A One Dimensional Inverse Problem For A Hyperbolic System With Complex Coefficients

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Abstract: Photoinduced fiber grating design leads to the following inverse problem. Suppose a complex valued $q(x) \in L^2[0, \infty)$ with q supported in $[0, X]$ represents the grating. Let A, B be the solution of the initial boundary value problem

$$\begin{aligned} A_x - A_t &= q(x)B, \qquad \forall x, \ \forall t \\ B_x + B_t &= \overline{q}(x)A, \qquad \forall x, \ \forall t \end{aligned}$$

with the initial conditions

$$A(x,t) = 0, \qquad B(x,t) = \delta(t-x), \qquad \text{for } t < 0 \ .$$

The response of the grating to the input, measured at the left end, is $A(x=0, t)$. We show that $q(x)$ may be recovered from a knowledge of $A(0,t)$ over the interval $[0, 2X]$. We also analyze the properties of the nonlinear map

$$q \mapsto A(0, .)_{[0,2X]} \ .$$

1 Introduction

The problem under consideration arises in the design of ultraviolet induced fiber gratings with specific response properties - see [5], [9], and in the study of solitons associated with the cubic Schrodinger equation - see [7].

The ultraviolet induced fiber grating is taken to be one dimensional and x measures the distance from the left end of the fiber. $q(x)$ is a complex valued function characterizing the grating - in [5]

$$q(x) = -\frac{i\pi}{\Lambda n_0}\frac{\triangle n(x)}{2}e^{-i\theta(x)}$$

where n_0 is the average refractive index, Λ is a reference period, $\triangle n(x)$ accounts for the local grating strength, and $\theta(x)$ will determine its phase variation and local period.

In our work below, we will allow a more general $q(x)$ - in fact q will be allowed to be any complex valued function in $L^2(-\infty,\infty)$. Further, we will assume $q(x) = 0$ for $x \leq 0$ and that the grating is finite so $q(x) = 0$ for $x > X$ for some known real number $X > 0$.

Consider the grating with a right moving wave(electric field) coming from the left end of the fiber. If $l(x,t)$ and $r(x,t)$ are the "left" and "right" components of the electric field in the fiber, then

$$\begin{aligned} l_x - l_t &= q(x)r, \qquad \forall\ t,\ \forall\ x & (1)\\ r_x + r_t &= \overline{q}(x)l, \qquad \forall\ t,\ \forall\ x & (2) \end{aligned}$$

with the initial conditions

$$l(x,t) = 0, \qquad r(x,t) = \phi(t-x), \qquad \text{for } t < 0\ . \tag{3}$$

with $\phi(t)$ zero for $t < 0$. So there is no wave coming from the right end of the grating. Note that because $q(x)$ is zero for $x \leq 0$ and $\phi(t)$ is zero for $t < 0$, the initial conditions do satisfy the PDE. Here $\overline{q}$ is the complex conjugate of q.

Using standard energy estimates for hyperbolic systems, one may show that if ϕ is in L^2_{loc} (i.e. square integrable on finite intervals) then (1) - (3) has a unique solution $l(x,t), r(x,t)$ which have L^2_{loc} traces on lines parallel to the x axis or the t axis, and l and r have L^2_{loc} traces on lines of slope 1 and -1 *respectively.* The solution l, r will satisfy the PDE in the weak sense - the solutions will satisfy the system of PDEs in the strong sense if q is smoother, say, C^1.

Since $q(x)$ is zero for $x \leq 0$ we have $l_x - l_t = 0$ in the region $x \leq 0$ for all t. Hence in the region $x \leq 0$, $l(x,t) = \psi(t+x)$ for some function $\psi(t)$, which will be supported in $[0,\infty)$ because $l(x,t)$ is zero for $t < 0$. Then $\psi(t+x)$ is the reflection of the input $\phi(t-x)$ by the fiber. The goal is to choose $q(x)$ so that for a given $\phi(t)$, $\psi(t)$ has the desired characteristics.

One may observe that $r(x = 0, t) = \phi(t)$ and $l(x = 0, t) = \psi(t)$, so we may define the reflection operator

$$\begin{aligned} \mathcal{R} \; : \; L^2[0,2X] & \;\rightarrow\; L^2[0,2X] \\ \phi(.) & \;\mapsto\; l(x{=}0,.) \; . \end{aligned}$$

Note that l is restricted to the time interval $[0, 2X]$ even though l may be non-zero for $t > 2X$. From the standard theory for initial value problems, $\mathcal{R}$ is a bounded linear operator. Here $L^2[0, 2X]$ is the space of all complex valued functions whose absolute value is square integrable.

We may write $\mathcal{R}$ as an integral operator with the help of the Green's function for the above initial value problem. For $q(x)$ smooth, let $A(x,t), B(x,t)$ be the solution of the initial value problem

$$A_x - A_t = q(x)B, \qquad \forall\, t,\ \forall\, x \tag{4}$$

$$B_x + B_t = \overline{q}(x)A, \qquad \forall\, t,\ \forall\, x \tag{5}$$

with the initial conditions

$$A(x,t) = 0, \qquad B(x,t) = \delta(t-x), \qquad \text{for } t < 0 \; . \tag{6}$$

Using the progressing wave expansion

$$A(x,t) = a(x,t)H(t-x) \tag{7}$$

$$B(x,t) = \delta(t-x) + b(x,t)H(t-x) \tag{8}$$

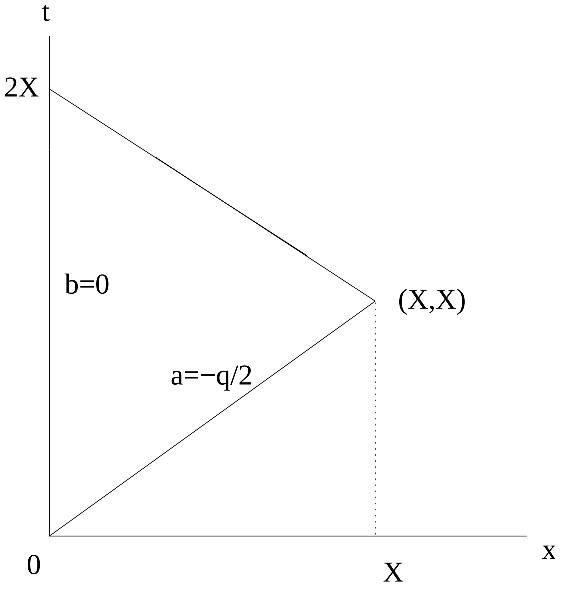
with $a(x,t), b(x,t)$ being solutions of the characteristic boundary value problem

$$a_x - a_t = q(x)b, \qquad 0 \le x \le t \tag{9}$$

$$b_x + b_t = \overline{q}(x)a, \qquad 0 \le x \le t \tag{10}$$

with the boundary conditions

$$a(x,x) = -\frac{q(x)}{2}, \qquad b(x{=}0,t) = 0 \qquad \text{for } x \ge 0,\ t \ge 0. \tag{11}$$



So for any $\phi(t)$ in $L^2[0,2X]$, the solution $l(x,t), r(x,t)$ of (1) - (3), in the region $x \geq 0$, $\forall t$, is given by

$$\begin{aligned} l(x,t) &= A(x,t) *_t \phi(t) = (a(x,t)H(t-x)) *_t \phi(t) \\ &= \int_0^{t-x} a(x,t-s)\phi(s)\, ds \qquad (12) \\ r(x,t) &= B(x,t) *_t \phi(t) = (\delta(t-x) + b(x,t)H(t-x)) *_t \phi(t) \\ &= \phi(t-x) + \int_0^{t-x} b(x,t-s)\phi(s)\, ds\ . \qquad (13) \end{aligned}$$

However, using standard techniques and energy estimates, one may show that, even if q is locally in L^2 (and not necessarily smooth), the characteristic boundary value problem (9) - (11) has a unique solution $a(x,t)$, $b(x,t)$ with these functions having L^2_{loc} traces on lines parallel to the x and t axes and $a(x,t)$ and $b(x,t)$ having L^2_{loc} traces on lines of slope 1 and -1 *respectively*. Note that $a(x,t)$ and $b(x,t)$ are supported in $0 \leq x \leq t$ and for $q \in L^2_{loc}$ will satisfy the PDE and the boundary conditions in the weak sense. Further, by applying a change of variable argument to the weak formulations of (1) - (3) and (9) - (11), one may show that the expressions for l and r in (12), (13), in terms of a, b is valid even when $q \in L^2_{loc}$ though the progressing wave argument breaks down. We chose to use the progressing wave argument for its simplicity even though the weak formulation argument is the one valid in the more general case.

Hence, the reflection operator may be written as the integral operator

$$\mathcal{R}\phi(t) = l(0,t) = \int_0^t a(x{=}0, t-s)\phi(s)\, ds\ .$$

So the problem may be rephrased in the following manner

> *Given a function $k(t)$ ($k(t-s)$ is the desired kernel of the reflection operator) supported in $[0,2X]$, determine a function $q(x) \in L^2[0,X]$ so that $a(x=0,t) = k(t)$ for all t in $[0,2X]$.*

The solution of this problem will require answering several questions. Consider the forward map mapping the fiber design q to the kernel $a(0,.)$ of the reflection operator $\mathcal{R}$.

$$\mathcal{F} \;:\; q \rightarrow a(x=0,.)|_{[0,2X]} \;.$$

Then we must answer the following questions

- What is the range of $\mathcal{F}$ i.e. what are the feasible candidates for the kernel of the reflection operator?
- Is $\mathcal{F}$ injective i.e. could there be more than one design q which generates the same kernel/reflection operator?
- How do we construct the inverse of $\mathcal{F}$ i.e. given a desired kernel $k(t-s)$ for the reflection operator, in the range of $\mathcal{F}$, how do we recover the design q which generates this response.
- Is $\mathcal{F}$ continuous and is its inverse continuous (assuming $\mathcal{F}$ is injective)? The continuity of the inverse of $\mathcal{F}$ would imply that small variations in the desired reflection operator kernel $k(.)$ would not result in dramatically different fiber designs q.

Note that $\mathcal{F}$ is a nonlinear map because the solution of the PDE (9) - (11) depends nonlinearly on q because of the qb and $\overline{q}a$ terms in the PDE. Hence answering these questions is nontrivial. We will prove

Theorem 1 *The forward map*

$$\begin{aligned} \mathcal{F} : L^2[0,X] &\rightarrow L^2[0,2X] \\ q &\mapsto a(x=0,.) \end{aligned}$$

is a continuous, injective, map whose range consists of functions $k(t)$ for which the (reflection) operator

$$\begin{aligned} \mathcal{R}_k \;:\; L^2[0,2X] &\rightarrow L^2[0,2X] \\ \phi(.) &\mapsto \int_0^t k(t-s)\phi(s)\,ds \end{aligned}$$

has norm strictly less than 1. Further, the inverse of $\mathcal{F}$ is continuous.

The necessity of the reflection operator having norm less than 1 seems plausible because, for a fiber, the reflected signal is always weaker than the incident signal because a part of the energy is transmitted through. The theorem asserts that this condition is also sufficient to generate a fiber design which generates this reflection operator.

The proof of the part of the theorem dealing with the range of $\mathcal{F}$ will involve the construction of the coefficient $q(x)$, hence the proof of this theorem will give us an inversion algorithm. Further, the continuity of the inverse of $\mathcal{F}$ implies that the algorithm is stable. A difficult part of this proof will be an upper bound on the norm of the design q in terms of the kernel $a(0,t)|_{[0,2X]}$ of the reflection operator. We state that separately as a theorem.

Theorem 2 *If $\mathcal{R}$ is the reflection operator corresponding to a coefficient q in $L^2[0,X]$ and $k(t-s)$ is the kernel of $\mathcal{R}$ (note $k(.) \in L^2[0,2X]$) then $\|R\| < 1$ and*

$$\|q\|^2 \leq \frac{2\|k\|^2}{1-\|R\|^2} \ .$$

An upper bound on $\|q\|$ in terms of the reflection operator $\mathcal{R}$ is important because this bound is used to prove that any of the algorithms proposed in the literature (including ours), for the reconstruction of q from $\mathcal{R}$, will terminate. The estimate we have provided is not sharp but is adequate for proving the termination of the algorithms. We will discuss sharp bounds on $\|q\|$ when discussing the frequency domain formulation of these problems below,

For any complex valued function $q(x) \in L^2[0,X]$ define a complex valued function $\eta(x)$ as the solution of the initial value problem

$$\eta' + q\overline{\eta} = 0, \qquad \eta(0) = 1 \ .$$

From the theory of linear ODEs we know this has a unique solution $\eta(x)$ in $[0,X]$ which will be non-zero everywhere (otherwise uniqueness would force $\eta = 0$ everywhere which would contradict $\eta(0) = 1$). Hence $q = -\eta'/\overline{\eta}$. For functions l, r satisfying (1) and (2), one may verify that

$$(\overline{\eta}l - \eta r)_t = (\overline{\eta}l + \eta r)_x \ .$$

So there is a complex valued function $w(x,t)$ so that

$$\overline{\eta}l - \eta r = 2w_x, \qquad \overline{\eta}l + \eta r = 2w_t \ .$$

Solving this system of algebraic equations for l and r we have

$$l = \frac{w_t + w_x}{\overline{\eta}}, \qquad r = \frac{w_t - w_x}{\eta} \ .$$

Substituting these into (1) or (2) and noting $q = -\eta'/\overline{\eta}$, we obtain

$$w_{tt} - w_{xx} - \overline{q}(w_x + w_t) - q(w_x - w_t) = 0 \ . \tag{14}$$

Symes in [10] studied the inverse problem associated with (14) but for the real valued q case and proved results related to Theorem 1 and Theorem 2. For the real valued q case the third and fourth terms in (14) may be combined into the one term $-2qw_x$. Much work has been done for the real valued case problem starting with the work of Gelfand-Levitan, Marchenko, Krein, Gopinath-Sondhi and others - [4] has a good summary of these works. The more recent book [6] studies many one dimensional inverse problems for hyperbolic PDEs for real valued coefficients, and some of the problems studied correspond to systems of equations where more than one coefficient is to be determined. Also [1] has a good exposition and interesting results for the real valued q case. Discussion of numerical methods for solving the inverse problems for the real valued q case may be found in [2], [3]. [6] is a good reference for a survey of theoretical results and numerical methods for one dimensional inverse problems in the time domain where several coefficients have to be determined. Also see [13] for general results on one dimensional inverse problems in the time domain.

The complex q problem has been studied by Feced et al. in [5], by Song and Shin in [9], and in [7]. In [9] and in [7], frequency domain techniques were used to recover the grating design q from the reflection data. These algorithms are inefficient since they are $O(n^3)$ algorithms. [5] contains a time domain algorithm which takes advantage of the causality to construct an $O(n^2)$ algorithm. The time domain algorithm in [5] is a downward continuation method which is a modification of the method for the real valued q case - see [3] for a discussion of this algorithm and its implementation in the real valued q case. However, these articles did not determine the range of $\mathcal{F}$ or study the continuity of the inverse of $\mathcal{F}$ or determine an upper bound on $\|q\|$ in terms of the data which is crucial in proving that the proposed algorithms will reconstruct the coefficient q. In this article our goal is to fill these gaps.

Our proof of Theorem 1, for the complex q case, imitates the proof in [10] for the real q case. For the real q case, $\mathcal{F}$ is actually a C^1 diffeomorphism onto an open subset of $L^2[0, 2X]$ as shown in [11] and we expect the same to be true for the complex q case though we have not attempted to prove that. Our proof of Theorem 2, for complex q, follows a path a little different from the one used in [10] for the real q case. In [10], for the real q case, the data is actually not the response operator $\mathcal{R}$ but the so called Neumann to Dirichlet map $\mathcal{B}$. These two operators are related via

$$\mathcal{B} = \frac{\mathcal{R} + I}{\mathcal{R} - I}$$

and results for the $\mathcal{B}$ problem imply results for the $\mathcal{R}$ problem and vice versa - see [1] for details. However, instead of imitating the work in [10] for the $\mathcal{B}$ problem and then examining the bridge with $\mathcal{R}$, we have chosen to work directly with $\mathcal{R}$ and this has resulted in a simpler proof of Theorem 2 for the complex q case.

The grating design problem may also be considered in the frequency domain as done in [9] and [7]. Taking the Fourier Transform of (1) and (2) in the time direction, and continuing to use the symbols $l(x, \omega)$ and $r(x, \omega)$ for the Fourier Transform of $l(x, t)$ and $r(x, t)$, the frequency domain version of the problem is the following. We have q as before, supported in $[0,X]$ and l,r are solutions of the system of ODEs

$$l_x - i\omega l = q\, r, \qquad r_x + i\omega r = \overline{q}\, l, \qquad -\infty < x < \infty$$

with the boundary conditions

$$\begin{aligned} r(x,\omega) &= e^{-i\omega x} && \text{for } x < 0 \\ l(x,\omega) &= 0 && \text{for } x > X \ . \end{aligned}$$

The boundary conditions assert that there is a right moving oscillatory wave coming from the left end and there are no left moving waves coming from the right end. The reflected signal is $l(x,w)$ for $x < 0$ which has the form $\mathcal{R}(\omega)e^{i\omega x}$ for $x < 0$. The goal is to recover q given the reflection coefficient $\mathcal{R}(\omega)$ for all real ω.

As proposed in [9] and [7], one defines a reflection coefficient $R(x,\omega)$, not just for $x < 0$ but for all x via

$$R(x,\omega) = \frac{l(x,\omega)}{r(x,\omega)} \ .$$

Then $R(x,\omega)$ is the solution of the initial value problem for the Ricatti equation

$$R_x = 2i\omega R - \overline{q}R^2 + q, \qquad x \geq 0$$

$$R(x=0,\omega) = 0, \qquad x > X \ .$$

Further, note that $R(x=0,\omega) = \mathcal{R}(\omega)$. So the inverse problem is the reconstruction of q given $\mathcal{R}(\omega)$.

In the frequency domain much work has been done for one dimensional inverse problems starting with the work of Gelfand and Levitan, Faddeev, Newton and others - see [8]. The algorithm proposed in [9] and [7] is the the layer stripping approach widely used for the real valued q problems. The real valued q case has been thoroughly analyzed in [12] with excellent results similar (but stronger) to our Theorems 1 and 2. The analysis for the complex valued q case in [9] and [7] is not as complete. However, we expect that the techniques in [12] will carry through to the complex q case and lead to results similar to the real q case. There is also a connection between the data for the finite time domain and frequency domain problems - see [1] for the results.

A problem similar to the problem we have studied, arises when the $\overline{q}l$ term in (2) is replaced by $-\overline{q}l$. This is the problem discussed in [7] when studying the cubic Schrodinger equation. Let q be as before, $\phi \in L^2[0,2X]$ and suppose l,r are the solutions of the initial value problem

$$\begin{aligned} l_x - l_t &= q(x)r, && \forall\ t,\ \forall\ x && (15) \\ r_x + r_t &= -\,\overline{q}(x)l, && \forall\ t,\ \forall\ x && (16) \end{aligned}$$

$$l(x,t) = 0, \qquad r(x,t) = \phi(t-x), \qquad \text{for } t < 0 \ . \tag{17}$$

Note the $-\overline{q}l$ term (instead of $\overline{q}l$) in (16). Using standard arguments, this initial value problem has a unique solution l, r which have locally L^2 traces on lines parallel to the axes. Hence, we may define (as before) the reflection operator

$$\begin{aligned} \underline{\mathcal{R}} \ : \ L^2[0,2X] & \rightarrow & L^2[0,2X] \\ \phi(.) & \mapsto & l(x{=}0,.) \ . \end{aligned}$$

As before, we may show that $\underline{\mathcal{R}}$ is an integral operator with

$$\underline{\mathcal{R}}\phi(t) = \int_0^t a(x{=}0, t-s)\phi(s) \, ds$$

where $a(x,t)$, $b(x,t)$ are the unique solution of the characteristic boundary value problem

$$a_x - a_t = q(x)b, \qquad 0 \leq x \leq t \tag{18}$$

$$b_x + b_t = -\,\overline{q}(x)a, \qquad 0 \leq x \leq t \tag{19}$$

with the boundary conditions

$$a(x,x) = -\frac{q(x)}{2}, \qquad b(x{=}0,t) = 0 \qquad \text{for } x \geq 0, \ t \geq 0. \tag{20}$$

Compare (20) with (11) - they are the same even though we replaced $\overline{q}a$ by $-\overline{q}a$ in (19).

The inverse problem consists of recovering $q(x)$ given $a(0,t)|_{[0,2X]}$. We define the forward map

$$\begin{aligned} \underline{\mathcal{F}} : L^2[0,X] & \rightarrow & L^2[0,2X] \\ q & \mapsto & a(x{=}0,.) \end{aligned}$$

and we are interested in questions similar to the ones we asked about $\mathcal{F}$. It turns out that just a change in the sign in (19) has a substantial impact on the range of $\underline{\mathcal{F}}$. One has

Theorem 3 *The forward map*

$$\begin{aligned} \underline{\mathcal{F}} : L^2[0,X] & \rightarrow & L^2[0,2X] \\ q & \mapsto & a(x{=}0,.) \end{aligned}$$

is a homeomorphism.

So compared to Theorem 1 the only change is that the range of $\underline{\mathcal{F}}$ is $L^2[0,2X]$ instead of being an open (strict) subset.

The proof proceeds exactly along the lines of the proof of Theorem 1 with the only change being the upper bound on $\|q\|$ in terms of the data (as in Theorem 2). For the new problem we have the following upper bound

Theorem 4 *Suppose $q \in L^2[0,X]$ and a, b are the solutions of (18) - (20) (so $a(0, t-s)$ is the kernel of $\underline{\mathcal{R}}$). Then*

$$\|q\|^2 \leq 2\|a(0,.)\|^2_{[0,2X]} \ .$$

Compare this with the statement of Theorem 2. It is the nature of the estimate in Theorem 4 which leads to the the surjectivity of $\underline{\mathcal{F}}$.

The proof of Theorem 4 is simpler and follows easily from integrating the identity (using Stokes's theorem)

$$(|a|^2 + |b|^2)_x + (|b|^2 - |a|^2)_t = 0$$

over the triangle OAB used in the proof of Theorem 2. Note that this identity is true only for a, b solving (18) - (20) and not (9) - (11). The corresponding identity for a, b satisfying (9) - (11) is

$$(|a|^2 - |b|^2)_x - (|a|^2 + |b|^2)_t = 0$$

and this is not enough to estimate $\|q\|$ in Theorem 2.

The statement concerning the range of $\underline{\mathcal{F}}$ in Theorem 3 is intriguing. Since the range of $\underline{\mathcal{F}}$ is all of $L^2[0,2X]$, we can find $q \in L^2[0,X]$ whose corresponding reflection operator has norm as large as we wish. In particular, for the new problem, we can find q so that the norm of the reflected signal at the $x = 0$ end is larger than the norm of the incident signal at $x = 0$. So the new problem is perhaps not a model of propagating optical signals in a fiber.

We would like to thank Greg Luther of Corning Inc. for bringing this problem to our attention, and to Rich Braun, David Edwards, and Peter Monk, for organizing the Mathematical Problems In Industry Workshop, in June 2000, at the University of Delaware, where this problem was presented by Greg Luther.

2 Proof of Theorem 2

We first prove Theorem 2 because it is shorter.

The definition of the reflection operator given in the Introduction may be restated as follows. Given a $q \in L^2[0,X]$, and the incoming wave $\phi(t-x)$ for a $\phi \in L^2[0,2X]$, consider the characteristic boundary value problem

$$\begin{aligned} l_x - l_t &= q(x)r, \qquad x \leq t \leq 2X - x, \quad 0 \leq x \leq X && (21) \\ r_x + r_t &= \overline{q}(x)l, \qquad x \leq t \leq 2X - x, \quad 0 \leq x \leq X && (22) \end{aligned}$$

with the boundary conditions

$$r(0,t) = \phi(t), \qquad 0 \leq t \leq 2X \qquad (23)$$

and

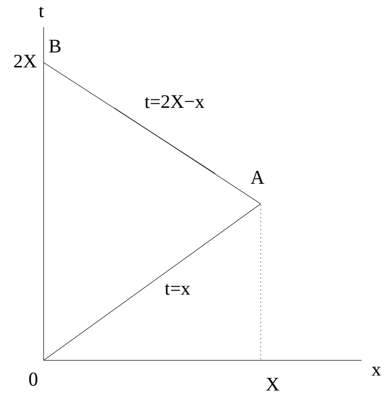
$$l(x,x) = 0, \qquad 0 \le x \le X \ . \tag{24}$$

Then from standard arguments this system has a unique solution in L^2 which has L^2 traces on lines parallel to the axes, and l, r have L^2 traces on lines of slope 1 and -1 respectively. The reflection operator corresponding to q is the map

$$\begin{aligned} \mathcal{R} \ : \ L^2[0,2X] &\rightarrow L^2[0,2X] \\ \mathcal{R}(\phi)(t) &= l(x{=}0,t) \ . \end{aligned}$$

In proving Theorem 2 we will also need an operator which measures the signal transmitted through - we call it the Transmission operator and it is defined to be

$$\begin{aligned} \mathcal{T} \ : \ L^2[0,2X] &\rightarrow L^2[0,X] \\ \mathcal{T}(\phi)(x) &= r(x, 2X-x) \ . \end{aligned}$$



So $\phi(.)$ is the signal coming in through OB, $\mathcal{R}\phi$ is the reflected signal leaving through OB and $\mathcal{T}\phi$ is the transmitted signal leaving through BA. Further, no signal is coming in through OA - that corresponds to (24). The well-posedness theory for hyperbolic systems shows that $\mathcal{R}$ and $\mathcal{T}$ are bounded (linear) operators.

From (21), (22) we have

$$\begin{aligned} (|l|^2 - |r|^2)_x - (|l|^2 + |r|^2)_t &= 2Re(\bar{l} l_x - r\bar{r}_x - \bar{l} l_t - r\bar{r}_t) \\ &= 2Re(\bar{l}(l_x - l_t) - r(\bar{r}_x + \bar{r}_t)) \\ &= 2Re(\bar{l}\, qr - r\, q\bar{l}) = 0 \ . \end{aligned} \tag{25}$$

Here $Re(z)$ is the real part of z. Integrating this over the region OAB and using Stokes's Theorem we obtain

$$0 = 2\int_{OA} |l|^2 \, dx - 2\int_{BA} |r|^2 \, dx + \int_{OB} |r|^2 - |l|^2 \, dt \ .$$

Substituting the values of l, r we obtain

$$\|\phi\|^2 = \|\mathcal{R}\phi\|^2 + 2\|\mathcal{T}\phi\|^2 \tag{26}$$

for all $\phi \in L^2[0, 2X]$ which implies

$$I = \mathcal{R}^*\mathcal{R} + 2\mathcal{T}^*\mathcal{T} \tag{27}$$

where $\mathcal{R}^*$ and $\mathcal{T}^*$ are the L^2 adjoints of $\mathcal{R}$ and $\mathcal{T}$ respectively.

For any $\phi \in L^2[0, 2X]$, if l, r is the (unique) solution of the characteristic boundary value problem (21) - (24) then let $\psi(x) = \mathcal{T}(\phi)(x) = r(x, 2X - x)$ for $x \in [0, X]$. Then the solution of the Goursat problem (21)-(22), (24), and the condition

$$r(x, 2X - x) = \psi(x), \qquad x \in [0, X] \tag{28}$$

(we replaced (23) by (28)) will be the same l, r and in particular $r(x = 0, t) = \phi(t)$ for $t \in [0, 2X]$. Note that from standard theory, the Goursat problem mentioned above has a unique solution for all $\psi \in L^2[0, X]$. Hence $\mathcal{T}$ is invertible with its inverse being

$$\begin{array}{rcl} \mathcal{T}^{-1} \; : \; L^2[0, X] & \rightarrow & L^2[0, 2X] \\ \mathcal{T}(\psi)(t) & = & r(x{=}0, t) \end{array}$$

where l, r is the unique solution of the characteristic value problem (21), (22) with the characteristic values

$$r(x, 2X - x) = \psi(x), \qquad l(x, x) = 0, \qquad 0 \leq x \leq X \; . \tag{29}$$

So $\mathcal{T}^{-1}$ is bounded, and from (26) we have

$$\|\phi\|^2 = \|\mathcal{R}\phi\|^2 + \frac{2\|\mathcal{T}^{-1}\|^2\|\mathcal{T}\phi\|^2}{\|\mathcal{T}^{-1}\|^2} \geq \|\mathcal{R}\phi\|^2 + \frac{2\|\phi\|^2}{\|T^{-1}\|^2}$$

which implies

$$\|R\|^2 \leq 1 - \frac{2}{\|T^{-1}\|^2} < 1 \; .$$

This proves one part of Theorem 2.

Now we must obtain an upper bound on $\|q\|$ in terms of the reflection operator $\mathcal{R}$. The kernel of $\mathcal{R}$ is $k(t - s)$ where $k(t) = a(0, t)$ with a, b being the solution of the characteristic boundary value problem (9) - (11). Now as shown above for l, r we have

$$(|a|^2 - |b|^2)_x - (|a|^2 + |b|^2)_t = 0 \; .$$

Integrating this over the triangular region OAB and using Stokes's theorem we have

$$0 = 2\int_{OA} |a|^2\, dx - 2\int_{BA} |b|^2\, dx + \int_{OB} |b|^2 - |a|^2\, dt\ .$$

Substituting the values of a and b from (11) and that $a(0,t) = k(t)$ we have

$$\frac{\|q\|^2}{2} = \|k\|^2 + 2\|h\|^2 \tag{30}$$

where $h(x) = b(x, 2X - x)$ is the value of b on BA. So to obtain an upper bound on $\|q\|$ we need an upper bound on $\|h\|$ in terms of $\mathcal{R}$. This will be done with the help of $\mathcal{T}^{-1}$.

Let l, r be the solution of the characteristic value problem (21), (22) with the characteristic conditions

$$r(x, 2X - x) = h(x), \qquad l(x,x) = 0, \qquad 0 \le x \le X\ . \tag{31}$$

Then from the definitions of the operators we have

$$r(0,.) = \mathcal{T}^{-1}(h), \qquad l(0,.) = \mathcal{R}\mathcal{T}^{-1}(h)\ . \tag{32}$$

Also, from the equations satisfied by a, b and l, r we have

$$\begin{aligned} 0 &= \bar{a}\, qr - r\, q\bar{a} - \bar{b}\, \bar{q}l + l\, \bar{q}\bar{b} \\ &= \bar{a}(l_x - l_t) - r(\bar{b}_x + \bar{b}_t) - \bar{b}(r_x + r_t) + l(\bar{a}_x - \bar{a}_t) \\ &= (\bar{a}\, l - \bar{b}\, r)_x \ - \ (\bar{a}\, l + r\bar{b})_t\ . \end{aligned}$$

Integrating this over the region OAB we have

$$0 = \int_{OB} (\bar{b}\, r - \bar{a}\, l)\, dt \ - \ 2\int_{BA} r\, \bar{b}\, dx \ + \ 2\int_{OA} \bar{a}\, l\, dx\ .$$

Substituting the boundary values of a, b, l, r we have

$$2\int_{BA} r\bar{b}\, dx = -\int_{OB} \bar{a}\, l$$

or equivalently

$$2\|h\|^2 = \ - \ \langle \mathcal{R}\mathcal{T}^{-1}(h), k\rangle$$

where $\langle , \rangle$ stands for the L^2 inner product. Hence

$$2\|h\|^2 \le \|\mathcal{R}\|\ \|\mathcal{T}^{-1}\|\ \|h\|\ \|k\|$$

which implies

$$2\|h\| \le \|\mathcal{R}\|\ \|\mathcal{T}^{-1}\|\ \|k\|\ . \tag{33}$$

Now from (27), we have

$$\|\mathcal{T}^{-1}\|^2 = \|\mathcal{T}^{-1}(\mathcal{T}^{-1})^*\| = \|(\mathcal{T}^*\mathcal{T})^{-1}\| = 2\|(I - \mathcal{R}^*\mathcal{R})^{-1}\| \ .$$

Hence combining this with (33) and (30) we have

$$\frac{\|q\|^2}{2} \leq \|k\|^2 + \|k\|^2\|\mathcal{R}\|^2\|(I - \mathcal{R}^*\mathcal{R})^{-1}\| \ \leq \ \frac{\|k\|^2}{1 - \|\mathcal{R}\|^2} \ .$$

because $\|(I - \mathcal{R}^*\mathcal{R})^{-1}\| \leq 1/(1 - \|\mathcal{R}\|^2)$.

QED

3 Local Inversion

The main step in the proof of Theorem 1 is the local reconstruction algorithm given in Proposition 1 below. For an $M > 0$, and $0 \leq \alpha < \beta \leq X$, define the complete metric space

$$K = \left\{ q \in L^2[\alpha, \beta] \ : \ \|q\| \leq M \ \right\} \ .$$

Note that the norm of q in K involves the values of $q(x)$ only over the interval $[\alpha, \beta]$ and not over $[0, X]$.

Proposition 1 *Given f, g in $L^2[\alpha, 2X - \alpha]$ and a large enough positive number M, there is a unique q in K, and L^2 functions $l(x,t)$, $r(x,t)$ which solve the boundary value problem*

$$\begin{aligned} l_x - l_t &= q(x) r, \qquad x \leq t \leq 2X - x, \qquad \alpha \leq x \leq \beta && (34) \\ r_x + r_t &= \overline{q}(x) l, \qquad x \leq t \leq 2X - x, \qquad \alpha \leq x \leq \beta && (35) \end{aligned}$$

$$l(\alpha, t) = f(t), \qquad r(\alpha, t) = g(t), \qquad \alpha \leq t \leq 2X - \alpha \ . \tag{36}$$

for which

$$l(x, x) = -\frac{q(x)}{2}, \qquad \alpha \leq x \leq \beta \tag{37}$$

provided $\beta - \alpha$ is small enough. Actually it is sufficient that $M \geq \sqrt{6(\|f\|^2 + \|g\|^2)}$ and $\beta - \alpha \leq \rho$ where

$$\rho = \min\{\frac{1}{4M^2}, \ X, \ \frac{1}{2500(\|f\|^2 + \|g\|^2)}\} \ . \tag{38}$$

The proof of Proposition 1 will need several energy type inequalities all of which are consequences of

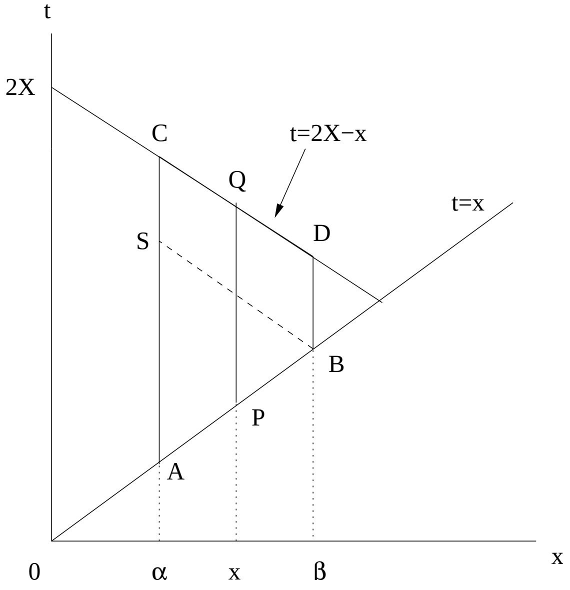
Lemma 1 *If u, v, f, g are L^2 functions with u, v having L^2 traces on lines parallel to the axes, u, v having L^2 traces on lines of slope 1 and -1 respectively, and*

$$\begin{aligned} u_x - u_t &= f, \qquad x \le t \le 2X - x, \qquad \alpha \le x \le \beta \\ v_x + v_t &= g, \qquad x \le t \le 2X - x, \qquad \alpha \le x \le \beta \end{aligned}$$

then for any x in $[\alpha, \beta]$

$$\int_{PQ} (|u|^2 + |v|^2)\, dt + 2\int_{CQ} |v|^2\, dx + 2\int_{AP} |u|^2\, dx = \int_{AC} (|u|^2 + |v|^2)\, dt + 2\iint_{APQC} Re(\overline{u}f + v\overline{g})$$

where $Re(z)$ is the real part of z.



The Lemma follows easily from an application of Stokes's theorem and the observation

$$(|u|^2 + |v|^2)_x + (|v|^2 - |u|^2)_t = 2Re(\overline{u}f + v\overline{g}) \ .$$

QED

We prove Proposition 1 using a contraction mapping argument. For a given q in K, using standard energy estimate arguments, the timelike Cauchy problem (34)-(36) has a unique solution $l(x,t)$, $r(x,t)$ in L^2 which has L^2 traces on lines parallel to the x and t axes, and l, r have L^2 traces on lines of slopes 1 and -1 respectively. So we may define the the map

$$\mathcal{C} : K \to K$$

$$\mathcal{C}(q)(x) = -2l(x,x) \ .$$

Note that $l(x,x)$ will be in $L^2[\alpha,\beta]$ but there is no guarantee that the $L^2[\alpha,\beta]$ norm of $2l(x,x)$ does not exceed M. So as it stands now it is not clear that $\mathcal{C}(q)$ is in K - we will show below that $\mathcal{C}(q)$ is in fact in K.

To prove the proposition we must find a unique q so that (37) is valid, that is show that $\mathcal{C}$ has a unique fixed point in K. We will prove that $\mathcal{C}$ is a contraction map if $\beta-\alpha$ is small enough and the proposition will follow.

We start by showing that $\mathcal{C}(q)$ is in K. Applying Lemma 1, to the functions l, r (in the definition of $\mathcal{C}(q)$), which satisfy (34) - (36), we have

$$\int_{PQ}(|l|^2+|r|^2)\,dt \;+\; 2\int_{CQ}|r|^2\,dx \;+\; 2\int_{AP}|l|^2 \;=\; \int_{AC}(|l|^2+|r|^2)\,dt \;+\; 4\iint_{APQC} Re(\bar{l}qr)\;.$$

For any x in $[\alpha,\beta]$, define

$$E(x) = \int_{PQ}(|l|^2+|r|^2) + 2\int_{AP}|l|^2 + 2\int_{CQ}|r|^2\;.$$

Then the above relation implies

$$\begin{aligned} E(x) &\le E(\alpha) + 4\iint_{APQC}|q|\,|l|\,|r| \;\le\; E(\alpha) + 2\int_{APQC}|q|\,(|l|^2+|r|^2) \\ &\le E(\alpha) + 2\int_\alpha^x |q(y)|\,E(y)\,dy \end{aligned}$$

So Gronwall's inequality implies that

$$E(x) \;\le\; E(\alpha)e^{2\int_\alpha^x |q(y)|\,dy} \;\le\; E(\alpha)e^{2\sqrt{x-\alpha}\,\|q\|}$$

for all x in $[\alpha,\beta]$. Hence

$$\begin{aligned} \|\mathcal{C}(q)\|^2 &= 4\int_\alpha^\beta |l(x,x)|^2 \;\le\; 2E(\beta) \;\le\; 2E(\alpha)e^{2\sqrt{\beta-\alpha}\,\|q\|} \;\le\; 2(\|f\|^2+\|g\|^2)e^{2\sqrt{\beta-\alpha}\,M} \\ &\le 2e^1(\|f\|^2+\|g\|^2) \;\le\; 6(\|f\|^2+\|g\|^2) \;\le\; M^2 \end{aligned}$$

Hence $\mathcal{C}(q)\in K$. Here we used $M^2 \ge 6(\|f\|^2+\|g\|^2)$ and $2M\sqrt{\beta-\alpha}\le 1$.

Next we show that $\mathcal{C}$ is a contraction map. Suppose q_1 and q_2 are in K and l_1, r_1 and l_2, r_2 are the corresponding solutions of (34)-(36). Let

$$p = q_2 - q_1, \qquad u = l_2 - l_1, \qquad v = r_2 - r_1\;.$$

Then subtracting the equations corresponding to q_1 and q_2 we have

$$u_x - u_t = p(x)r_2 + q_1(x)v, \qquad x \le t \le 2X - x, \qquad \alpha \le x \le \beta \tag{39}$$

$$v_x + v_t = \overline{p}(x)l_2 + \overline{q}_1(x)u, \qquad x \le t \le 2X - x, \qquad \alpha \le x \le \beta \tag{40}$$

Also note that

$$\begin{aligned} u(\alpha,.) &= l_2(\alpha,.) - l_1(\alpha,.) = f(.) - f(.) = 0 & (41)\\ v(\alpha,.) &= r_2(\alpha,.) - r_1(\alpha,.) = g(.) - g(.) = 0\ . & (42) \end{aligned}$$

For any x in $[\alpha,\beta]$, we redefine (to avoid using too many symbols)

$$E(x) = \int_{PQ} (|u|^2 + |v|^2) + 2\int_{AP} |u|^2 + 2\int_{CQ} |v|^2, \qquad E^*(x) = \sup_{a\le y\le x} E(y)\ .$$

Then noting that

$$\mathcal{C}(q_2)(x) - \mathcal{C}(q_1)(x) = 2(l_1(x,x) - l_2(x,x)) = -2u(x,x)$$

we have

$$\|\mathcal{C}(q_1) - \mathcal{C}(q_2)\|^2 = \int_\alpha^\beta |\mathcal{C}(q_1)(y) - \mathcal{C}(q_2)(y)|^2\, dy \ = \ 4\int_{AB} |u|^2\, dy \ \le\ 4E(\beta)\ . \tag{43}$$

So $\mathcal{C}$ will be a contraction for small $\beta-\alpha$ if we can estimate $E(\beta)$ in terms of $\beta-\alpha$ and $\|p\|$.

Applying the Lemma to (39) - (40), we have $f = pr_2 + q_1 v$ and $g = \overline{p} l_2 + \overline{q}_1 u$ and hence

$$\overline{u} f + v\overline{g} = 2q_1 \overline{u} v + p(r_2\overline{u} + \overline{l}_2 v).$$

Hence the Lemma gives

$$\begin{aligned} E(x) &\le E(\alpha) + 2\iint_{APQC} |q_1(y)|\, |u|\, |v| \ + \ |p(y)|\, |r_2|\, |u| + |p(y)|\, |l_2|\, |v| \\ &\le E(\alpha) + 2\iint_{ABDC} (|p(y)|\, |r_2|\, |u| + |p(y)|\, |l_2|\, |v|) \\ &\quad + \int_\alpha^x |q_1(y)|\, dy \int_y^{2X-y} (|u|^2 + |v|^2)(y,t)\, dt \\ &\le E(\alpha) + 2\iint_{ABDC} |p(y)|(|r_2|\, |u| + |l_2|\, |v|) + \int_\alpha^x |q_1(y)|\, E(y)\, dy \end{aligned}$$

So from Gronwall's inequality

$$\begin{aligned} E(x) &\le \left(E(\alpha) + 2\iint_{ABDC} |p(y)|(|r_2|\, |u| + |l_2|\, |v|) \right) e^{\int_\alpha^x |q_1(y)|\, dy} & (44)\\ &\le 2e^{M\sqrt{\beta-\alpha}} \iint_{ABDC} |p(y)|(|r_2|\, |u| + |l_2|\, |v|) \\ &\le 6\iint_{ABDC} |p(y)|(|r_2|\, |u| + |l_2|\, |v|) & (45) \end{aligned}$$

with the last two relations following from $\|q_1\| \leq M$ (because $q_1 \in K$), $E(\alpha) = 0$ because of (41) - (42), and that $2M\sqrt{\beta - \alpha} \leq 1$.

Now

$$\begin{aligned}\left(\iint_{ABDC} |p|\ |r_2|\ |u|\right)^2 &\leq \iint_{ABDC} |pr_2|^2 \iint_{ABCD} |u|^2 \\ &\leq \|pr_2\|^2_{ABDC} \int_\alpha^\beta E(x)\ dx \\ &\leq \|pr_2\|^2_{ABDC}\ (\beta - \alpha)E^*(\beta)\ .\end{aligned}$$

Similarly

$$\left(\iint_{ABDC} |p|\ |l_2|\ |v|\right)^2 \leq \|pl_2\|^2_{ABDC}\ (\beta - \alpha)E^*(\beta)\ .$$

Using these two relations in (45) we obtain

$$E(x) \leq 6\sqrt{\beta - \alpha}\sqrt{E^*(\beta)}\ (\|p\ l_2\|_{ABDC} + \|p\ r_2\|_{ABDC}), \qquad \alpha \leq x \leq \beta\ .$$

and taking the supremum over all x in $[\alpha, \beta]$ we have

$$E^*(\beta)\ \leq 6\sqrt{\beta - \alpha}\ \sqrt{E^*(\beta)}\ (\|p\ l_2\|_{ABDC} + \|p\ r_2\|_{ABDC})$$

implying

$$\sqrt{E^*(\beta)}\ \leq 6\sqrt{\beta - \alpha}\ (\|p\ l_2\|_{ABDC} + \|p\ r_2\|_{ABDC}) \tag{46}$$

- this $\sqrt{\beta - \alpha}$ term will be crucial in proving that $\mathcal{C}$ is a contraction.

Now it remains to estimate $\|pl_2\|_{ABDC}$ and $\|pr_2\|_{ABDC}$ in terms of $\|p\|$. We have

$$\|pl_2\|^2_{ABDC} = \int_\alpha^\beta |p(x)|^2\ dx \int_x^{2X-x} |l_2(x,t)|^2\ dt\ \leq \int_\alpha^\beta |p(x)|^2\ E_2(x)\ dx \tag{47}$$

where $E_2(x)$ is $E(x)$ with u, v replaced by l_2, r_2. If we repeat the calculations used in the derivation of (44), but for l_2, r_2 instead of u, v, we would arrive at a relation similar to (44) except that the $|pr_2u|$ and $|pl_2v|$ terms will not be there but $E(\alpha)$ will not be zero unlike the u, v case. Hence

$$E_2(x) \leq E_2(\alpha)e^{M\sqrt{\beta - \alpha}}\ \leq\ 3E_2(\alpha) \qquad \alpha \leq x \leq \beta \tag{48}$$

because $2M\sqrt{\beta - \alpha} \leq 1$. Using this in (47) we obtain

$$\|pl_2\|^2\ \leq\ 3E_2(\alpha) \int_a^b p^2(x)\ dx\ =\ 3\|p\|^2\ E_2(\alpha)$$

and similarly

$$\|pr_2\|^2\ \leq\ 3\|p\|^2\ E_2(\alpha)\ .$$

Using this in (46) we obtain

$$\sqrt{E^*(\beta)} \ \leq \ 24\sqrt{\beta-\alpha}\ \sqrt{E_2(\alpha)}\ \|p\| \ .$$

Using this in (43) we obtain

$$\|\mathcal{C}(q_1)-\mathcal{C}(q_2)\| \ \leq \ 48\sqrt{\beta-\alpha}\ \sqrt{E_2(\alpha)}\ \|p\| \ = \ 48\sqrt{\beta-\alpha}\ \sqrt{E_2(\alpha)}\ \|q_2-q_1\| \ .$$

Noting that

$$E_2(\alpha) = \int_{\alpha}^{2X-\alpha} |f(t)|^2 + |g(t)|^2 \, dt = \|f\|^2 + \|g\|^2$$

we conclude that $\mathcal{C}$ is a contraction mapping if $|\beta-\alpha| \leq \rho$ provided

$$2500\ \rho\ (\|f\|^2 + \|g\|^2) \leq 1 \ .$$

QED

4 Proof of Theorem 1

From the well-posedness theory for (9) - (11), using energy estimates, one may show that $\mathcal{F}(q)$ is in $L^2[0,2X]$, and further, an application of the Lemma to the difference of two solutions, for the case $\alpha=0, \beta=X$, combined with Gronwall's inequality, will show that $\mathcal{F}$ is locally Lipschitz continuous.

Now we prove that $\mathcal{F}$ is injective and that the range of $\mathcal{F}$ consists of $k \in L^2[0,2X]$ for which $\|\mathcal{R}_k\| < 1$. Suppose $k(.) \in L^2[0,2X]$ so that $\mathcal{R}_k$, the operator with kernel $k(t-s)$, has norm less than 1. We will show that there is a unique $q \in L^2[0,X]$ and functions a, b so that

$$\begin{aligned} a_x - a_t &= q(x)b, \qquad x \leq t \leq 2X-x, \qquad 0 \leq x \leq X && (49)\\ b_x + b_t &= \bar{q}(x)a, \qquad x \leq t \leq 2X-x, \qquad 0 \leq x \leq X && (50)\end{aligned}$$

$$a(0,t) = k(t), \qquad b(0,t) = 0, \qquad 0 \leq t \leq 2X \tag{51}$$

for which

$$a(x,x) = -\frac{q(x)}{2}, \qquad 0 \leq x \leq X \tag{52}$$

proving the claim about the range of $\mathcal{F}$ and the injectivity of $\mathcal{F}$. The existence of a unique a, b, q, satisfying the above boundary value problem, will be done by a repeated application of Proposition 1 in steps of size $\rho_0 > 0$ over the interval $[0,X]$, where ρ_0 will be determined below.

Define the positive number μ_0 by

$$\mu_0^2 = \frac{2\|k\|^2}{1-\|\mathcal{R}_k\|^2} \ .$$

Then based on Theorem 2 we expect that $\|q\| \leq \mu_0$ for the unique solution mentioned in the previous paragraph.

The ρ_0 will be the ρ of Proposition 1, but one which will work for all α and for all f, g which will arise when we start with $f = k$ and $g = 0$ at $\alpha = 0$. Now, for any a, b satisfying (49) - (51), one can show that for $0 \leq x \leq X$

$$\int_x^{2X-x} (|a(x,t)|^2 + |b(x,t)|^2) \, dt \ \leq \ e^{\|q\|\sqrt{X}} \int_0^{2X} (|a(0,t)|^2 + |b(0,t)|^2) \, dt \ = \ e^{\mu_0\sqrt{X}} \|k\|^2 \ . \quad (53)$$

- see (48) in the proof of Proposition 1.

This, combined with Proposition 1, suggests how we define the M_0 and ρ_0 that we hope will work for all α in $[0, X]$. Define M_0, ρ_0 by

$$\begin{aligned} M_0^2 &= 6e^{\mu_0\sqrt{X}}\|k\|^2 \\ \rho_0 &= \min\left\{ \frac{1}{4M_0^2}, \ X, \ \frac{e^{-\mu_0\sqrt{X}}}{2500\|k\|^2} \right\} \ . \end{aligned}$$

Now we proceed to prove the existence of a unique q, a, b satisfying (49) - (52). We start by applying Proposition 1 with $\alpha = 0$, $\beta = \rho_0$, $f = k$, $g = 0$. Since $M_0^2 \geq 6\|k\|^2$, we use $M = M_0$ and since

$$\frac{1}{2500\|k\|^2} \geq \rho_0$$

we may use $\rho = \rho_0$. So there is a unique $q \in L^2[0, \rho_0]$, and a, b, which satisfy (49) - (52) over the interval $[0, \rho_0]$ (and not $[0, X]$). So we have managed to recover q over the interval $[0, \rho_0]$.

Proposition 1 guarantees that $\|q\|_[0, \rho_0] \leq M_0$ but we can do better. A little thought (and the causality of solutions to (49) - (51)) should convince us that the reflection operator associated with the coefficient $q(x)$ on $[0, \rho_0]$ will be $k'(t-s)$ where $k'(.)$ is the restriction of $k(.)$ to the interval $[0, 2\rho_0]$. Hence $\|k'\| \leq \|k\|$ and some thought will convince us that $\|\mathcal{R}_{k'}\| \leq \|\mathcal{R}_k\|$. So from Theorem 2 applied to the $X = \rho_0$ case gives us that

$$\|q\|_{[0,\rho_0]} \leq \frac{2\|k'\|^2}{1-\|\mathcal{R}_{k'}\|^2} \leq \frac{2\|k\|^2}{1-\|\mathcal{R}_k\|^2} = \mu_0 \ .$$

Now we show the induction step. Suppose for some $x_0 < X$ we have found a unique $q \in L^2[0, x_0]$ with $\|q\|_{[0,x_0]} \leq \mu_0$ and a, b, which satisfy (49) - (52) over the region $x \leq t \leq$

$2X - x,\ 0 \le x \le x_0$ (and not $0 \le x \le X$). Then, as in (53), we have

$$\int_\alpha^{2X-\alpha} |a(x_0,t)|^2 + |b(x_0,t)|^2 \, dt \le e^{\|q\|\sqrt{x_0}} \|k\|^2 \le e^{\mu_0 \sqrt{X}} \|k\|^2 \ .$$

Apply Proposition 1 to the case where $\alpha = x_0,\ \beta = x_0 + \rho_0,\ f = a(x_0,.)$ and $g = b(x_0,.)$. Since

$$M_0^2 = 6 e^{\mu_0 \sqrt{X}} \|k\|^2 \ge 6(\|a(x_0,.)\|^2 + \|b(x_0,.)|^2)$$

we may take $M = M_0$. Further, since

$$\frac{1}{2500(\|a(x_0,.)\|^2 + \|b(x_0,.)\|^2)} \ge \frac{e^{-\mu_0\sqrt{X}}}{2500\|k\|^2} \ge \rho_0$$

we may use $\rho = \rho_0$. So there is a unique q in $L^2[x_0, x_0+\rho_0]$ and l, r, which satisfy (34) - (37) with $l(x_0,.) = a(x_0,.)$ and $r(x_0,.) = b(x_0,.)$. So we have found an extension for a, b, q so that (49) - (52) is valid over $[0, x_0 + \rho_0]$ instead of just $[0, x_0]$. Further, arguing as before we can show that $\|q\|_{[0,x_0+\rho_0]} \le \mu_0$.

So in each induction step, we recover q over an interval which is ρ_0 units longer than the previous interval and hence we will cover $[0, X]$ in a finite number of steps. Note that the size of ρ_0 was completely determined by our data $k(.)$ - this was the crucial contribution of Theorem 2. Without this estimate we would not have been able to show that the induction step terminates in a finite number of steps.

We now prove the continuity of the inverse of $\mathcal{F}$. Firstly, we show that the range of $\mathcal{F}$ is an open subset of $L^2[0, 2X]$. Given a k in the range of $\mathcal{F}$, so $k \in L^2[0, 2X]$ with $\|\mathcal{R}_k\| < 1$, we can find a ball of radius ϵ around k in $L^2[0, 2X]$ which is still in the range of $\mathcal{F}$. In fact, the linearity of $\mathcal{R}_k$ in k and the fact that (from properties of the Fourier Transform)

$$\|\mathcal{R}_k\| \le \sqrt{2X} \|k\|$$

we may take ϵ to be any number positive number less than $(1 - \|\mathcal{R}_k\|)/\sqrt{2X}$.

We will show that the inverse of $\mathcal{F}$ is locally Lipschitz continuous in the sense that given $N > 0$ and $\delta > 0$,

$$\|\mathcal{F}^{-1}(k_1) - \mathcal{F}^{-1}(k_2)\| \le C \|k_1 - k_2\|$$

for any $k_i \in L^2[0, 2X]$, with $\|k_i\| < N$ and $1 - \|\mathcal{R}_{k_i}\| > \delta,\ i = 1, 2$. Here C depends only on N, δ and X.

The above inequality is proved, as in the inversion algorithm above, by first proving a local result, and then generating a result valid over $[0, X]$. We skip the proof which is very similar to the proof in [10].

QED

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