - (\eta(x)v_x)_x = 0$$

to the Plasma equation (1) with $q = (\sqrt{\eta})''/\sqrt{\eta}$. However, the map $\eta \mapsto q$ is not surjective in any reasonable class of functions e.g. there is no bounded positive η , over the interval $[0, 2\pi]$, for which $(\sqrt{\eta})''/\sqrt{\eta} = -1$. Thus the results in [6] do not imply the results in this article.

Theorem 1 requires an upper bound on $\|q\|_\infty$ - no such coefficient information was required in [6] because of the estimates by Symes in [8] for the inversion from reflection data for the Webster's Horn equation. Similar estimates have yet to be proved for the Plasma equation.

Other earlier works on inverse transmission problems for the wave equation are [2] and [5] - they have been discussed in detail in [6]. We have also discussed there the importance of the right choice for the boundary conditions and the length of the time interval over which the transmission data must be measured.

The main step in the proof is to relate the data of the transmission problem to that of a "reflection" problem for which one can then apply the downward continuation method. We state this more carefully. Consider the Goursat problem

$$F_{tt} - F_{xx} + qF = 0, \quad |t| \leq x \tag{4}$$

$$F(x, \pm x) = \frac{1}{4} \int_0^x q(\sigma) d\sigma, \quad x \geq 0 \tag{5}$$

If q is continuous, then we may show that (4), (5) has a unique solution in C^1 by using the change of variables $r = t + x, s = t - x$ and appealing to

Proposition 2 *Consider the Goursat problem*

$$w_{rs} + q(r, s)w = f(r, s) \quad 0 \leq r \leq R, \quad 0 \leq s \leq S \tag{6}$$

$$w(r, 0) = a(r) \quad 0 \leq r \leq R \tag{7}$$

$$w(0, s) = b(s) \quad 0 \leq s \leq S \tag{8}$$

If $q, f \in C([0, R] \times [0, S])$, $a \in C^1[0, R]$, $b \in C^1[0, S]$, and $a(0) = b(0)$, then (6) - (8) has a unique solution which is C^1 .

Define

$$m(t) \equiv u(X, t) - H(t - X) \quad (9)$$

Theorem 2 Assume $q \in C[0, \infty)$ and q is zero for $x \geq X$. Then for $|t| \leq X$,

$$F(X, t) = \frac{1}{2}\{\alpha(X + t) + \alpha(X - t) - \alpha(2X)\}, \quad (10)$$

$$F_x(X, t) = \frac{1}{2}\{\alpha'(X + t) + \alpha'(X - t)\}, \quad (11)$$

where

$$m'(t + X) + \alpha(t) + \int_0^t m'(X + t - s)\alpha(s) ds = 0, \quad 0 \leq t \leq 2X. \quad (12)$$

2 Proofs of Theorems

Proof of Theorem 1 Given $u(X, t)$ for $X \leq t \leq 3X$ we can recover $\alpha(t)$ for $0 \leq t \leq 2X$, from (12) because it is a Volterra equation. So if Theorem 2 is true then we can recover $F(X, t)$ and $F_x(X, t)$ for $|t| \leq X$. Then the theorem follows from Theorems 1 and 2 of [7] with minor modifications for the 'even' case.

Proof of Theorem 2 For $q \in C[0, \infty)$, consider the Goursat problems

$$F_{tt} - F_{xx} + qF = 0, \quad G_{tt} - G_{xx} + qG = -\frac{q}{2}, \quad |t| < x \quad (13)$$

$$F(x, \pm x) = \frac{1}{4} \int_0^x q(\sigma) d\sigma, \quad G(x, \pm x) = 0 \quad (14)$$

Then from Proposition 2, each of the above equations has a unique solution in C^1 . We claim that if u solves (1)-(3) then

Proposition 3

$$u(x, t) = \frac{1}{2} \{u_0(t - x) + u_0(t + x) - H(t + x) + H(t - x)\} + \int_{t-x}^{t+x} F(x, t - s)u_0(s) ds - G(x, t)$$

where $u_0(t) = u(0, t)$ and G is defined to be zero for $|t| > x$.

Define

$$p_0(t) \equiv u(0, t) - H(t) = u_0(t) - H(t)$$

Then $p_0(t)$ and $m(t + X)$ are zero when $t < 0$, and the derivatives of $p_0(t)$ and $m(X + t)$ have a jump type singularity across $t = 0$. Writing the conclusion of Proposition 3 in terms of p_0 , we have

$$u(x, t) - H(t - x) = \frac{1}{2} \{p_0(t - x) + p_0(t + x)\} + \int_{t-x}^{t+x} F(x, t - s)p_0(s) ds + Q(x, t) \quad (15)$$

where - assuming F is defined to be zero for $|t| > x$ -

$$Q(x, t) = \int_{t-x}^{t+x} F(x, t - s)H(s) ds - G(x, t) = \int_{-\infty}^t F(x, s) ds - G(x, t) \quad (16)$$

$$= \int_{-\infty}^t \{F(x, s) - G_t(x, s)\} ds = \{F(x, t) - G_t(x, t)\} * H(t) \quad (17)$$

Note that $Q(x, t)$ is continuous everywhere and its derivatives have a jump type singularity across $|t| = x$. So from (15)

$$\begin{aligned} m(t) &= \frac{1}{2} \{p_0(t - X) + p_0(t + X)\} + \int_{t-X}^{t+X} F(X, t - s)p_0(s) ds + Q(X, t) \\ &= a(t) * p_0(t) + b(t) * H(t) \end{aligned} \quad (18)$$

where

$$a(t) = \frac{1}{2} \{\delta(t + X) + \delta(t - X)\} + F(X, t), \quad b(t) = F(X, t) - G_t(X, t) \quad (19)$$

Now $q = 0$ for $x \geq X$, so u satisfies the constant coefficient wave equation in the region $x \geq X$, $t \in \mathcal{R}$. Also, $u = 0$ for $t < 0$, hence in the region $x \geq X$ the waves will come from the left and continue to move right - there will be no left going waves in $x \geq X$. So $u_x + u_t = 0$ in $x \geq X$, in particular when $x = X$. But from (15)

$$\begin{aligned} u_x(X, t) + \delta(t - X) &= \frac{1}{2} \{p'_0(t + X) - p'_0(t - X)\} + Q_x(X, t) \\ &\quad + F(X, X) \{p_0(t + X) + p_0(t - X)\} + \int_{t-X}^{t+X} F_x(X, t - s)p_0(s) ds \\ u_t(X, t) - \delta(t - X) &= \frac{1}{2} \{p'_0(t + X) + p'_0(t - X)\} + Q_t(X, t) \\ &\quad + F(X, X) \{p_0(t + X) - p_0(t - X)\} + \int_{t-X}^{t+X} F_t(X, t - s)p_0(s) ds \end{aligned}$$

So

$$\begin{aligned} 0 &= u_x(X, t) + u_t(X, t) \\ &= p'_0(t + X) + \{Q_x + Q_t\}(X, t) + 2F(X, X)p_0(t + X) + \int_{t-X}^{t+X} \{F_x + F_t\}(X, t - s)p_0(s) ds \end{aligned}$$

Integrating this with respect to t , and noting that p_0 is zero for $t < 0$, we obtain

$$0 = p_0(t + X) + 2F(X, X) \int_{-\infty}^t p_0(s + X) ds + \int_{-\infty}^t \{Q_x + Q_t\}(X, s) ds$$

$$\begin{aligned}
& + \int_{-\infty}^t \int_{r-X}^{r+X} (F_x + F_t)(X, r-s) p_0(s) ds dr \\
= & p_0(t) * \delta(t+X) + 2F(X, X) p_0(t) * H(t+X) + \{Q_x + Q_t\}(X, t) * H(t) \\
& + \int_{-\infty}^t \int_{-\infty}^{\infty} (F_x + F_t)(X, r-s) p_0(s) ds dr \\
= & c(t) * p_0(t) + d(t) * H(t)
\end{aligned} \tag{20}$$

where F_x and F_t are defined to be zero for $|t| \geq x$, and

$$c(t) = \delta(t+X) + 2F(X, X)H(t+X) + \int_{-\infty}^t (F_t + F_x)(X, r) dr \tag{21}$$

$$d(t) = Q_x(X, t) + Q_t(X, t) \tag{22}$$

Note that, there seems a conflict in the definition of F_x and F_t on (and only on) $|t| = x$ because F is defined to be zero for $|t| > x$ which automatically determines F_x and F_t on $|t| = x$ - in fact they should have a δ type singularity. We chose to define F_x and F_t to be zero on $|t| \geq x$ for convenience of notation. We could have instead used new symbols for the functions which were F_x or F_t in $|t| \leq x$ and zero outside but this would make the notation more cumbersome. What we have done will not lead to any errors as long as we write all integrals of the new F_x or F_t , on any region, as a sum of integrals over the part lying in $|t| \leq x$ and over the part lying in $|t| > x$, and use the fact that F, F_x, F_t are zero in $t > |x|$.

Eliminating the unknown p_0 from (18) and (20) we obtain

$$c(t) * m(t) = \{b(t) * c(t) - a(t) * d(t)\} * H(t) \tag{23}$$

Define

$$P(x, t) = 2F(x, x)H(t+x) + \int_{-\infty}^t (F_t + F_x)(x, r) dr \tag{24}$$

then we claim

Proposition 4

$$b(t) * c(t) - a(t) * d(t) = -P(X, t-X) \quad \forall t \in \mathcal{R}$$

In light of this, using (21) and (23), the definitions of P and c , we obtain for $t \geq 0$,

$$\begin{aligned}
0 & = m(t) * [\delta(t+X) + P(X, t)] + P(X, t-X) * H(t) \\
& = m(t+X) + [m(t+X) + H(t)] * P(X, t-X) \\
& = m(t+X) + \int_{-\infty}^{\infty} [m(t+X-s) + H(t-s)] P(X, s-X) ds \\
& = m(t+X) + \int_0^t [m(t+X-s) + 1] P(X, s-X) ds
\end{aligned}$$

Here we have used the fact that $m(t)$ is zero for $t < X$ and $P(x, t)$ is zero for $t < -x$. Differentiating the above equation with respect to t and noting that $m(X) = 0$, we obtain

$$m'(t+X) + P(X, t-X) + \int_0^t m'(t+X-s) P(X, s-X) ds = 0, \quad \text{for } t \geq 0 \quad (25)$$

This is a Volterra equation in $P(X, t-X)$ and if $m'(t)$ is given for $X \leq t \leq 3X$ then we can recover $P(X, t)$ over the interval $|t| \leq X$. But from (24)

$$P_t(x, t) = F_t(x, t) + F_x(x, t) \quad \text{for } |t| \leq x$$

and since $F(x, t)$ is even in t , we have

$$F_x(X, t) = \frac{P_t(X, t) + P_t(X, -t)}{2}, \quad F_t(X, t) = \frac{P_t(X, t) - P_t(X, -t)}{2} \quad \text{for } |t| \leq X \quad (26)$$

Hence

$$F(X, t) = F(X, -X) + \frac{1}{2} \{P(X, t) + P(X, -t) - P(X, -X) - P(X, X)\} \quad (27)$$

Now, from Proposition 1, in the region $t \geq x \geq 0$, we have $u(x, x) = 1$, $u_x(0, t) = 0$. So

$$u_t(x, x) + u_x(x, x) = 0 \quad (28)$$

and (1) implies

$$0 = u_{tt}(x, x) - u_{xx}(x, x) + q(x)u(x, x) = \frac{d}{dx} \{u_t(x, x) - u_x(x, x)\} + q(x) = 0$$

So integrating and using $u_t(0, t) = 0$, $u_x(0, t) = 0$ we have

$$u_t(x, x) - u_x(x, x) = - \int_0^x q(\sigma) d\sigma$$

Combining this with (28) we obtain

$$u_t(x, x) = -\frac{1}{2} \int_0^x q(\sigma) d\sigma$$

So $m'(X) = -2F(X, -X)$. Also from (25), $P(X, -X) = -m'(X)$, hence (27) implies

$$F(X, t) = \frac{1}{2} \{P(X, t) + P(X, -t) - P(X, X)\} \quad (29)$$

So defining $\alpha(t) \equiv P(X, t-X)$, Theorem 2 follows from (25), (26), and (29).

3 Proof of Propositions

Proposition 1 *Suppose $q(x) \in C[0, \infty)$ and is zero for $x > X$ for some known positive number X . Then (1) - (3) has a unique solution $u(x, t)$ which is locally bounded. Further, $u(x, t) - H(t-x)$ is a continuous function which is differentiable in the region $t > x$, is zero in $t < x$, and its derivatives have a jump type singularity across $t = x$.*

Proof of Proposition 1

The Proposition should follow easily from standard techniques for the well-posedness of hyperbolic initial boundary value problems. However, we know no reference where this Proposition is explicitly stated so we give a brief outline of its proof.

A function $u(x, t) \in L_{loc}^\infty(R_+ \times R)$ which is zero for $t \ll 0$, is a solution of (1)-(3) in the weak sense, if

$$\int_0^\infty \int_{-\infty}^\infty u(x, t) (\phi_{tt} - \phi_{xx} + q\phi)(x, t) dt dx = \phi(0, 0) \quad (30)$$

for all $\phi(x, t) \in C^\infty(R \times R)$ with $\phi_x(0, t) = 0$, and which are zero for large t . The progressing wave expansion suggests that the most singular term in $u(x, t)$ is $H(t - x)$. In light of that, suppose $v(x, t)$ is a C^1 function which is the unique weak solution of the Goursat problem in the conical region $t \geq |x|$

$$v_{tt} - v_{xx} + q(|x|)v = -q(|x|) \quad t \geq |x| \quad (31)$$

$$v(\pm t, t) = 0 \quad t \geq 0 \quad (32)$$

Note that $v(x, t)$ is even in x - hence $v_x(0, t) = 0$. Define $v(x, t)$ to be zero over the region $t < |x|$. Then one may verify that $v(x, t) + H(t - x)$ satisfies (30) and is the unique locally bounded solution of (1)-(3). Thus to prove Proposition 1 it is enough to prove that (31),(32) has a unique C^1 solution. The change of variables $r = t + x$, $s = t - x$ transforms the wave operator to $\partial_r \partial_s$ up to a constant. Hence Proposition 1 will follow from

Proposition 2 *If $q, f \in C([0, R] \times [0, S])$, $a \in C^1[0, R]$, $b \in C^1[0, S]$, and $a(0) = b(0)$, then (6) - (8) has a unique solution which is C^1 .*

Proof of Proposition 2

Integrating (6) over the rectangle $[0, r] \times [0, s]$ and using (7) and (7), one may show that a solution of (6)-(8) must be a solution of

$$(I + T)w(r, s) = a(r) + b(s) - a(0) + \int_0^s \int_0^r f, \quad 0 \leq r \leq R, \quad 0 \leq s \leq S \quad (33)$$

where

$$\begin{aligned} T : C([0, R] \times [0, S]) &\rightarrow C([0, R] \times [0, S]) \\ Tw(r, s) &= \int_0^s \int_0^r q(\xi, \eta) w(\xi, \eta) d\xi d\eta \end{aligned}$$

Under the sup norm, $\|T\| \leq RS\|q\|_\infty$, so (33) has a unique continuous solution provided $RS\|q\|_\infty < 1$. Therefore, if $S^* = 1/(2R\|q\|_\infty)$, then we have solved (33) if $S \leq S^*$. However, we can repeat the above for the rectangle $[0, R] \times [S^*, 2S^*]$, etc. and prove that (33) has a unique solution for any positive R and S . Further, since Tw is differentiable if w is continuous, (33) also implies that w is C^1 .

Proposition 3 *If u solves (1)-(3) then*

$$u(x, t) = \frac{1}{2} \{u_0(t-x) + u_0(t+x) - H(t+x) + H(t-x)\} + \int_{t-x}^{t+x} F(x, t-s)u_0(s) ds - G(x, t)$$

where F and G solve (13), (14), $u_0(t) = u(0, t)$ and G is defined to be zero for $|t| > x$.

Proof of Proposition 3

If $u(0, \cdot) = u_0$ and $u_x(0, \cdot) = v_0$ and u_0, v_0 and q were smooth then just from straightforward differentiation one may show that

$$\begin{aligned} u(x, t) = & \frac{1}{2} \left\{ u_0(t-x) + u_0(t+x) + \int_{t-x}^{t+x} v_0(s) ds \right\} \\ & + \int_{t-x}^{t+x} F(x, t-s)u_0(s) ds + \int_{t-x}^{t+x} G(x, s-t)v_0(s) ds \end{aligned} \quad (34)$$

satisfies (1) and that $u(0, \cdot) = u_0$ and $u_x(0, \cdot) = v_0$. In our case, $v_0 = -\delta(t)$. Substituting this formally into (34) we obtain the expression in Proposition 3. A more rigorous proof may be obtained by a direct verification of the weak version of (1)-(3).

Proposition 4

$$b(t) * c(t) - a(t) * d(t) = -P(X, t-X) \quad \forall t \in \mathcal{R}$$

Proof of Proposition 4

From (16)

$$\begin{aligned} Q_x(x, t) &= F(x, x)H(t-x) + F(x, -x)H(t+x) + \int_{-\infty}^t F_x(x, s) ds - G_x(x, t) \\ Q_t(x, t) &= F(x, t) - G_t(x, t) \end{aligned}$$

Hence using the even nature of F we have

$$\begin{aligned} d(t) &= Q_x(X, t) + Q_t(X, t) = \{F(X, t) + F(X, X)H(t-X) - F(X, X)H(t+X)\} \\ &\quad + 2F(X, X)H(t+X) + \int_{-\infty}^t F_x(X, r) dr - (G_x + G_t)(X, t) \\ &= 2F(X, X)H(t+X) + \int_{-\infty}^t (F_t + F_x)(X, r) dr - (G_x + G_t)(X, t) \\ &= (P - G_x - G_t)(X, t) \end{aligned}$$

Also, from (21) and (24)

$$c(t) = \delta(t+X) + P(X, t)$$

Hence from (19)

$$\begin{aligned}
& b(t) * c(t) - a(t) * d(t) + P(X, t - X) \\
&= P(X, t - X) + (F - G_t)(X, t) * [\delta(t + X) + P(X, t)] \\
&\quad - \left[\frac{1}{2} \{ \delta(t + X) + \delta(t - X) \} + F(X, t) \right] * (P - G_x - G_t)(X, t) \\
&= \frac{1}{2} (G_x - G_t)(X, t + X) + \frac{1}{2} (G_x + G_t)(X, t - X) + (F - \frac{P}{2})(X, t + X) + \frac{P}{2}(X, t - X) \\
&\quad + (F - P) * G_t(X, t) + F * G_x(X, t)
\end{aligned}$$

From (24), integrating F_t we have

$$P(x, t) = F(x, t) + F(x, x)[H(t + x) + H(t - x)] + F_x(x, t) * H(t)$$

Also,

$$\begin{aligned}
G_t(x, t) * H(t - x) &= \int_{-\infty}^{\infty} ds G_t(x, s) H(t - s - x) = \int_{-\infty}^{t-x} ds G_t(x, s) = G(x, t - x) \\
G_t(x, t) * H(t + x) &= G(x, t + x) \\
G_t * F_x * H &= F_x * G_t * H = F_x * G
\end{aligned}$$

So defining

$$\phi(x, t) = \frac{1}{2} [G_x(x, t + x) - G_t(x, t + x) + G_x(x, t - x) + G_t(x, t - x)] \quad (35)$$

we have

$$\begin{aligned}
& b(t) * c(t) - a(t) * d(t) + P(X, t - X) \\
&= \phi(X, t) + \frac{1}{2} [F(X, t + X) - F(X, X)\{H(t + 2X) + H(t)\} - F_x(X, t) * H(t + X)] \\
&\quad + \frac{1}{2} [F(X, t - X) + F(X, X)\{H(t) + H(t - 2X)\} + F_x(X, t) * H(t - X)] \\
&\quad + (F * G_x - F_x * G)(X, t) - F(X, X)\{G(X, t + x) + G(X, t - X)\} \\
&= \phi(X, t) + \left[F * G_x(X, t) - F_x(X, t) * \left\{ G(X, t) + \frac{H(t + X) - H(t - X)}{2} \right\} \right] \\
&\quad - F(X, X) \left[G(X, t + X) + G(X, t - X) + \frac{H(t + 2X) - H(t - 2X)}{2} \right] \\
&\quad + \frac{1}{2} [F(X, t + X) + F(X, t - X)]
\end{aligned}$$

If we introduce the symbol

$$L(x, t) \equiv G(x, t) + \frac{1}{2} [H(t + x) - H(t - x)] = \begin{cases} G(x, t) + \frac{1}{2} & |t| \leq x \\ 0 & |t| > x \end{cases} \quad (36)$$

Then

$$L_{tt} - L_{xx} + qL = 0 \quad \text{for } |t| \leq x \quad (37)$$

$$L(x, \pm x) = \frac{1}{2} \quad (38)$$

Further, $G_x = L_x$, $G_t = L_t$ if L_x and L_t are defined to be zero outside $|t| \leq x$. So

$$b * c(t) - a * d(t) + P(X, t - X) = \phi(X, t) + \psi(X, t) + \chi(X, t)$$

where

$$\begin{aligned} \phi(x, t) &= \frac{1}{2} [L_x(x, t+x) - L_t(x, t+x) + L_x(x, t-x) + L_t(x, t-x)] \\ \psi(x, t) &= [L_x * F - L * F_x](x, t) \\ \chi(x, t) &= -F(x, x) [L(x, t+x) + L(x, t-x)] + \frac{1}{2} [F(x, t+x) + F(x, t-x)] \end{aligned}$$

So if we prove that $\phi(x, t) + \psi(x, t) + \chi(x, t) = 0$ for $x \geq 0$ then the Proposition will have been proved.

Since $F(x, t)$ and $L(x, t)$ are even in t and supported in the region $|t| \leq x$, one may show that ϕ , ψ , χ are even in t and supported in the region $|t| \leq 2x$. So the Proposition follows if $\phi(x, t) + \psi(x, t) + \chi(x, t) = 0$ in the region $0 \leq t \leq 2x$. Now, in the region $0 \leq t \leq 2x$, using the support of F and L ,

$$\begin{aligned} \phi(x, t) &= \frac{1}{2} [L_x(x, t-x) + L_t(x, t-x)] \\ \psi(x, t) &= \int_{-x}^x [L_x(x, s) F(x, t-s) - L(x, s) F_x(x, t-s)] ds \\ &= \int_{t-x}^x [L_x(x, s) F(x, t-s) - L(x, s) F_x(x, t-s)] ds \\ \chi(x, t) &= -F(x, x)L(x, t-x) + \frac{1}{2}F(x, t-x) \end{aligned}$$

So using $L(x, x) = 1/2$

$$\begin{aligned} \phi(x, 2x) &= \frac{1}{2} [L_x(x, x) + L_t(x, x)] = \frac{1}{2} \frac{d}{dx} L(x, x) = 0 \\ \psi(x, 2x) &= \int_x^x \{L_x(x, s)F(x, 2x-s) - L(x, s)F_x(x, 2x-s)\} ds = 0 \\ \chi(x, 2x) &= -F(x, x)L(x, x) + \frac{1}{2}F(x, x) = 0 \end{aligned}$$

So to establish our claim it will be enough to show that $\phi_x(x, t) + \psi_x(x, t) + \chi_x(x, t) = 0$ in the region $0 < t < 2x$. In this region

$$\begin{aligned} \phi_x(x, t) &= \frac{1}{2} [L_{xx}(x, t-x) - L_{tt}(x, t-x)] = \frac{q(x)}{2} L(x, t-x) \\ \chi_x(x, t) &= \frac{1}{2} \frac{\partial}{\partial x} \{F(x, t-x)\} - F(x, x) \frac{\partial}{\partial x} \{L(x, t-x)\} - L(x, t-x) \frac{d}{dx} \{F(x, x)\} \\ \psi_x(x, t) &= L_x(x, x) F(x, t-x) - L(x, x) F_x(x, t-x) + L_x(x, t-x) F(x, x) - L(x, t-x) F_x(x, x) \end{aligned}$$

$$\begin{aligned}
& + \int_{t-x}^x [L_{xx}(x, s) F(x, t-s) - L(x, s) F_{xx}(x, t-s)] ds \\
= & \text{stuff} + \int_{t-x}^x [L_{tt}(x, s) F(x, t-s) - L(x, s) F_{tt}(x, t-s)] ds \\
= & \text{stuff} + \int_{t-x}^x \frac{d}{ds} [L_t(x, s) F(x, t-s) + L(x, s) F_t(x, t-s)] ds \\
= & L_x(x, x) F(x, t-x) - L(x, x) F_x(x, t-x) + L_x(x, t-x) F(x, x) - L(x, t-x) F_x(x, x) \\
& + L_t(x, x) F(x, t-x) + L(x, x) F_t(x, t-x) - L_t(x, t-x) F(x, x) - L(x, t-x) F_t(x, x) \\
= & F(x, t-x) \frac{d}{dx} \{L(x, x)\} - L(x, x) \frac{\partial}{\partial x} \{F(x, t-x)\} + F(x, x) \frac{\partial}{\partial x} \{L(x, t-x)\} \\
& - L(x, t-x) \frac{d}{dx} \{F(x, x)\}
\end{aligned}$$

Noting that

$$L(x, x) = \frac{1}{2}, \quad F(x, x) = \frac{1}{4} \int_0^x q(s) ds$$

we obtain $\phi_x(x, t) + \psi_x(x, t) + \chi_x(x, t) = 0$ in the region $0 < t < 2x$.

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