AN INVERSE IMPEDANCE TRANSMISSION PROBLEM FOR THE WAVE EQUATION

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1 Introduction

Suppose Ω is a bounded open set in R^n, $n \geq 2$, with smooth boundary. Let D be a strictly convex open subset of Ω, with smooth boundary, with $d(\partial\Omega, D) > 0$. Define the impedance

$$\eta(x) = \begin{cases} 1 & x \in R^n \setminus \overline{D} \\ k & x \in D \end{cases}$$

with $k \neq 1$, $k > 0$. Consider the transmission problem

$$\begin{aligned} \eta u_{tt} - \nabla_x \cdot (\eta \nabla_x u) &= 0 \quad on \quad R^n \times (-\infty, T] && (1) \\ u(x,t) &= (t - x_n)_+^2 \quad for \quad t << 0 && (2) \end{aligned}$$

Here

$$(s)_+^2 = \begin{cases} s^2 & s > 0 \\ 0 & s \leq 0 \end{cases}$$

Let u^i be the function u restricted to (the interior) $D \times (-\infty, T]$, and u^e be the function u restricted to (the exterior) $(R^n \setminus \overline{D}) \times (-\infty, T]$. Then, in fact u satisfies (see [9])

$$u^i{}_{tt} - \triangle_x u^i = 0 \qquad on \quad D \times (-\infty, T] \tag{3}$$

$$u^e{}_{tt} - \triangle_x u^e = 0 \qquad on \quad (R^n \setminus \overline{D}) \times (-\infty, T] \tag{4}$$

$$u^i = 0, \quad u^e = (t - x_n)_+^2 \qquad for \quad t << 0 \tag{5}$$

and u^e, u^i are coupled via

$$u^i = u^e, \quad k\frac{\partial u^i}{\partial \nu} = \frac{\partial u^e}{\partial \nu} \quad on \quad \partial D \times (-\infty, T] \tag{6}$$

Here ν is the outward pointing unit normal to D.

Using standard techniques for initial boundary value problems for hyperbolic systems with zero boundary conditions, one can show that $u \in H^1_{loc}(R^n \times (-\infty, T])$. However, their is a jump in the normal derivative of u, across ∂D. Further $u^i \in H^2(D \times (-\infty, T])$ and $u^e \in H^2_{loc}(\ (R^n \backslash \overline{D}) \times (-\infty, T])$. We have included an outline of the proof for the reader's convenience - see Lemma 4.

Our aim is to study the inverse problem i.e. to recover D given $\partial u / \partial \nu$ on $\partial \Omega \times (-\infty, T]$. We prove the following

Theorem *Suppose Ω is a bounded open subset of R^n, with smooth boundary, $n \geq 2$. Let D be a strictly convex, open, subset of Ω, with smooth boundary, with $d(\partial\Omega, D) > 0$. If*

$$T - \inf_{x \in D} x_n > 6 \ diam(\Omega)$$

then D is uniquely determined from a knowledge of $\partial u / \partial \nu$ on $\partial \Omega \times (-\infty, T]$.

The theorem asserts that if the medium is probed by a plane wave, and the response is measured on a surface surrounding D, for a long enough time, then D is uniquely determined.

Majda and Taylor in [12] studied an inverse velocity transmission problem for the wave equation. They recovered convex domains D from high frequency scattering data. However, their result does not cover the situation in our theorem. Loosely speaking, they need data of the form $u(k, \theta, \omega)$ for large frequencies k, and $(\theta, \omega) \in V$, where V is a subset of $S^{n-1} \times S^{n-1}$, so that the map

$$\begin{array}{rcl} h \ : \ V & \longrightarrow & S^{n-1} \\ (\theta, \omega) & \longmapsto & \dfrac{\omega - \theta}{|\omega - \theta|} \end{array}$$

is surjective. Here θ represents the direction of the incoming wave and ω the direction in which the response is measured. In particular they can deal with the backscatter case where

$$V = \left\{ \, (\theta, -\theta) : \; \theta \in S^{n-1} \, \right\}$$

However, the situation dealt with in our theorem corresponds to

$$V = \left\{ \, (e_n, \theta) : \; \theta \in S^{n-1} \, \right\}$$

where $e_n = (0, \cdots, 0, 1)$, and one observes that $h(V) \neq S^{n-1}$.

Hansen in [6] studies a more general inverse velocity transmission problem in a bounded region. The velocity is assumed to be smooth except for a jump across a hypersurface. The medium is probed by point sources generated at all points on the boundary and the response is measured over the whole boundary. He shows that one may recover the location of the jump and the size of the jump across this discontinuity.

Isakov in [8] studies domain recovery problems for the wave equation, where the domain in question occurs as the support of the source term on the right hand side of the equation.

We attempt to recover D from the response of the medium to only one experiment. The unknown D may be described by a function of $n-1$ variables, where as the data $\partial u/\partial\nu$ depends on n parameters - so the problem is a little overdetermined. If one has the response of the medium to plane waves coming from all possible directions i.e. the Dirichlet to Neumann map, then the problem is highly overdetermined and uniqueness may be proved as in [14].

Schiffer considered an inverse obstacle problem for the constant speed wave equation, where the obstacle is impenetrable i.e. u is zero on ∂D. In this case uniqueness follows fairly quickly from unique continuation for the constant speed wave equation (see Lemma 1), though he considered the case $T = \infty$ and appealed to unique continuation results for elliptic equations - see [10] for details.

Friedman and Isakov in [5] considered an inverse transmission problem for an elliptic equation where they proved uniqueness provided D was polygonal and was far enough from $\partial\Omega$. They used unique continuation for Laplace's equation and special properties of solutions of Laplace's equation in a wedge shaped domain. Note that the elliptic problem is harder, because one attempts to recover an object depending on $n-1$ parameters with data also depending only on $n-1$ parameters - though Isakov and Friedman's result is only for polygonal domains, which are determined if the vertices of the polygon are known. However, they allow their input disturbance to be any non-zero function, unlike the specialised nature of the incoming wave for our theorem.

We close the introduction with a few remarks on some unsolved transmission problems for the wave equation. Firstly, it would be interesting to study the uniqueness question when η is not piecewise constant but some function smooth everywhere except for a jump across D. Further, instead of assuming an impulsive wave as the incoming disturbance, is uniqueness valid for any non-zero incoming wave? Finally, the inverse velocity transmission problem has not been resolved, even for the impulsive incoming wave case. Most of the ideas from this article carry over but we were unable to complete the proof.

2 LEMMAS

In this section we present lemmas, on the following four topics, needed in the proof of the Theorem.

- A unique continuation result for the constant speed wave equation.
- An analysis of the wave front set of u.
- The regularity of u.
- Green's theorem for domains which may not have a smooth boundary.

2.1

We prove a unique continuation result for the constant speed wave equation. Here $y \in R^{n-1}$, $z, t \in R$.

Lemma 1 *Suppose ρ, l, T are fixed positive real numbers, and $u(y,z,t)$ is a distribution on R^{n+1} satisfying*

$$u_{tt} - \triangle_y u - u_{zz} = 0$$

on

$$\{\ (y,z,t)\ : |y| < \rho,\ 0 \leq z < l,\ |t| < T\ \}$$

If u and u_z are zero on

$$\{\ (y,0,t)\ : |y| < \rho,\ |t| < T\ \}$$

then u is zero on

$$\{\ (y,z,t)\ : |y| < \rho,\ 0 \leq z < l,\ 2z + |t| < T\ \}$$

This lemma asserts that if u satisfies the wave equation in a cylindrical neighbourhood of the line

$$\{\ (0,z,0)\ :\ 0 \leq z < l\ \}$$

and u and u_z are zero on one end of this cylindrical neighbourhood then u is zero on a tapered cylindrical neighbourhood of the above line, the tapering occurring in the t direction. Further, since the laplacian is rotation and translation invariant, the above result is valid for any line segment in (y,z) space not just for a segment of the z axis.

The proof of Lemma 1 follows fairly quickly from the unique continuation theorem for the timelike cauchy problem for the constant coefficient wave equation in [1]. Let us state this result for completeness. Here $y \in R^{n-1}$ and $z, t \in R$.

Lemma 2 (John) *Suppose ϵ and H are positive real numbers, and $u(y,z,t)$ is a smooth function satisfying*

$$u_{tt} - \triangle_y u - u_{zz} = 0$$

on

$$\{\ (y,z,t)\ :\ z \geq 0,\ |(y-a,z)| + |t| < H,\ |a| < \epsilon\ \}$$

If u and u_z are zero on

$$\{\ (y,0,t)\ : |y| < \epsilon,\ |t| < H\ \}$$

then u is zero on

$$\{\ (y,z,t)\ :\ z \geq 0,\ |(y,z)| + |t| < H\ \}$$

Note that wave equation is translation invariant so that this result is valid over translations of the above region.

Proof of Lemma 1 First we prove the lemma when u is smooth.

Choose a small positive δ. Choose a point $b \in R^{n-1}$ and a $t_0 \in R$ with $|b| < \rho - \delta$ and $|t_0| < T - \delta$. Then using Lemma 2 with δ playing the role of H we have u is zero on

$$\{ \ (y,z,t) \ : \ z \geq 0, \ |(y-b,z)| + |t-t_0| < \delta \ \}$$

for all b and t_0 with $|b| < \rho - \delta$ and $|t_0| < T - \delta$. Taking $y = b$ in the above region we have u is zero on

$$\{ \ (y,z,t) \ : \ |y| < \rho - \delta, \ z + |t| < T, \ 0 \leq z < \delta \ \}$$

Choose a z_0 in $[0,\delta)$. Then u and u_y are zero on

$$\{ \ (y,z_0,t) \ : \ |y| < \rho - \delta, \ |t| < T - z_0, \ \}$$

Choose b and t_0 with $|b| < \rho - \delta$ and $|t_0| < T - z_0 - \delta$. Then from Lemma 2 we have u is zero on

$$\{ \ (y,z,t) \ : \ z \geq z_0, \ |(y-b,z-z_0)| + |t-t_0| < \delta, \ \}$$

for all z_0, b, t_0 with $0 \leq z_0 < \delta$, $|b| < \rho - \delta$, and $|t_0| < T - z_0 - \delta$. Taking $z = z_0$ and $t = t_0$ in this region we have u and u_z are zero on

$$\{ \ (y,z_0,t_0) \ : \ |y-b| < \delta \ \}$$

Hence u is zero on

$$\{ \ (y,z,t) \ : \ 0 \leq z < \delta, \ |y| < \rho, \ z + |t| < T - \delta, \ \}$$

This is true for all sufficiently small δ so we may conclude that u is zero on

$$\{ \ (y,z,t) \ : \ 0 \leq z < \delta, \ |y| < \rho, \ 2z + |t| < T \ \}$$

So we have progressed a distance δ in the z direction. If we repeat this procedure several times the lemma follows for the smooth u case.

To prove the lemma in the non-smooth u case we extend the region of definition of u to include $z < 0$ by taking u to be zero for $z < 0$. Since u and

u_z are zero on $z = 0$, u will satisfy the wave equation even for $z < 0$. Now if we choose a smooth function $\phi(y, z, t)$ with *supp* $\phi \subset B_\epsilon(0, -\epsilon, 0)$ then $u * \phi$ is a smooth function satisfying the hypothesis of the lemma with ρ, l, and T replaced by $\rho - \epsilon$, $l - \epsilon$, and $T - \epsilon$. So from the result for the smooth case, $u * \phi$ is zero on the appropriate region. Noting that this is valid for all ϕ and all small ϵ we have u is zero on the required region.

QED

2.2

We now give a description of the wave front set of the solutions u^i, u^e of (3) - (6). One can perhaps intuitively see that the wave front set will consist of incident, transmitted, and reflected bicharacteristics - we shall not need the behavior near grazing points. We attempt to state it precisely in Lemma 3.

Since D is strictly convex and smooth, there exist exactly two points where the tangent plane to ∂D is parallel to $x_n = 0$. Let $p = (p', p_n)$, ($p' \in R^{n-1}$, $p_n \in R$) be the point which has the smaller *nth* coordinate. Then in (3) - (6), p is the first point on ∂D to feel the influence of the incident wave, and the wave reaches p at time p_n. We shall need the wave front set of u^i, u^e only near $(x = p, t = p_n)$. The direction of propagation of the incident wave is transversal to ∂D near p - hence we can gladly avoid considering grazing rays.

Let N be a small neighbourhood of p in R^n, and $\epsilon > 0$, so that no line parallel to the x_n axis is tangential to $\partial D \cap N$, and the incoming wave does not reach any point of $\partial D \setminus N$ before $t = p_n + \epsilon$. Let $\nu(a)$ be the outward pointing unit normal to $\partial D \cap N$ at a. We define the reflected vector $r(a)$ at a to be the unit vector in R^n, in the plane determined by $\nu(a)$ and $e_n = (0, \cdots, 0, 1)$, so that

$$-e_n \cdot \nu(a) = \nu(a) \cdot r(a)$$

i.e. $r(a)$ and $-e_n$ make equal angles with $\nu(a)$. See Figure 1.

We use the notation $x = (x', x_n)$, where $x \in R^n$, $x' \in R^{n-1}$. A point in the cotangent bundle of $R^n \times R$ will be written as $[x, t; \xi, \tau]$ with x, $\xi \in R^n$, and t, $\tau \in R$. Define the projection

$$\begin{array}{rrcl} \pi : & T^*(R^n_x \times R^1_t) & \longmapsto & R^n_x \times R^1_t \\ & [x, t; \xi, \tau] & \longmapsto & (x, t) \end{array}$$

We define the set of incident and transmitted bicharacteristics

$$\mathcal{IT} = \{ \; [\, x, x_n \,; (0, \sigma), \sigma \,] \quad : \quad x \in R^n, \quad \sigma \in R \; \}$$

and the set of reflected bicharacteristics near $N \times (p_n, p_n + \epsilon)$

$$\mathcal{R} = \{ \; [\, a + (t - a_n) r(a), t \,; \sigma r(a), \sigma \,] \; : \; a \in N \cap \partial D, \; a_n < t < a_n + \epsilon, \; \sigma \in R \; \}$$

Then we have

Lemma 3

$$\begin{aligned} WF(\, u^e|_{N \times (-\infty, p_n + \epsilon)}) &= (\mathcal{IT} \cup \mathcal{R}) \cap \pi^{-1}(N \times (-\infty, p_n + \epsilon)) \\ WF(\, u^i|_{N \times (-\infty, p_n + \epsilon)}) &= \mathcal{IT} \cap \pi^{-1}(N \times (-\infty, p_n + \epsilon)) \end{aligned}$$

Remark: Note the equality in the statement of the Lemma (assuming $k \neq 1$). From results for hyperbolic systems of initial boundary value problems (e.g. in [15]), one can quickly see that the LHS is contained in the RHS. In the Lemma we make the stronger claim that equality holds i.e u^i and u^e are indeed singular in the directions prescribed in the Lemma. A more general result is proved in [7] using considerable microlocal machinery but we have included the proof of our specialised result because of its elementary nature.

Proof of Lemma 3

We prove the Lemma by constructing the most singular part of u^e and u^i. We construct this by writing u^e and u^i as a progressing wave as done in [4]. We shall do this construction only for $n = 2$, the higher dimensional case proceeds along the same lines - the notation gets a little cumbersome.

To avoid too many subscripts, just for this Lemma, we shall use $x = (y, z)$, $y, z \in R$. The incoming wave will be $(t - z)_+^2$. There is no loss of generality in assuming that the first point on ∂D to feel the incoming wave is the origin, and near the origin ∂D is given by $z = \rho(y)$ with $\rho(0) = 0$ and $\rho(y)$ is strictly convex near $y = 0$. Throughout this proof, the constructions are valid only for $t \in (-\infty, \epsilon)$.

PICTURE GOES HERE

We first construct the reflected front. See Figure 2. Let $\vec{r}(s)$ be the unit reflected vector at $(s, \rho(s))$. Then

$$\vec{r}(s) = \sin 2\theta \mathbf{j} - \cos 2\theta \mathbf{k}$$

where $\tan\theta = \rho'(s)$. Therefore at time $t > 0$ the reflected front has parametric equations

$$y = s + (t - \rho(s)) \sin 2\theta, \quad z = \rho(s) - (t - \rho(s)) \cos 2\theta \tag{7}$$

with s varying over a neighborhood of 0.

Let us write the equation of the reflected front, at time t, in the form $t = \sigma(y, z)$. Then

$$\sigma(y, z) = \rho(y) \quad on \;\; \partial D \tag{8}$$

Further, $t - \sigma(y, z)$ satisfies the eikonal equation corresponding to the wave equation i.e.

$$1 = \sigma_y^2 + \sigma_z^2$$

We now express u^e and u^i as progressing waves. We will only be concerned with the most singular term, so the following equations will be correct up to a smoother term.

We seek the most singular part of u^e and u^i in the form

$$\begin{aligned} u^e(y, z, t) &= (t - z)_+^2 + a(y, z, t)(t - \sigma(y, z))_+^2 && (9) \\ u^i(y, z, t) &= b(y, z, t)(t - z)_+^2 && (10) \end{aligned}$$

where a and b must satisfy the transport equations

$$a_t + \sigma_y a_y + \sigma_z a_z + (\sigma_{yy} + \sigma_{zz})a = 0 \tag{11}$$

$$b_t + b_z = 0 \tag{12}$$

if u^i and u^e are to satisfy the wave equation (3), (4) with C^1 error. Further, $\sigma(y, z)$ is bounded for $(y, z) \in N$, so u^i and u^e also satisfy the initial condition (5). It remains for u^i and u^e to satisfy the transmission condition (6).

(11) may be thought of as an ode along the reflected bicharacteristics, and (12) may be thought of as an ode along a transmitted bicharacteristic. The initial values for a and b will be determined by the values of $a(y, z, t)$ and $b(y, z, t)$ for $(y, z) \in \partial D$.

We shall use (6) to show that a, b are non-zero for all (y,z) on ∂D near $(0,0)$, for $t \geq z$. From this it follows that a is non-zero for all (y,z,t) near $(0,0,0)$, for $t \geq \sigma(y,z)$, because a obey the homogeneous transport equation. Further, $b(y,z,t)$ will be non-zero for $t > z$, for all (y,z) near $(0,0)$. Hence the Lemma will follow from an analysis of the wave front set of (9), (10).

So it remains to prove that a, b are non-zero on ∂D, near the origin, for $t \geq z$. Using (6), we have $u^e = u^i$ and $u^e_z = k u^i_z$ on ∂D. Hence, matching the most singular part, on ∂D, we have

$$\begin{aligned} (t-z)_+^2 + a(y,z,t)(t-\sigma(y,z))_+^2 &= b(y,z,t)(t-z)_+^2 \\ -2(t-z)_+ - 2a(y,z,t)(t-\sigma(y,z))_+\sigma_z(y,z) &= -kb(y,z,t)(t-z)_+ \end{aligned}$$

Noting that ∂D is given by $z = \rho(y)$, and $\sigma(y,z) = \rho(y)$ on ∂D, we have

$$\begin{aligned} 1 + a(y,z,t) &= b(y,z,t) \\ -2 - 2a(y,z,t)\sigma_z(y,z) &= -2kb(y,z,t) \end{aligned}$$

on ∂D, when $t \geq z$. Hence, on ∂D, for $t \geq z$

$$a(y,z,t) = \frac{1-k}{k-\sigma_z(y,z)}, \quad b(y,z,t) = \frac{1-\sigma_z(y,z)}{k-\sigma_z(y,z)} \tag{13}$$

Now $t = \sigma(y,z)$ is obtained by solving (7) for s, t in terms of y, z. Using implicit differentiation, one can show that $\sigma_z(0,0) = -1$. Therefore, noting that $k \neq 1$, (13) implies that a, b are non-zero at $(0,0,0)$. Therefore continuity implies a and b are non-zero on ∂D, near $(0,0,0)$, for $t \geq z$. This proves the Lemma.

QED

2.3

Here we show that u has the regularity which was claimed in the introduction.

Lemma 4 *The system (3) - (6) has a unique solution with* $u^i \in H^2(D \times (-\infty, T])$ *and* $u^e \in H^2_{loc}(\,(R^n \setminus \overline{D}) \times (-\infty, T])$.

If we identify a neighbourhood of ∂D in $R^n \setminus D$ with a neighbourhood of ∂D in $\overline{D}$, then the above system may be thought of as a system of 2nd order hyperbolic pde with homogeneous boundary condition. So we can imitate the proofs of results for such systems to prove our Lemma. The ideas used in proving the existence and uniqueness may be found in [11]. The ideas used in proving the regularity are in [13]. For our and the reader's convenience we have included an outline of the proof.

Outline of Proof of Lemma 4

Using finiteness of speed of propagation, it is clear that in the period $(-\infty, T]$ only a compact subset of R^n will feel the effect of the reflection. Therefore if $v = u - (t - x_n)_+^2$, then there is no loss of generality if we study the following problem

$$
\begin{aligned}
&{v^i}_{tt} - \triangle_x v^i = f^i \qquad on \quad D \times [0, T] && (14)\\
&{v^e}_{tt} - \triangle_x v^e = f^e \qquad on \quad (R^n \setminus \overline{D}) \times [0, T] && (15)
\end{aligned}
$$

$$
\begin{aligned}
&v^i(.,0) = 0, \ \ v^i_t(.,0) = 0 \qquad on \qquad D && (16)\\
&v^e(.,0) = 0, \ \ v^e_t(.,0) = 0 \qquad on \qquad R^n \setminus D && (17)
\end{aligned}
$$

and v^e, v^i are coupled via

$$
v^i = v^e, \quad k\frac{\partial v^i}{\partial \nu} = \frac{\partial v^e}{\partial \nu} \quad on \quad \partial D \times [0, T] \tag{18}
$$

Here f is a function in $L^2(R^n \times [0, T])$.

Define the class $\mathcal{T}$ of test functions as those $\phi(x, t)$ with

- $\phi \in C(R^n \times [0, T])$
- ϕ smooth on $\overline{D} \times [0, T]$ and $(R^n \setminus D) \times [0, T]$
- $\phi = 0$ if $|x|$ is large
- $\phi(., 0) = \phi_t(., 0) = 0$ on R^n
- $\partial\phi^e/\partial\nu = k\ \partial\phi^i/\partial\nu$ on $\partial D \times [0, T]$

and define the operator P

$$
Pv \equiv \eta v_{tt} - \nabla_x \cdot (\eta \nabla_x v) \equiv \begin{cases} k(v_{tt} - \triangle_x v) & in \quad D \times [0, T] \\ v_{tt} - \triangle_x v & in \quad (R^n \setminus \overline{D}) \times [0, T] \end{cases}
$$

Then using standard techniques one can prove an energy estimate for the transmission problem. It takes the form

$$\|\phi\|_1 \leq C\|P\phi\|_0 \qquad \forall \phi \in \mathcal{T}$$

The proof of existence of a unique solution of (14)-(18), in $H^1(R^n \times [0,T])$, follows exactly along the lines of the proof of Theorem 3.2 in [11] - one shows $P(\mathcal{T})$ is dense in $L^2(R^n \times [0,T])$ because of the energy estimate satisfied by functions in $\mathcal{T}$ etc.

Hence we have existence of a unique u in $H^1_{loc}(R^n \times [0,T])$. To prove that u^e and u^i are actually in H^2_{loc}, one notes that u is in H^2_{loc} for $t << 0$, and imitates the proof of Theorem 3.1 in [13].

QED

2.4

We state a version of Green's Theorem valid for special open sets in R^n - the boundary need not be smooth. We have included the relevant definitions and the statement of the Theorem for the reader's convenience. Further details may be found in [2] or [3].

We first give a definition of the exterior normal for any subset of R^n. Suppose A is a subset of R^n and x a point in R^n. A unit vector p is defined to be an exterior normal to A at x if

$$\begin{aligned} \lim_{r\to 0^+} r^{-n}\, \mu\{\, y : |y-x| < r,\ (y-x)\cdot p < 0,\ \ y \notin A \,\} &= 0 \\ \lim_{r\to 0^+} r^{-n}\, \mu\{\, y : |y-x| < r,\ (y-x)\cdot p > 0,\ \ y \in A \,\} &= 0 \end{aligned}$$

Here μ is the Lebesgue measure on R^n. It is shown in [3] that if such a unit vector exists (for a given A and x) then it is unique. We denote this unit vector by $\nu(A,x)$. If no such unit vector exists then we set $\nu(A,x) = 0$. One may verify that if A is a subset of R^n, then $\nu(A,x)$ is zero if x is in the interior of A or $R^n \setminus A$. Further, if x is on the boundary of A, and for some disk U around x,

$$U \cap A = \{\ (y', y_n) \in U\ :\ y' \in R^{n-1},\ y_n \in R,\ y_n \leq f(y')\ \}$$

for some smooth function f from R^{n-1} to R i.e. near x, A is the region on one side of a graph, then $\nu(A,x)$ is the usual outward pointing normal. So $\nu(A,x)$ extends the notion of an outward pointing normal to arbitrary subsets of R^n.

Next we give a definition of the $n-1$ dimensional Hausdorff measure on R^n. For a subset S of R^n define

$$\gamma(S) = vol(n-1)\ 2^{n-1}\ (diam\ S)^{n-1}$$

where $vol(n-1)$ is the volume of the $n-1$ dimensional unit ball. For any positive δ, define

$$\phi_\delta(S) = \inf_{\mathcal{F}} \sum_{U \in \mathcal{F}} \gamma(U)$$

where $\mathcal{F}$ is a countable open cover of S, with each set in $\mathcal{F}$ having diameter less than δ. Now $\phi_\delta(S)$ is an increasing function of δ so we may define

$$\Phi(S) = \lim_{\delta \to 0^+}\ \phi_\delta(S)$$

It is shown in [2] that Φ is a measure on the σ algebra of all Borel subsets of R^n. Further, if a surface S is the graph of a smooth function from an open subset of R^{n-1} to R, then $\Phi(S)$ equals the usual surface area of S (Section 3.3.4 in [2]). So the Hausdorff measure generalises the notion of surface area to Borel subsets of R^n. Now we may state the Gauss-Green theorem for open sets as in [3].

Lemma 5 (Federer) *Let M be a bounded open subset of R^n with boundary S, $\Phi(S) < \infty$, and $f \in C^1(\overline{M})$. Then*

$$\int_M \frac{\partial f}{\partial x_j}\, dx = \int f(x)\ \nu_j(M,x)\ d\Phi \qquad\qquad j = 1, 2, \cdots, n$$

Here $\nu_j(M,x)$ is the jth component of $\nu(M,x)$.

3 PROOF OF THE THEOREM

Suppose D_j, $j = 1,2$ are two smooth, strictly convex, open subsets of Ω with $d(\partial\Omega, D_j) > 0$. Let u_j, $j = 1,2$ be the corresponding solutions of (1) - (2). We are to show that if

$$u_1 = u_2 \quad on \quad \partial\Omega \times (-\infty, T] \tag{19}$$

then $D_1 = D_2$.

From (3) - (6), for $j = 1, 2$

$$\begin{aligned} u^i_{j,tt} - \triangle_x u^i_j = 0 \qquad & on \quad D_j \times (-\infty, T] \\ u^e_{j,tt} - \triangle_x u^e_j = 0 \qquad & on \quad (R^n \setminus \overline{D_j}) \times (-\infty, T] \\ u^i_j = 0, \quad u^e_j = (t - x_n)^2_+ \qquad & for \quad t << 0 \end{aligned}$$

and u^e_j, u^i_j are coupled via

$$u^i_j = u^e_j, \quad k\frac{\partial u^i_j}{\partial \nu} = \frac{\partial u^e_j}{\partial \nu} \quad on \quad \partial D_j \times (-\infty, T] \tag{20}$$

So u^e_1 and u^e_2 have the same initial condition, satisfy the wave equation on $(R^n \setminus \overline{\Omega}) \times (-\infty, T]$, and from (19) $u^e_1 = u^e_2$ on $\partial\Omega \times (-\infty, T]$. Hence, from uniqueness for the initial boundary value problem, we have $u^e_1 = u^e_2$ on $(R^n \setminus \overline{\Omega}) \times (-\infty, T]$.

Using the convexity of D_1 and D_2 we can show that through every point in $R^n \setminus \overline{D_1 \cup D_2}$ there is a half line contained in $R^n \setminus \overline{D_1 \cup D_2}$. Now u^e_1, u^e_2 satisfy the same wave equation in $(R^n \setminus \overline{D_1 \cup D_2}) \times (-\infty, T]$, and $u^e_1 = u^e_2$ on $(R^n \setminus \overline{\Omega}) \times (-\infty, T]$. So using Lemma 1 we conclude that

$$u^e_1 = u^e_2 \ \ on \ \ (R^n \setminus \overline{D_1 \cup D_2}) \times (-\infty, T_1] \tag{21}$$

Here $T_1 = T - 2diam(\Omega)$.

Let pj be the unique point on $\overline{D_j}$ with the smallest x_n coordinate. We may assume without loss of generality that $p1_n \leq p2_n$. If $p1 \neq p2$, then for $t > p1_n$, t close to $p1_n$, $WF(u^e_1)$ will contain a reflected bicharacteristic starting at $(x = p1, t = p1_n)$, while $WF(u^e_2)$ will not have a reflected bicharacteristic near $(x = p1, t = p1_n)$ (from Lemma 3) - in fact u^e_2 will be smooth near $(x = p1, t = p1_n)$. But u^e_1 and u^e_2 agree outside $\overline{D_1 \cup D_2}$, hence their wave front sets must agree outside that region. So we must have $p1 = p2$ - let us call it p. So p lies on $\partial D_1 \cap \partial D_2$, and is the first point where reflection occurs.

From (21), u^e_1 and u^e_2 must agree on

$$\{ \ (x, t) \ : \ x_n < p_n, \ p_n < t < T_1 \ \}$$

In particular their wave front sets must agree there. But from Lemma 3, one can recover the shape of the reflector D_j, near p, from $WF(u^e_j)$ - just trace backwards the reflected bicharacteristics. So ∂D_1 and ∂D_2 match up near p i.e. there is ball B in R^n, around p, so that

$$B \cap \partial D_1 = B \cap \partial D_2 \equiv S$$

We have

$$u_1^e - u_2^e = 0 \quad on \quad (R^n \setminus (D_1 \cup D_2)) \times (-\infty, T_1]$$

and from (6)

$$u_1^i = u_2^i, \quad \frac{\partial u_1^i}{\partial \nu} = \frac{\partial u_2^i}{\partial \nu} \quad on \quad S \times (-\infty, T]$$

So $u_1 - u_2$ is H^2 across $S \times (-\infty, T_1)$ and *zero* on one side of it. Further $u_1 - u_2$ obeys the wave equation in $(D_1 \cap D_2) \times (-\infty, T_1]$. Hence, using the connectedness of $D_1 \cap D_2$ (because they are convex), from Lemma 1, $u_1^i - u_2^i = 0$ is zero on $(D_1 \cap D_2) \times (-\infty, T_2]$, where $T_2 = T - 4\, diam(\Omega)$.

If $D_1 \neq D_2$ then let us consider the region $D_1 \setminus \overline{D_2}$ - we may assume this is non-empty. It is an open set and its boundary may not be a smooth submanifold of R^n. This forces us to analyse the boundary of $D_1 \setminus \overline{D_2}$ more carefully.

Let x be on the boundary of $D_1 \setminus \overline{D_2}$ then $x \in \partial D_1 \cup \partial D_2$. If such an x is not in $\partial D_1 \cap \partial D_2$, then either $x \in \partial D_1 \cap (\overline{D_2})^c$ or $x \in \partial D_2 \cap D_1$. We call such points *regular* boundary points. Note that near a regular boundary point, the boundary of $D_1 \setminus \overline{D_2}$ consists of either a patch from ∂D_1 or from ∂D_2 but not from both; hence the boundary of $D_1 \setminus \overline{D_2}$ is smooth near such a point. On the other hand, if x is on the boundary of $D_1 \setminus \overline{D_2}$ and $x \in \partial D_1 \cap \partial D_2$, then using the terminology of the previous section,

(i) if $\nu(D_1, x) = -\nu(D_2, x)$, then D_1, D_2 are on opposite sides of the tangent plane to D_1 at x, implying $D_1 \cap D_2 = \emptyset$. This contradicts what has been said earlier. So no such x exists.

(ii) if $\nu(D_1, x) \neq -\nu(D_2, x)$ then one may show that $\nu(D_1 \setminus \overline{D_2}, x) = 0$. Such points will be called *singular* boundary points.

Further note that $\partial(D_1 \setminus \overline{D_2}) \subset \partial D_1 \cup \partial D_2$, and ∂D_1 and ∂D_2 have finite surface area, so the boundary of $D_1 \setminus \overline{D_2}$ has finite Hausdorff measure. Hence Green's theorem is valid in $D_1 \setminus \overline{D_2}$ from Lemma 5.

Let x be a *regular* boundary point of $\partial(D_1 \setminus \overline{D_2})$. ($\nu$ is outward normal to $D_1 \setminus \overline{D_2}$). Then either $x \in \partial D_2 \cap D_1$, and there

$$u_1^i = u_2^i = u_2^e$$

$$k\,\frac{\partial u_1^i}{\partial \nu} = k\,\frac{\partial u_2^i}{\partial \nu} = \frac{\partial u_2^e}{\partial \nu}$$

or $x \in \partial D_1 \cap (\overline{D_2})^c$, and there

$$u_1^i = u_1^e = u_2^e$$

$$k\,\frac{\partial u_1^i}{\partial \nu} = \frac{\partial u_1^e}{\partial \nu} = \frac{\partial u_2^e}{\partial \nu}$$

So at every regular point on $\partial(D_1 \setminus \overline{D_2}) \times (-\infty, T_2]$, we have

$$u_1^i = u_2^e, \quad k\,\frac{\partial u_1^i}{\partial \nu} = \frac{\partial u_2^e}{\partial \nu} \tag{22}$$

and at every singular boundary point x, $\nu(D_1 \setminus \overline{D_2}, x) = 0$. Further, since $D_1 \setminus \overline{D_2}$ is an open set, if x is not on $\partial(D_1 \setminus \overline{D_2})$ then $\nu(D_1 \setminus \overline{D_2}, x) = 0$.

On $(D_1 \setminus \overline{D_2}) \times (-\infty, T_2]$, u_1^i and u_2^e satisfy the same wave equation, have the same initial condition, and (from (22)) the same Dirichlet boundary condition. Hence using uniqueness for the initial boundary value problem, we have $u_1^i = u_2^e$ on $(D_1 \setminus \overline{D_2}) \times (-\infty, T_2]$. Note, here we need energy estimates for zero BC, which are valid because we may use Green's theorem over the set $D_1 \setminus \overline{D_2}$.

Repeating this argument with the Neumann boundary condition instead of the Dirichlet boundary condition and noting (22), we have $ku_1^i = u_2^e$ on $(D_1 \setminus \overline{D_2}) \times (-\infty, T_2]$. Since $k \neq 1$, this is only possible if $u_1^i = 0$ on $(D_1 \setminus \overline{D_2}) \times (-\infty, T_2]$. So using Lemma 1, u_1^i is zero on $D_1 \times (-\infty, T_3]$, where $T_3 = T - 6diam(\Omega)$. This contradicts Lemma 3 because we are given that $T_3 > p_n$.

QED

ACKNOWLEDGEMENTS
We wish to thank Victor Isakov, Ralph Kleinman, and Bill Symes for discussions on this topic, and the referee for pointing out an error in the Introduction.

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