

AN INVERSE PROBLEM FOR THE WAVE EQUATION
IN THE HALF PLANE

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Abstract

Let R_+^n denote the half plane. Consider

$$\begin{aligned}u_{tt} - \Delta_x u + q(x)u &= 0 & (x, t) \in R_+^n \times [0, T] \\u(x, 0) = 0, u_t(x, 0) &= 0 & x \in R_+^n \\ \frac{\partial u}{\partial \nu}(x, t) &= f(x, t) & (x, t) \in (\partial R_+^n) \times [0, T]\end{aligned}$$

Define the Neumann to Dirichlet map

$$\begin{aligned}\Lambda_q : L^2(R^{n-1} \times [0, T]) &\longrightarrow H^1(R^{n-1} \times [0, T]) \\ f &\longmapsto u|_{(\partial R_+^n) \times [0, T]}\end{aligned}$$

We show that Λ_q uniquely determines q provided T is large enough and q is constant outside a ball of known radius. This result extends the uniqueness result for bounded domains to the half plane case.

We use the notation $x = (y, z)$ where $x \in R^n$, $y \in R^{n-1}$, $z \in R$, and we use R_+^n for the the half plane i.e.

$$R_+^n = \{ x \in R^n : z > 0 \}$$

Suppose $n > 1$ and $q(x)$ is a bounded function on the half plane R_+^n . Consider the initial boundary value problem

$$u_{tt} - \Delta_x u + q(x)u = 0 \quad (x, t) \in R_+^n \times [0, T] \quad (1)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0 \quad x \in R_+^n \quad (2)$$

$$u_z(y, 0, t) = f(y, t) \quad \text{for } (y, t) \in R^{n-1} \times [0, T] \quad (3)$$

The above is a well posed problem and we may define the Neumann to Dirichlet map

$$\begin{aligned} \Lambda_q : L^2(R^{n-1} \times [0, T]) &\longrightarrow H^1(R^{n-1} \times [0, T]) \\ f(y, t) &\longmapsto u(y, 0, t) \end{aligned}$$

We are interested in determining q given Λ_q .

In [6]¹ we showed that Λ_q does determine q uniquely provided T was large enough and we work with a bounded domain instead of the half plane as here. The half plane case is important because of its relevance to problems in geophysics where impulses are sent in from the surface of the earth, and the response is recorded on the surface of the earth. In this article we show that Λ_q uniquely determines q even in the half plane case, provided T is large enough and q is constant outside a ball, i.e.

Theorem $\Lambda_{q_1} = \Lambda_{q_2}$ implies $q_1 = q_2$ provided

- (i) $q_1, q_2 \in L^\infty(R_+^n)$, and are constant (possibly different) on $\{x : |x| > r\}$
- (ii) $T > (\pi + 1)r$

The proof for the bounded domain case in [6] (or [5]) does not apply to the half plane case because the ability to measure on all sides of the domain and the ability to send pulses from all sides of the domain plays an important role in the proof. However, while reading [3] and [7], it struck us that we need not send pulses from all sides because of results on the Xray transform in [2], and even in the half plane case we can recover measurements on all sides by appealing to unique continuation results for the timelike Cauchy problem for the constant coefficient wave equation. So a modification of the proof in [6] should work here. We state the above mentioned results as Lemmas.

¹See [5] for a shorter proof

Lemma 1 (Hamaker, Smith, Solmon, Wagner) *Suppose $p(x)$ is a bounded function on R^n with compact support, and A an infinite subset of R^n bounded away from the convex hull of the support of p . If the Xray transform*

$$\int_0^\infty p(a + s\omega) ds = 0, \quad \forall a \in A, \forall \omega \in R^n, \|\omega\| = 1$$

then $p = 0$.

The proof of Lemma 1 may be found in [2].

Next we state a unique continuation result for the timelike Cauchy problem for the constant speed wave equation, in a form which will be useful to us. Here $y \in R^{n-1}$, $z, t \in R$.

Lemma 2 *Suppose k is a real number, ρ, l, T are positive real numbers, and $u(y, z, t)$ is a distribution on R^{n+1} satisfying*

$$u_{tt} - \Delta_y u - u_{zz} + ku = 0$$

on

$$\{ (y, z, t) : |y| < \rho, 0 \leq z < l, |t| < T \}$$

If u and u_z are zero on

$$\{ (y, 0, t) : |y| < \rho, |t| < T \}$$

then u is zero on

$$\{ (y, z, t) : |y| < \rho, 0 \leq z < l, 2z + |t| < T \}$$

This lemma asserts that if u satisfies the wave equation in a cylindrical neighbourhood of the line

$$\{ (0, z, 0) : 0 \leq z < l \}$$

and u and u_z are zero on one end of this cylindrical neighbourhood then u is zero on a tapered cylindrical neighbourhood of the above line, the tapering occurring in the t direction. Further, since the Laplacian is rotation and translation invariant, the above result is valid for *any line segment* in (y, z) space not just for a segment of the z axis.

The proof of Lemma 2 in the $k = 0$ case is given in [4] and follows from unique continuation results for the timelike Cauchy problem for the constant

speed wave equation in [1]. The $k \neq 0$ case may be reduced to the $k = 0$ case by using the method of descent i.e. define

$$v(y, a, z, t) = e^{ia\sqrt{k}}u(y, z, t) \quad y \in R^{n-1}, \quad a, z, t \in R$$

where \sqrt{k} may be complex. Then $v(y, a, z, t)$ satisfies the wave equation with no lower order terms, t as the time variable, and y, a, z as the space variables. Further all the conditions of Lemma 2 are also satisfied.

Unique continuation across $R_+^n \times [0, T]$ may not be valid for the wave equation if its coefficients are not constants, at least when the coefficients depend on x and t - see [8] for examples; the case where the coefficients depend on x but are independent of t has not been resolved. In the theorem this is what forces us to restrict q to functions which are constant outside a ball.

Proof of Theorem

Suppose q_i , $i = 1, 2$, are bounded functions on R_+^n which are constant (possibly different) outside the ball of radius r centered at the origin. Given $\Lambda_{q_1} = \Lambda_{q_2}$, we are to show that $q_1 = q_2$ everywhere.

Throughout the article we will be using Neumann data f which have compact support for a fixed t , so solutions to (1)-(3) will have compact support for a fixed t .

We first claim that $q_1 = q_2$ if $\|x\| > r$ i.e. q_1 and q_2 are the same constant in $\|x\| > r$. This is really a simpler version of the claim of our theorem - here we claim that constant potentials q may be uniquely determined from the Neumann to Dirichlet map. This follows from Lemma 5 which we prove later. So we assume from here on that $q_1 = q_2$ in $\|x\| > r$.

Next we prove an integral identity. For a given *smooth* Neumann data f let u_i denote the solution to (1)-(3) with $q(x)$ replaced by $q_i(x)$. Let $w = u_1 - u_2$ and $p(x) = q_2(x) - q_1(x)$. Then noting that u_i satisfy (1)-(3) and $\Lambda_{q_1} = \Lambda_{q_2}$, we obtain

$$w_{tt} - \Delta_x w + q_1(x)w = p(x)u_2 \quad (x, t) \in R_+^n \times [0, T] \quad (4)$$

$$w(x, 0) = 0, \quad w_t(x, 0) = 0 \quad x \in R_+^n \quad (5)$$

$$w(y, 0, t) = 0, \quad w_z(y, 0, t) = 0 \quad (y, t) \in R^{n-1} \times [0, T] \quad (6)$$

Since $q_1 = q_2$ outside the ball of radius r , we have $p(x)$ is zero outside the

ball of radius r . Also $q_1 = k$, for some constant k , outside the ball of radius r . Hence

$$\begin{aligned} w_{tt} - \Delta_x w + kw &= 0 & \text{for } x \in R_+^n, \|x\| > r, t \in [0, T] \\ w(y, 0, t) = 0, w_z(y, 0, t) &= 0 & \text{for } \|y\| > r, t \in [0, T] \end{aligned}$$

Now, we may take w to be zero for $t < 0$ and still satisfy the wave equation. Let

$$\epsilon \equiv (T - (\pi + 1)r)/6$$

Then any point in $r < \|x\| < r + \epsilon$ may be joined to some point in $\|(y, 0)\| > r$, by a piecewise linear curve (with consecutive line segments not perpendicular), contained in $\|x\| > r$, of length less than $\pi(r + \epsilon)/2$. Hence, appealing to Lemma 2 for each line segment of this curve, we have

$$w(x, t) = 0, \quad \text{for } r \leq \|x\| \leq r + \epsilon, t \in [0, T^*], z > 0$$

where $T^* = T - \pi(r + \epsilon)$. In particular, using (6)

$$w(x, t) = 0, \quad \frac{\partial w}{\partial \nu}(x, t) = 0, \quad \text{for } (x, t) \in \partial H \times [0, T^*] \quad (7)$$

where H is the hemiball

$$H \equiv \{ x \in R_+^n : \|x\| \leq r \}$$

Now suppose $v(x, t)$ is a smooth function satisfying

$$v_{tt} - \Delta_x v + q_1(x)v = 0 \quad (x, t) \in H \times [0, T^*] \quad (8)$$

$$v(x, T^*) = 0, \quad v_t(x, T^*) = 0 \quad x \in H \quad (9)$$

Then using (4) and the divergence theorem

$$\begin{aligned} & \int_{H \times [0, T^*]} dx dt p(x) u_2(x, t) v(x, t) \\ &= \int_{H \times [0, T^*]} dx dt (\square w + q_1 w) v \\ &= \int_{H \times [0, T^*]} dx dt w (\square v + q_1 v) + \int_{\partial H \times [0, T^*]} dS_x dt (w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu}) \\ &+ \int_H dx (w_t v - w v_t) \Big|_{t=0}^{T^*} \end{aligned}$$

So using (5), (7), (8), and (9) we have

$$\int_{H \times [0, T^*]} dx dt p(x) u_2(x, t) v(x, t) = 0 \quad (10)$$

for every f in $C_0^\infty(R^{n-1} \times (0, T])$, for all u_2 satisfying (1)-(3) (with $q = q_2$), and for all smooth v satisfying (8), (9).

Now we construct special solutions u_2 and v which are concentrated near a line. Using these in (10) we will show that the Xray transform of p is zero along certain lines. Then appealing to Lemma 1 we will be able to claim that $p(x) = 0$. The special solutions will be of the form given in

Lemma 3 *Suppose $\omega \in R^n$ is a unit vector, σ a positive real number, and $\chi \in C^\infty(R^n)$ with*

$$\text{supp } \chi \subset \{x \in R^n : x_n < -\epsilon, \|x\| < 2\epsilon\} \equiv N_\epsilon \quad (11)$$

Then for any locally bounded function $q(x)$ we can construct $u(x, t)$ and $v(x, t)$ of the form

$$\begin{aligned} v(x, t) &= \chi(x + t\omega)e^{-i\sigma(x \cdot \omega + t)} + R_1(x, t), & \|R_1\|_{L^2(H \times [0, T^*])} &\leq \frac{C}{\sigma} \\ u(x, t) &= \chi(x + t\omega)e^{i\sigma(x \cdot \omega + t)} + R_2(x, t), & \|R_2\|_{L^2(R_+^n \times [0, T])} &\leq \frac{C}{\sigma} \end{aligned}$$

such that u solves (1)-(3), for a suitably chosen f , and v satisfies (8)-(9). Here C depends only on T , χ , and q .

The above construction is very similar to the construction used in [6] and we give an outline of its proof after the proof of the Theorem is complete. So we continue with the proof of the Theorem.

Proof of Theorem continued

We may take u_2 and v to have the special form given in Lemma 3. Using these special solutions in (10) we obtain

$$\begin{aligned} 0 &= \int_{H \times [0, T^*]} p(x) u_2(x, t) v(x, t) dx dt \\ &= \int_{H \times [0, T^*]} p(x) \chi^2(x + t\omega) dx dt + \text{Remainder} \end{aligned}$$

where

$$\begin{aligned} \text{Remainder} &= \int_{H \times [0, T^*]} R_1(x, t) \chi(x + t\omega) e^{i\sigma(x \cdot \omega + t)} \\ &\quad + \int_{H \times [0, T^*]} R_2(x, t) \chi(x + t\omega) e^{-i\sigma(x \cdot \omega + t)} + \int_{H \times [0, T^*]} R_1(x, t) R_2(x, t) \end{aligned}$$

Using the Cauchy-Schwarz inequality and the L^2 estimates for R_1, R_2 , we obtain

$$|Remainder| \leq \frac{C}{\sigma}$$

with C independent of σ . So we have

$$\begin{aligned} 0 &= \lim_{\sigma \rightarrow \infty} \int_{H \times [0, T^*]} p(x, t) u_2(x, t) v(x, t) dx dt \\ &= \int_{H \times [0, T^*]} p(x) \chi^2(x + t\omega) dx dt \\ &= \int_{R^n \times [0, T^*]} p(x) \chi^2(x + t\omega) dx dt \\ &= \int_{R^n} \chi^2(x) dx \int_0^{T^*} p(x - t\omega) dt \end{aligned}$$

Here we have used $p(x)$ is zero outside H .

The class of functions $\chi^2(x)$ satisfying (11) is complete in $L^2(N_\epsilon)$ (N_ϵ defined in (11)). So we may conclude that

$$\int_0^{T^*} p(a - t\omega) dt = 0, \quad \forall a \in N_\epsilon, \forall \omega \in R^n, \|\omega\| = 1$$

Any point which is at a distance greater than T^* from some point of N_ϵ outside H . Since p is zero outside H we may conclude that

$$\int_0^\infty p(a - t\omega) dt = 0, \quad \forall a \in N_\epsilon, \forall \omega \in R^n, \|\omega\| = 1$$

Now appealing to Lemma 1 we conclude that $p = 0$.

QED

Outline of proof of Lemma 3

Choose R_2 as the solution of the following well posed initial boundary value problem

$$\begin{aligned} (\square + q)R_2 &= -(\square + q) \left(\chi(x + t\omega) e^{i\sigma(x \cdot \omega + t)} \right) & (x, t) \in R_+^n \times [0, T^*] \\ R_2(x, 0) &= 0, \quad \frac{\partial R_2}{\partial t}(x, 0) = 0 & x \in R_+^n \\ \frac{\partial R_2}{\partial z}(y, 0, t) &= 0 & (y, t) \in R^{n-1} \times [0, T^*] \end{aligned}$$

Then imitating the arguments used in [6] one may prove

$$\|R_2\|_{L^2(R_+^n \times [0, T^*])} \leq \frac{C}{\sigma}$$

with C independent of σ . Now if we take

$$u(x, t) = \chi(x + t\omega)e^{i\sigma(x \cdot \omega + t)} + R_2(x, t)$$

then this u satisfies whatever was sought of it in Lemma 3.

Again, choose R_1 as the solution of the following well posed initial boundary value problem

$$\begin{aligned} (\square + q)R_1 &= -(\square + q)\left(\chi(x + t\omega)e^{-i\sigma(x \cdot \omega + t)}\right) & (x, t) \in H \times [0, T^*] \\ R_1(x, T^*) &= 0, \quad \frac{\partial R_1}{\partial t}(x, T^*) = 0 & x \in H \\ \frac{\partial R_1}{\partial \nu}(x, t) &= 0 & (x, t) \in \partial H \times [0, T^*] \end{aligned}$$

Then imitating the arguments used in [6] one may prove

$$\|R_1\|_{L^2(H \times [0, T^*])} \leq \frac{C}{\sigma}$$

with C independent of σ . Now if we take

$$v(x, t) = \chi(x + t\omega)e^{-i\sigma(x \cdot \omega + t)} + R_1(x, t)$$

then

$$\begin{aligned} \text{supp } v(\cdot, T^*) &\subset \text{supp } \chi(x + T^*\omega) \cap H \\ &\subset (N_\epsilon - T^*\omega) \cap H \\ &= \emptyset \end{aligned}$$

Hence $v(\cdot, T^*)$ (and similarly $v_t(\cdot, T^*)$) are zero on H . Now one may verify that v satisfies whatever was sought of it in Lemma 3.

QED

In the proof of the theorem we used Lemma 5 which claims that constant potentials may be recovered from the Neumann to Dirichlet map. Here we give a proof of it. We first prove in the one dimensional case that constant potentials may be recovered from an arbitrary single experiment (instead of

the Neumann to Dirichlet map). Even in higher dimensions, for the half plane case, one may show, using arguments very similar to the one dimensional case, that constant potentials may be recovered from an arbitrary single experiment (and that would be enough for our purpose), but the proof involves complicated expressions so we have chosen not to use it here.

Lemma 4 *Suppose $k_j, j = 1, 2$ are real constants, and for $j = 1, 2$*

$$\begin{aligned} u_{tt}^j - u_{zz}^j + k_j u^j &= 0 & z \geq 0, t \in [0, T] \\ u^j(z, 0) = 0, u_t^j(z, 0) &= 0 & z \geq 0 \\ \frac{\partial u^j}{\partial z}(0, t) &= f(t) & t \in [0, T] \end{aligned}$$

Then $u^1(0, t) = u^2(0, t)$ for $t \in [0, T]$ implies $k_1 = k_2$ provided $f \in C^2[0, T]$ and f is non-zero.

Proof of Lemma 4

We shall prove the Lemma by working with explicit formulas for u^j and comparing the boundary values. Consider

$$\begin{aligned} u_{tt} - u_{zz} + ku &= 0 & z \geq 0, t \in [0, T] \\ u(z, 0) = 0, u_t(z, 0) &= 0 & z \geq 0 \\ \frac{\partial u}{\partial z}(0, t) &= f(t) & t \in [0, T] \end{aligned}$$

Then $e^{ia\sqrt{k}}u(z, t)$ solves the two space dimensional wave equation with no lower order terms. Then using the method of descent and the Green's function for the half plane in three space dimensions, we can show that

$$u(z, t) = \frac{-1}{2\pi} \int_{\|(m_1, m_2, z)\| \leq t} dm_1 dm_2 \frac{e^{im_1\sqrt{k}} f(t - \|(m_1, m_2, z)\|)}{\|(m_1, m_2, z)\|}$$

Hence

$$\begin{aligned} u(0, t) &= \frac{-1}{2\pi} \int_{\|m\| \leq t} dm_1 dm_2 \frac{e^{im_1\sqrt{k}} f(t - \|m\|)}{\|m\|} \\ &= \frac{-1}{2\pi} \int_0^t dr f(r) \int_0^{2\pi} d\theta e^{i(t-r)\cos\theta\sqrt{k}} \\ &= \int_0^t dr f(r) \phi((t-r)\sqrt{k}) \end{aligned}$$

where

$$\phi(s) \equiv \frac{-1}{2\pi} \int_0^{2\pi} d\theta e^{is \cos \theta} \quad (12)$$

Noting that $\phi(0) = -1$, $\phi'(0) = 0$, and $\phi''(0) = 1/2$, we obtain

$$u_{ttt}(0, t) = -f''(t) + \frac{k}{2}f(t) + k^{3/2} \int_0^t dr f(r) \phi'''((t-r)\sqrt{k})$$

Now, if $u^1(0, t) = u^2(0, t)$ then $u_{ttt}^1(0, t) = u_{ttt}^2(0, t)$. Hence

$$\begin{aligned} \frac{k_1}{2}f(t) + k_1^{3/2} \int_0^t dr f(r) \phi'''((t-r)\sqrt{k_1}) = \\ \frac{k_2}{2}f(t) + k_2^{3/2} \int_0^t dr f(r) \phi'''((t-r)\sqrt{k_2}) \end{aligned}$$

Noting that $\phi'''(s)$ is bounded if s lies in a bounded subset of the complex plane we obtain

$$|k_2 - k_1| |f(t)| \leq C \int_0^t dr |f(r)|, \quad t \in [0, T]$$

with C depending only on k_1, k_2, T . If $k_1 \neq k_2$ then

$$|f(t)| \leq C \int_0^t dr |f(r)|, \quad t \in [0, T]$$

with C depending only on k_1, k_2, T . Hence $|f(t)| = 0$ for $t \in [0, T]$ from Gronwall's inequality, contradicting the hypothesis of the Lemma.

QED

Lemma 5 *Suppose k_j , $j = 1, 2$ are real constants and for $j = 1, 2$*

$$\begin{aligned} u_{tt}^j - \Delta_y u^j - u_{zz}^j + k_j u^j &= 0 & y \in R^{n-1}, z \geq 0, t \in [0, T] \\ u^j(y, z, 0) = 0, u_t^j(y, z, 0) &= 0 & y \in R^{n-1}, z \geq 0 \\ \frac{\partial u^j}{\partial z}(y, 0, t) &= f(y, t) & y \in R^{n-1}, t \in [0, T] \end{aligned}$$

If $u^1(0, 0, t) = u^2(0, 0, t)$ for $t \in [0, T]$, for all $f \in C^2(R^{n-1} \times [0, T])$ with $\text{supp } f \subset \{y : \|y\| < 2T\} \times [0, T]$, then $k_1 = k_2$.

Proof of Lemma 5

Suppose $u_1(0, 0, t) = u_2(0, 0, t)$ for $t \in [0, T]$ for all the f mentioned in the above Lemma. Now $u^j(0, 0, t)$ for $t \in [0, T]$ is influenced by the value of $f(y, t)$ only if $\|y\| \leq T$. Hence $u_1(0, 0, t) = u_2(0, 0, t)$ for $t \in [0, T]$ for all f which are independent of y i.e. for all $f \in C^2[0, T]$. But if f is independent of y then so are u^j , and we are now in the one space dimensional case. We now apply Lemma 4 and Lemma 5 follows.

QED

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