Inverse problems for the wave equation with a single coincident source-receiver pair

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Abstract

We prove uniqueness for inverse problems for the operator $\partial_t^2 - \Delta_x - q(x)$ for data coming from a single coincident source-receiver pair. We prove uniqueness when $q_1 \geq q_2$ or when q is a product of an unknown spherically symmetric function and a known function of the angular variables but the source receiver pair is NOT at the center.

Keywords Inverse problem; wave equation.

AMS Subject Classification: 35R30, 35L10

1 Introduction

The problems considered in this article are motivated by problems arising in geophysics. The earth is probed by acoustic waves generated by explosions (impulsive point sources) on the surface of the earth and the earth's response is measured on the surface for a certain time period. Geophysicists wish to determine the properties of the interior of the earth, such as the density or elasticity parameters, from the response measured on the surface of the earth. They have access to data generated by many such experiments - corresponding to different source locations; our aim is to study whether the earth's properties can be determined from data from one experiment. Because of the configuration of the source and the receivers for such experiments, there are known difficulties (folklore) which suggest this inversion is not feasible in any stable manner in the general case. In this article we consider the uniqueness question when the earth is assumed to be either spherically symmetric (except the source is not at the center of symmetry) or when we compare two mediums where the value of the physical parameter of one medium exceeds that of the second medium.

In this article we consider the earth as occupying a ball in $\mathbb{R}^3$. We probe the earth by an acoustic wave generated by a point source on the surface of the ball and the response of the medium is measured at the same location as the source for a certain time interval. One is interested in the recovery of the function representing the medium parameter from this data. We prove two uniqueness results for this inverse problem.

Though we should model the forward problem as a bounded domain initial boundary value problem, we have chosen a free space model to focus on the difficulties associated with the nonlinear aspects of the problem. We have also assumed that the acoustic waves propagate according to the far simpler plasma wave equation instead of the more realistic equations of elasticity because this simple model is difficult enough.

Below $e = (0,0,1)$ and we use the standard notation $\square := \partial_t^2 - \Delta_x$. To every point x in $\mathbb{R}^3$ we associate its spherical coordinates (ρ, ϕ, θ) where $\rho = |x|$, $\rho \cos\phi = x \cdot e$ and θ is the angle associated with the polar coordinates of the first two coordinates of x. For our first result $q(x)$ is a product of a spherically symmetric function $a(.)$ and a function $b(.)$ of the angular variables ϕ, θ, that is $q(x) = a(|x|)b(x/|x|)$ for some function $b(.)$ on the unit sphere in $\mathbb{R}^3$.

Theorem 1. *Suppose $q_i(x)$, $i = 1, 2$ are C^3 functions on $\mathbb{R}^3$, which are supported on $|x| \leq 1 - \delta$ for some $\delta > 0$, and $q_i(x) = a_i(|x|)\, b(x/|x|)$ for some C^3 functions $a_i(.)$ on $[0, \infty)$ and $b(\cdot)$ on the unit sphere on $\mathbb{R}^3$. Let u_i be the solution of the IVP*

$$(\square - q_i)u_i = \delta(x - e, t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1}$$

$$u_i(\cdot, t) = 0 \qquad t < 0. \tag{2}$$

If $u_1(e,t) = u_2(e,t)$ for all $t \in (0, 2]$ then $q_1 = q_2$ provided $b(e) \neq 0$ and $\partial_\phi b/\sin\phi$ is bounded.

In particular the theorem is valid when the q_i are spherically symmetric - note that the source receiver pair is not at the center of symmetry.

The second uniqueness result is for general potentials q_1, q_2 but requires that $q_1(\cdot) \geq q_2(\cdot)$.

Theorem 2. *Suppose q_i, $i = 1, 2$ are C^3 functions on $\mathbb{R}^3$ and $q_1(x) \geq q_2(x)$ for all $x \in \mathbb{R}^3$. Let u_i be the solution of*

$$(\square - q_i)u_i = \delta(x, t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{3}$$

$$u_i(\cdot, t) = 0 \qquad t < 0. \tag{4}$$

If $u_1(0,t) = u_2(0,t)$ for all $t \in (0, T]$ then $q_1(x) = q_2(x)$ for all x with $|x| \leq T/2$.

In Theorem 1, if q was spherically symmetric and the source and the receiver were at the origin, at the center of symmetry, instead of being off centered, then $u(x,t)$ would also be spherically symmetric and the problem would reduce to an inverse problem for the one dimensional wave equation. This problem has been thoroughly analyzed starting with the work of Gelfand, Levitan, Marchenko, Krein and others - see [1], [2] for surveys of these methods and references. For the off centered case, even if $q(x)$ is spherically symmetric, $u(x,t)$ is not spherically symmetric and we have been unable to reduce this problem to an inverse problem for a one dimensional wave equation. Compare this with the problem studied in [5], where a horizontally stratified medium problem with a point source, in three dimensions, was reduced to a one dimensional problem even though the solution of the PDE was dependent on all the three space variables.

If the coefficients are not spherically symmetric then uniqueness results for multidimensional inverse problems for hyperbolic PDEs, for formally determined data (data and coefficients depend on the same number of parameters) are rarer. Bukhgeim and Klibanov initiated the use of Carleman estimates for formally determined multidimensional hyperbolic inverse problems but required initial data which was non-zero at every point in the domain. Romanov, Iskaov, Lavrentiev, Yamamoto, Sacks and Symes, the author and others have studied formally determined inverse problems with point sources or planar sources but uniqueness has been established only for families of q with small norms or other very specialized families of q. Please consult [7], [9] and [10] and Chapter 4 of [8] for the results, methods, and references. The interesting article [4] does have results where one of the q does not have to satisfy the smallness condition but the result is still not strong enough for general uniqueness. In light of these comments, Theorem 2 is somewhat surprising considering the simplicity of the proof of Theorem 2. Theorem 2 and its proof were motivated by a result of Stefanov in [11], where he proved a similar result in the frequency domain with the source being a plane wave instead of a point source and the data was the far field pattern instead of near field measurements.

Section 2 contains the simple proof of Theorem 2, section 3 contains the much more complicated proof of Theorem 1, and section 4 contains the proofs of two propositions used in the proof of Theorem 1. We will use the symbol $\preccurlyeq$ to mean 'less than or equal to a constant multiple'.

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2 Proof of Theorem 2

If q is a C^3 function on $\mathbb{R}^3$ and $u(x,t)$ the solution of the IVP

$$(\Box - q)u(x,t) = \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{5}$$

$$u = 0 \qquad \text{for } t < 0 \tag{6}$$

then it can be shown (see [6]) that

$$u(x,t) = \frac{1}{4\pi}\frac{\delta(t-|x|)}{|x|} + v(x,t) \tag{7}$$

where $v(x,t)$ is zero for $t < |x|$ and, in the region $t \geq |x|$, $v(x,t)$ is a C^2 solution of the characteristic boundary value problem (Goursat Problem)

$$(\Box - q)v(x,t) = 0, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \quad t \geq |x| \tag{8}$$

$$v(x,|x|) = \frac{1}{8\pi}\int_0^1 q(sx)\,ds. \tag{9}$$

Let u_1, u_2 be the solutions corresponding to q_1, q_2 with v_1, v_2 the parts associated to (7). For any $\sigma > 0$ we note that $u_2(x, 2\sigma - t) = 0$ for $t > 2\sigma$ and

$$(\Box - q_2)(u_2(x, 2\sigma - t)) = (\Box - q_2)(u_2)(x, 2\sigma - t)) = \delta(x, 2\sigma - t).$$

Hence

$$\begin{aligned}
u_1(0,2\sigma) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}} u_1(x,t)\,\delta(x,2\sigma-t)\,dx\,dt \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}} u_1(x,t)\,(\Box - q_2)(u_2(x,2\sigma-t))\,dx\,dt \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}} u_2(x,2\sigma-t)\,(\Box - q_1)(u_1(x,t)) + (q_1(x)-q_2(x))u_1(x,t)\,u_2(x,2\sigma-t)\,dx\,dt \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}} u_2(x,2\sigma-t)\,\delta(x,t) + (q_1(x)-q_2(x))u_1(x,t)\,u_2(x,2\sigma-t)\,dx\,dt \\
&= u_2(0,2\sigma) + \int_{\mathbb{R}^3}\int_{\mathbb{R}} (q_1(x)-q_2(x))u_1(x,t)\,u_2(x,2\sigma-t)\,dx\,dt.
\end{aligned}$$

Hence from the hypothesis, for all $\sigma \in (0, T/2]$, we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}} (q_1(x)-q_2(x))u_1(x,t)\,u_2(x,2\sigma-t)\,dx\,dt = 0. \tag{10}$$

Define $q := q_1 - q_2$ and let

$$Q(\sigma) := \frac{1}{4\pi\sigma^2}\int_{|x|=\sigma} q(x)\,dS_x$$

be the mean value of q on $|x| = \sigma$. Then using the expansion (7) in (10), for any $\sigma \in (0, T/2]$, we have

$$
\begin{aligned}
0 &= \frac{1}{16\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}} q(x) \, \frac{\delta(t-|x|)\,\delta(2\sigma - t - |x|)}{|x|^2} \, dx \, dt + \int_{\mathbb{R}^3} \int_{\mathbb{R}} q(x) \, v_1(x,t) \, v_2(x, 2\sigma - t) \, dx \, dt \\
&\quad + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}} q(x) \left(\frac{\delta(t-|x|)}{|x|} \, v_2(x, 2\sigma - t) + \frac{\delta(2\sigma - t - |x|)}{|x|} \, v_1(x,t) \right) \, dx \; dt \\
&= \frac{1}{16\pi^2\sigma^2} \int_{R^3} q(x) \, \delta(2\sigma - 2|x|) \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}} q(x) \, v_1(x,t) \, v_2(x, 2\sigma - t) \, dx \, dt \\
&\quad + \frac{1}{4\pi} \int_{\mathbb{R}^3} q(x) \, \frac{(v_1 + v_2)(x, 2\sigma - |x|)}{|x|} \, dx \\
&= \frac{1}{8\pi} Q(\sigma) + \int_{|x| \leq \sigma} q(x) \int_{|x|}^{2\sigma - |x|} v_1(x,t) \, v_2(x, 2\sigma - t) \, dt \, dx + \frac{1}{4\pi} \int_{|x| \leq \sigma} q(x) \frac{(v_1 + v_2)(x, 2\sigma - |x|)}{|x|} \, dx
\end{aligned}
$$

where we used the supports of $v_1(x,t)$ and $v_2(x, 2\sigma - t)$ and the definition of $Q(\sigma)$ in the last step. Now v_1 and v_2 are bounded on compact subsets, and $q(x) \geq 0$; hence for any $\sigma \in (0, T/2]$, we have

$$
\begin{aligned}
Q(\sigma) &\preccurlyeq \int_{|x| \leq \sigma} |q(x)| (1 + |x|^{-1}) \, dx = \int_0^\sigma (1 + r^{-1}) \int_{|x| = r} q(x) \, dS_x \, dr \\
&\preccurlyeq \int_0^\sigma (r^2 + r) Q(r) \, dr, \quad \preccurlyeq \int_0^\sigma Q(r) \, dr \qquad \forall \sigma \in [0, T/2].
\end{aligned} \tag{11}
$$

Since $q \geq 0$, so $Q(\sigma) \geq 0$ for all $\sigma \geq 0$, hence (11) implies $Q(\sigma) = 0$ for all $\sigma \in [0, T/2]$, which implies $q(x) = 0$ if $|x| \leq T/2$. **QED**

Remark The derivation of (10) and other steps in this section were done with a formal computation to simplify the presentation. These steps can be made rigorous if we use only v_1 and v_2, start with an integration on the double conical region $|x| \leq t \leq 2\sigma - |x|$ in x, t space (See Figure 1) and use the divergence theorem. We would have to use (9); further on the surfaces $t = 2\sigma - |x|$ and $t = |x|$ we would use integration along lines originating from the vertices to cancel out a differentiation.

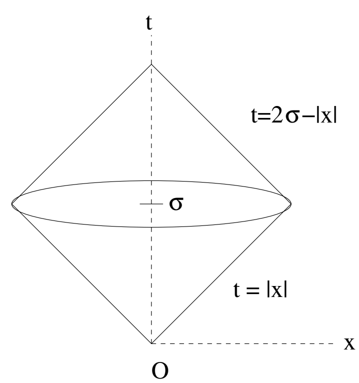

Figure 1: Double conical region

3 Proof of Theorem 1

For this section, it will be convenient to extend the definition of $b(.)$ to $\mathbb{R}^3 \setminus \{0\}$ as a homogeneous function of degree 0.

For any non-zero $y \in \mathbb{R}^3$, we define $\hat{y} := y/|y|$; for any function $h(x)$ on $\mathbb{R}^3$, h^e will denote the function defined by $h^e(x) := h(x + \mathrm{e})$. From [3], we define the operators

$$\Omega_{i,j} := y_i \partial_j - y_j \partial_i, \qquad \Omega := \frac{1}{2} \sum_{i,j=1}^{3} \Omega_{i,j}^2. \tag{12}$$

$\Omega_{i,j}$ correspond to differentiation in directions tangential to the spheres $|y| =$ constant and Ω is the Laplacian on these spheres. Further, in $\mathbb{R}^3$ (similar formulas for general $\mathbb{R}^n$)

$$|y|^2 \Delta = (y \cdot \nabla)(y \cdot \nabla) + (y \cdot \nabla) + \Omega \tag{13}$$

and

$$\Omega_{ij}(|y|) = 0. \tag{14}$$

Finally, since $\Omega_{1,2} f(y) = y \cdot (\nabla \times [0, 0, f(y)])$ with similar relations for other i, j, an application of Stokes's theorem shows that

$$\int_{|y|=\rho} (\Omega_{ij} f)(y)\, g(y)\, dS_y = -\int_{|y|=\rho} f(y)\, (\Omega_{ij} g)(y)\, dS_y. \tag{15}$$

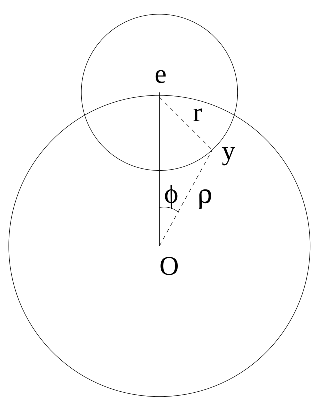

Figure 2: Spherical coordinates

To every point y in $\mathbb{R}^3$ we associate its spherical coordinates (ρ, ϕ, θ) (see figure 2) and the sphere $|y - e| = r$ may be parametrized by ρ and θ with the relations

$$2\rho \cos\phi = \rho^2 + 1 - r^2, \qquad dS_y = r\rho\, d\rho\, d\theta. \tag{16}$$

Since $q(y) = a(\rho)\, b(y)$ and q is supported on the unit ball, we have

$$(Mq)(e,r) = \frac{1}{4\pi r^2} \int_{|y-e|=r} q(y)\, dS_y = \frac{1}{4\pi r} \int_{1-r}^{1+r} \int_0^{2\pi} \rho\, a(\rho)\, b(y)\, d\theta\, d\rho$$
$$= \frac{1}{4\pi r} \int_{1-r}^{1} \int_0^{2\pi} \rho\, a(\rho)\, b(y)\, d\theta\, d\rho. \tag{17}$$

From (16) we have $\rho \sin\phi\, \partial_r \phi = r$, so

$$\partial_r (r (Mq)(e,r)) = \frac{(1-r) a(1-r)}{4\pi} \int_0^{2\pi} b(e)\, d\theta + \frac{1}{4\pi} \int_{1-r}^{1} \int_0^{2\pi} \rho\, a(\rho)\, (\partial_\phi b)(y)\, \partial_r \phi\, d\theta\, d\rho$$
$$= \frac{b(e)}{2} (1-r) a(1-r) + \frac{1}{4\pi r} \int_{1-r}^{1} \int_0^{2\pi} a(\rho)\, (\partial_\phi b)(y)\, \frac{r}{\sin\phi}\, d\theta\, d\rho. \tag{18}$$

It will be convenient to translate variables so that the source receiver pair at e moves to the origin. Since the u_i are solutions of (1), (2), we have

$$(\Box - q_i^e(x)) u_i^e(x,t) = \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{19}$$
$$u_i^e(\cdot, t) = 0 \qquad t < 0. \tag{20}$$

Then it is shown in [6] that

$$u_i^e(x,t) = \frac{1}{4\pi} \frac{\delta(t - |x|)}{|x|} + v_i(x,t) \tag{21}$$

where $v_i(x,t)$ is zero for $t < |x|$ and in the region $t \geq |x|$ is the C^2 solution of the Goursat problem

$$(\Box - q_i^e(x)) v_i(x,t) = 0 \qquad t \geq |x| \tag{22}$$
$$v_i(x, |x|) = \frac{1}{8\pi} \int_0^1 q_i^e(sx)\, ds. \tag{23}$$

Define

$$a := a_1 - a_2, \quad q := q_1 - q_2, \quad u := u_1 - u_2, \quad v := v_1 - v_2.$$

Note that $q = q_1 - q_2 = a_1 b - a_2 b = ab$. Subtracting (19) for $i = 1, 2$ from each other we have

$$(\Box - q_1^e(x)) u^e(x,t) = q^e(x) u_2^e(x,t) \qquad (x,t) \in R^3 \times R \tag{24}$$
$$u^e(x,t) = 0 \qquad \text{for } t < 0 \tag{25}$$

Then $u^e(x,t)$ is zero for $t < |x|$ and $u^e(x,t) = v(x,t)$ on $t \geq |x|$ where $v(x,t)$ is the solution of the Goursat problem

$$(\Box - q_1^e(x)) v(x,t) = q^e(x) v_2(x,t) \qquad t \geq |x| \tag{26}$$
$$v(x, |x|) = \frac{1}{8\pi} \int_0^1 q^e(sx)\, ds. \tag{27}$$

We are given that $v(0,t)=0$ for all $t \in [0,2]$ and we have to show that $q(.)=0$.

If we bring the $q_1^e u^e$ to the RHS of (24) then using the standard solution for initial value problems for the wave equation, the solution of (24), (25) may be written as

$$
\begin{aligned}
u^e(x,t) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{\delta(t-s-|x-y|)}{|x-y|}(q_1^e u^e + q^e u_2^e)(y,s)\,ds\,dy \\
&= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(q_1^e u^e)(y,t-|x-y|)}{|x-y|}\,dy \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}} q^e(y)\frac{\delta(t-s-|x-y|)}{|x-y|}\left(\frac{1}{4\pi}\frac{\delta(s-|y|)}{|y|} + v_2(y,s)\right)ds\,dy \\
&= \frac{1}{16\pi^2}\int_{R^3} q^e(y)\frac{\delta(t-|y|-|x-y|)}{|y|\,|x-y|}\,dy + \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(q_1^e u^e)(y,t-|x-y|)}{|x-y|}\,dy \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(q^e v_2)(y,t-|x-y|)}{|x-y|}\,dy.
\end{aligned}
$$

Since $u^e = v$ on $t \geq |y|$ and u^e is zero on $t < |y|$, we have, that for $t \geq 0$,

$$
\begin{aligned}
v(0,t) &= \frac{1}{16\pi^2}\int_{R^3}\frac{q^e(y)}{|y|^2}\,\delta(t-2|y|) + \frac{1}{4\pi}\int_{|y|\leq t/2}\frac{(q^e v_2 + q_1^e v)(y,t-|y|)}{|y|}\,dy \\
&= \frac{1}{32\pi^2}\int_{|y|=t/2}\frac{q^e(y)}{|y|^2}\,dS_y + \frac{1}{4\pi}\int_{|y|\leq t/2}\frac{(q^e v_2 + q_1^e v)(y,t-|y|)}{|y|}\,dy \\
&= \frac{1}{8\pi}(Mq)(e,t/2) + \frac{1}{4\pi}\int_{|y|\leq t/2}\frac{(q^e v_2 + q_1^e v)(y,t-|y|)}{|y|}\,dy.
\end{aligned}
$$

Hence

$$
8\pi v(0,2t) = (Mq)(e,t) + 2\int_{|y|\leq t}\frac{(q^e v_2 + q_1^e v)(y,2t-|y|))}{|y|}dy.
$$

Since $v(0,t)=0$ for $t \in [0,2]$, we have for all $t \in [0,1]$ that

$$
\begin{aligned}
-\partial_t t(Mq)(e,t) &= 2\partial_t t\int_{|y|\leq t}\frac{(q^e v_2 + q_1^e v)(y,2t-|y|)}{|y|}\,dy \\
&= 2\int_{|y|\leq t}\frac{(q^e v_2 + q_1^e v)(y,2t-|y|)}{|y|}\,dy + 2t\partial_t\int_{|y|\leq t}\frac{(q^e v_2 + q_1^e v)(y,2t-|y|)}{|y|}\,dy
\end{aligned}
$$

Now, noting that

$$
\int_{|y|\leq t} f(y,t)\,dy = \int_0^t\int_{|y|=r} f(y,t)\,dS_y,
$$

and using (18) we have

$$
\begin{aligned}
-\frac{b(e)}{2}(1-t)a(1-t) = 2\int_{|y|\leq t} \frac{(q^e v_2 + q_1^e v)(y, 2t-|y|)}{|y|}\, dy + 2t \int_{|y|=t} \frac{(q^e v_2 + q_1^e v)(y, 2t-|y|)}{|y|}\, dS_y \\
+ 4t \int_{|y|\leq t} \frac{(q^e \partial_t v_2 + q_1^e \partial_t v)(y, 2t-|y|)}{|y|}\, dy \\
+ \frac{1}{4\pi t}\int_{1-t}^{1}\int_0^{2\pi} a(\rho)\,(\partial_\phi b)(y)\, \frac{t}{\sin\phi}\, d\theta\, d\rho.
\end{aligned}
$$

Hence, using the boundedness of q_1^e, v_2 and their derivatives, the boundedness of $\partial_\phi b/\sin\phi$, that $a(1-|y|)$, $q_1^e(y)$, $q^e(y)$ are zero for $|y|\leq\delta$, and $b(e)\neq 0$, we have for all $t\in(0,1)$ that

$$
\begin{aligned}
(1-t)^2|a(1-t)|^2 \preccurlyeq \int_0^t |a(1-s)|^2\, ds + \int_{|y|\leq t} |q(y+e)|^2\, dy \\
+ \int_{|y|=t} |q(y+e)|^2 + |v(y,|y|)|^2\, dS_y \\
+ \int_{|y|\leq t} |v(y, 2t-|y|)|^2\, dy + \left| \int_{|y|\leq t} \frac{(q_1^e\, v_t)(y, 2t-|y|)}{|y|}\, dy \right|^2.
\end{aligned}
\tag{28}
$$

We are given that $a(r)=0$ for $r\geq 1-\delta$. Our goal is to show that each term on the RHS of (28) is smaller than a constant times $\int_0^t |a(1-r)|^2\, dr$ for all $t\in(0,\delta_1)$ for some $\delta_1>\delta$, $\delta_1<1$. Then, from Gronwall's inequality, we have $a(r)=0$ for all $r\geq 1-\delta_1$ and we have proved that q is zero over an annulus larger than what was given. Note that this can be done for an arbitrary $\delta\in(0,1)$. This is enough to show that $q(.)=0$ on $(0,1)$ because if δ^* is the infimum of all δ such that $q(.)$ is zero on $[\delta,1]$ then the continuity of q implies $q=0$ on $[\delta^*,1]$ and then the above result implies that $\delta^*=0$. The term which will be the hardest to estimate is the last integral in (28).

To estimate the first and the second integrals in (28) we note that $a(r)=0$ for $r\geq 1-\delta$. Using (17), for all $t\in(0,1)$ we have

$$
\int_{|y|=t} |q(y+e)|^2\, dS_y = 4\pi t^2 M(q^2)(e,t) \preccurlyeq \int_0^t |a(1-s)|^2\, ds. \tag{29}
$$

$$
\begin{aligned}
\int_{|y|\leq t} |q(y+e)|^2\, dy = \int_0^t \int_{|y|=r} |q(y+e)|^2\, dS_y\, dr = 4\pi \int_0^t r^2 M(q^2)(e,r)\, dr \\
\preccurlyeq \int_0^t \int_0^r |a(1-s)|^2\, ds\, dr \preccurlyeq \int_0^t |a(1-s)|^2\, ds.
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\int_{|y|=t} |v(y,|y|)|^2 \, dS_y &\leq \frac{1}{64\pi^2} \int_{|y|=t} \int_0^1 |q(sy+e)|^2 \, ds \, dS_y \\
&= \frac{t^2}{64\pi^2} \int_0^1 \int_{|\omega|=1} |q(e+st\omega)|^2 \, d\omega \, ds \\
&= \frac{t^2}{16\pi} \int_0^1 M(q^2)(e,st) \, ds \preccurlyeq \int_0^1 \frac{1}{s^2} \int_0^{st} |a(1-r)|^2 \, dr \, ds \\
&\preccurlyeq \int_0^t |a(1-r)|^2 \int_{r/t}^1 \frac{1}{s^2} \, ds \, dr \\
&\preccurlyeq \int_0^t |a(1-r)|^2 (1+t/r) \, dr \preccurlyeq \frac{1}{\delta} \int_0^t |a(1-r)|^2 \, dr. \qquad (31)
\end{aligned}
$$

The last two integrals in (28) will be estimated using the following proposition about Goursat problems, which gives the value of a first order transverse derivative on a characteristic cone and also the standard energy estimates for Goursat problems.

Proposition 1. *Suppose on the region $(y,t) \in \mathbb{R}^3 \times \mathbb{R}$ with $t \geq |y|$, the functions $p(y)$, $F(y,t)$ are continuous, $f(y)$ is C^1, and $w(y,t)$ is a C^2 solution of the Goursat problem*

$$
(\Box - p) w(y,t) = F(y,t), \qquad (y,t) \in \mathbb{R}^3 \times \mathbb{R}, \quad t \geq |y|, \qquad (32)
$$

$$
w(y,|y|) = f(y), \qquad y \in \mathbb{R}^3. \qquad (33)
$$

Then for any y with $|y| \leq \tau$ we have

$$
\begin{aligned}
(|y| w_t + y \cdot \nabla w + w)(y, \tau - |y|) = \int_{|y|}^{\tau/2} & \frac{1}{s} (\Omega w)(s\hat{y}, \tau - s) - s (\Box w)(s\hat{y}, \tau - s) \, ds \\
& + (z \cdot \nabla + 1)(v(z,|z|))|_{z = \tau\hat{y}/2}. \qquad (34)
\end{aligned}
$$

We also have the following energy estimates for any $0 < \tau \leq T$.

$$
\begin{aligned}
\text{(a)} \quad & \int_{|y| \leq \min(\tau, T-\tau)} |w_t(y,\tau)|^2 + |\nabla w(y,\tau)|^2 + |w(y,\tau)|^2 \, dy \\
& \preccurlyeq \int_{|y| \leq \min(\tau, T/2)} |\nabla f(y)|^2 + |f(y)|^2 \, dy + \iint_{D_{\tau,T}} |F(y,t)|^2 \, dy \, dt, \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
\text{(b)} \quad & \int_{|y| \leq T/2} |\nabla (w(y, T-|y|))|^2 + |w(y, T-|y|)|^2 \\
& \preccurlyeq \int_{|y| \leq T/2} |\nabla f(y)|^2 + |f(y)|^2 \, dy + \iint_{D_{T/2,T}} |F(y,t)|^2 \, dy \, dt. \qquad (36)
\end{aligned}
$$

Here $D_{\tau,T} = \{(y,t) \in \mathbb{R}^3 \times \mathbb{R} \, : \, |y| \leq t \leq \min(\tau, T - |y|)\}$.

Observe that this proposition does NOT give an estimate for $\int_{|y| \leq T/2} |w_t(y, T-y)|^2 \, dy$. In fact estimating this would require two derivatives of f - which will not be a useful estimate. This is the main difficulty in estimating the last integral in (28).

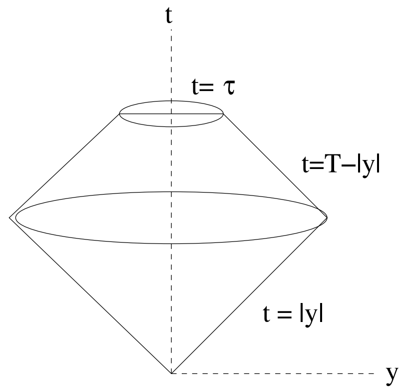

Figure 3: $D_{\tau,T}$ when $\tau > T/2$

Proof of Theorem 1 continued
We next estimate the last integral in (28).

$$\begin{aligned} 2|y|v_t(y, 2t-|y|) &= (|y|v_t + y\cdot\nabla v)(y, 2t-|y|) + (|y|v_t - y\cdot\nabla v)(y, 2t-|y|) \\ &= (|y|v_t + y\cdot\nabla v)(y, 2t-|y|) - y\cdot\nabla(v(y, 2t-|y|)). \end{aligned}$$

Hence, using the fact that $q_1^e(y) = 0$ for $|y| \le \delta$, we have

$$\begin{aligned} \left| \int_{|y|\le t} \frac{q_1^e(y)}{|y|}\, v_t(y, 2t-|y|)\, dy \right|^2 \preccurlyeq &\left| \int_{|y|\le t} \frac{q_1^e(y)}{|y|^2}\, (|y|v_t + y\cdot\nabla v)(y, 2t-|y|)\, dy \right|^2 \\ &+ \int_{\delta\le|y|\le t} |\nabla(v(y, 2t-y))|^2\, dy. \end{aligned} \tag{37}$$

Now

$$8\pi\,(y\cdot\nabla + 1)(v(y,|y|)) = (y\cdot\nabla+1)(\int_0^1 q^e(sy)\,ds) = \int_0^1 \frac{d}{ds}(sq^e(sy))\,ds = q^e(y).$$

Using this, from (34) in Proposition 1, applied to v (note v satisfies (26), (27)), we have

$$\begin{aligned} &(|y|v_t + y\cdot\nabla v)(y, 2t-|y|) \\ &\quad= \int_{|y|}^t \frac{1}{s}(\Omega v)(s\hat{y}, 2t-s) - s(\Box v)(s\hat{y}, 2t-s)\,ds - v(y, 2t-|y|) + (z\cdot\nabla+1)(v(z,|z|))|_{z=t\hat{y}} \\ &\quad= \int_{|y|}^t \frac{1}{s}(\Omega v)(s\hat{y}, 2t-s) - s(q_1^e v + q^e v_2)(s\hat{y}, 2t-s)\,ds - v(y, 2t-|y|) + \frac{q^e(y)}{8\pi}. \end{aligned}$$

Using this in (37) and noting that $q_1^e(y)$ and $q^e(y)$ are zero for $|y| \le \delta$, for all $t \in (0,1)$ we have

$$
\begin{aligned}
&\left| \int_{|y|\le t} \frac{q_1^e(y)}{|y|}\, v_t(y, 2t-|y|)\, dy \right|^2 \\
&\quad \preccurlyeq \left| \int_{|y|\le t} \frac{1}{|y|^2} \int_{|y|}^t \frac{q_1^e(y)}{s}\, (\Omega v)(s\hat{y}, 2t-s)\, ds\, dy \right|^2 + \int_{\delta\le|y|\le t} \int_{|y|}^t s^2 (|v|^2 + |q^e|^2)(s\hat{y}, 2t-s)\, ds\, dy \\
&\qquad + \int_{|y|\le t} |\nabla(v(y, 2t-|y|))|^2 + |v(y,2t-|y|)|^2 + |q^e(y)|^2\, dy \\
&\quad \preccurlyeq \int_0^t |a(1-r)|^2\, dr + \left| \int_{|y|\le t} \frac{1}{|y|^2} \int_{|y|}^t \frac{q_1^e(y)}{s}\, (\Omega v)(s\hat{y}, 2t-s)\, ds\, dy \right|^2 \\
&\qquad + \int_{|y|\le t} |\nabla(v(y, 2t-|y|))|^2 + |v(y,2t-|y|)|^2\, dy
\end{aligned}
\tag{38}
$$

where the last step followed from the fact that we have already estimated all the other terms - the other ds integral is handled in a manner as done below. Now $q_1^e(y)$ is zero for $|y| \le \delta$, so using (14) and (15) we have

$$
\begin{aligned}
\pm \int_{|y|\le t} \frac{1}{|y|^2} \int_{|y|}^t \frac{q_1^e(y)}{s}\, (\Omega v)(s\hat{y}, 2t-s)\, ds\, dy &= \pm \sum_{ij} \int_\delta^t \int_r^t \int_{|y|=r} \frac{q_1^e(y)}{2sr^2}\, (\Omega_{ij}^2 v)(s\hat{y}, 2t-s)\, dS_y\, ds\, dr \\
&= \pm \sum_{ij} \int_\delta^t \int_r^t \int_{|y|=r} \frac{s\, q_1^e(y)}{2r^4}\, \Omega_{ij}^2 (v(s\hat{y}, 2t-s))\, dS_y\, ds\, dr \\
&= \pm \sum_{ij} \int_\delta^t \int_r^t \int_{|y|=r} \frac{s\, \Omega_{ij}^2 q_1^e(y)}{2r^4}\, v(s\hat{y}, 2t-s)\, dS_y\, ds\, dr \\
&\preccurlyeq \int_\delta^t \int_r^t \int_{|y|=r} |v(s\hat{y}, 2t-s)|\, dS_y\, ds\, dr \preccurlyeq \int_\delta^t \int_r^t \int_{|\theta|=1} r^2\, |v(s\theta, 2t-s)|\, d\theta\, ds\, dr \\
&\qquad = \int_\delta^t \int_{|\theta|=1} \int_\delta^s r^2\, |v(s\theta, 2t-s)|\, dr\, d\theta\, ds \;\preccurlyeq \int_{|y|\le t} |v(y, 2t-|y|)|\, dy.
\end{aligned}
$$

Hence for all $t \in (0,1)$, we have

$$
\left| \int_{|y|\le t} \frac{1}{|y|^2} \int_{|y|}^t \frac{q_1^e(y)}{s}\, (\Omega v)(s\hat{y}, 2t-s)\, ds\, dy \right|^2 \preccurlyeq \int_{|y|\le t} |v(y, 2t-|y|)|^2\, dy. \tag{39}
$$

So going back to (28) and using (29), (30), (31), (38) and (39), we have shown that for all $t \in (0,1)$

$$
(1-t)^2 |a(1-t)|^2 \preccurlyeq \int_0^t |a(1-r)|^2\, dr + \int_{|y|\le t} |\nabla(v(y, 2t-|y|))|^2 + |v(y,2t-|y|)|^2\, dy. \tag{40}
$$

To estimate the second integral in (40), we apply (b) of Proposition 1 to $v(y,t)$ with $T = 2t$; note $v(y,t)$ is the solution of the Goursat problem (26), (27) so $p = q_1$, $F = q^e v_2$, and $f = v(y, |y|)$.

Then

$$
\begin{aligned}
\int_{|y|\leq t} & |\nabla(v(y, 2t-|y|))|^2 + |v(y, 2t-y)|^2 \, dy \\
& \preccurlyeq \int_{|y|\leq t} |\nabla(v(y,|y|))|^2 + |v(y,|y|)|^2 \, dy + \int_{|y|\leq t} \int_{|y|}^{2t-|y|} |(q^e v_2)(y,s)|^2 \, ds \, dy \\
& \preccurlyeq \int_{|y|\leq t} |\nabla(v(y,|y|))|^2 + |v(y,|y|)|^2 \, dy + \int_{|y|\leq t} |q^e(y)|^2 \, dy \\
& \preccurlyeq \int_{|y|\leq t} |\nabla(v(y,|y|))|^2 \, dy + \int_0^t |a(1-r)|^2 \, dr
\end{aligned}
$$

with the last step following from (30), (31) and the estimate

$$
\begin{aligned}
\int_{|y|\leq t} v^2(y,|y|) \, dy = \int_0^t \int_{|y|=s} v^2(y,|y|) dS_y \, ds & \preccurlyeq \int_0^t \int_0^s |a(1-r)|^2 \, dr \\
& \preccurlyeq \int_0^t |a(1-r)|^2 \, dr.
\end{aligned}
$$

Finally, taking the gradient of (27), we have

$$
\begin{aligned}
8\pi \nabla(v(y,|y|)) &= \nabla \int_0^1 q(e+sy) \, ds = \nabla \int_0^1 a(|e+sy|) \, b(e+sy) \, ds \\
&= \int_0^1 a'(|e+sy|) \, b(e+sy) \, \frac{s(e+sy)}{|e+sy|} \, ds + \int_0^1 sa(|e+sy|) \, (\nabla b)(e+sy) \, ds
\end{aligned}
$$

hence, from (40) we have shown that for all $t \in (0,1)$

$$
\begin{aligned}
(1-t)^2 |a(1-t)|^2 \preccurlyeq & \int_0^t |a(1-r)|^2 + \int_{|y|\leq t} \int_0^1 s^2 |a(|e+sy|)|^2 \, ds \, dy \\
& + \int_{|y|\leq t} \left| \int_0^1 a'(|e+sy|) \, b(e+sy) \, \frac{s(e+sy)}{|e+sy|} \, ds \right|^2 dy. \qquad (41)
\end{aligned}
$$

We need to estimate the second and the third integrals on the RHS of (41) by the L^2 norm of a. The third integral has a derivative of a and we will cancel the derivative by the integration in s - this will be the harder integral to estimate. Estimating both the integrals will require a change of variable $r = |e+sy|$ and we want that this change of variable have no singularity; this is not true if t is allowed to be an arbitrary number in $(0,1)$. The following proposition guarantees that this change of variable has no singularity for all $t \in (0, \delta_1)$ for some $\delta_1 > \delta$. Below $B_\rho(p)$ will denote the closed ball of radius ρ centered at the point $p \in \mathbb{R}^3$ and for points $a, b \in \mathbb{R}^3$, $[a,b]$ will denote the line segment joining a to b.

Proposition 2. *For any $\rho \in (0,1)$ there is a $\overline{\rho} > \rho$ (with $\overline{\rho} < 1$) and a $\sigma > 0$ so that*

- *if $z \in B_{\overline{\rho}}(e) \setminus B_{1-\rho}(0)$ then $[e,z] \cap B_{1-\rho}(0) = \emptyset$;*

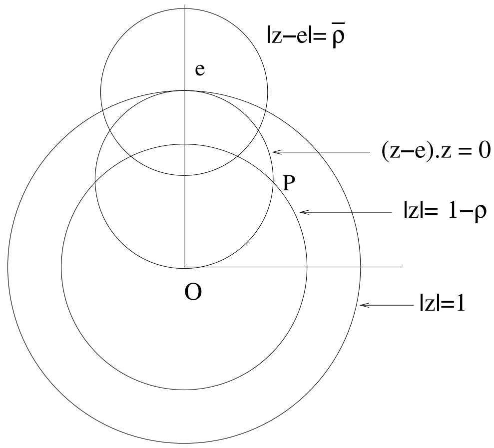

Figure 4: Spheres for Proposition 2

- *if* $z \in B_{\overline{\rho}}(e) \cap B_{1-\rho}(0)$ *then* $[e, z]$ *may be split into* $[e, x^*]$ *and* $[x^*, z]$ *where* $[e, x^*) \cap B_{1-\rho}(0) = \emptyset$ *and* $[x^*, z] \subseteq B_{1-\rho}(0)$*; further* $(z - e) \cdot x \leq -\sigma$ *for all* $x \in [x^*, z]$.

Proof of Theorem 1 continued
We have $q(r) = 0$ for $r \geq 1 - \delta$ for some $\delta \in (0, 1)$. Applying Proposition 2 with $\rho = \delta$ there is a $\delta_1 = \bar{\rho}$ in $(\delta, 1)$ and a $\sigma > 0$ with the property guaranteed by Proposition 2. We will obtain the appropriate estimate for the third integral on the RHS of (41) for all $t \in (0, \delta_1)$.

We first estimate

$$\int_0^1 a'(|e + sy|)\, b(e + sy)\, \frac{s(e + sy)}{|e + sy|}\, ds \tag{42}$$

for all $y \in B_{\delta_1}(0)$. Take $y \in B_{\delta_1}(0)$, take $z = e + y$; note $z \in B_{\delta_1}(e)$. If $[e, e + y]$ is outside $B_{1-\delta}(0)$ then the integral (42) is zero. So assume that there is an $s^* \in (0, 1)$ so that $[e, e + s^* y)$ is outside $B_{1-\delta}(0)$ and $[e + s^* y, e + y] \subseteq B_{1-\delta}(0)$; then $(e + sy) \cdot y \leq -\sigma$ for all $s \in [s^*, 1]$. Hence

$$\int_0^1 a'(|e + sy|)\, b(e + sy)\, \frac{s(e + sy)}{|e + sy|}\, ds = \int_{s^*}^1 a'(|e + sy|)\, b(e + sy)\, \frac{s(e + sy)}{|e + sy|}\, ds.$$

Introduce the change of variables $r = |e + sy|$. Noting that $r^2 = |e + sy|^2$ implies

$$\frac{ds}{r} = \frac{dr}{y \cdot (e + sy)}$$

and that $b(e + sy)$ is independent of $r = |e + sy|$, we obtain

$$\begin{aligned}\int_{s^*}^1 a'(|e + sy|)\, b(e + sy)\, \frac{s(e + sy)}{|e + sy|}\, ds &= \int_{|e+s^*y|}^{|e+y|} a'(r)\, b(e + sy)\, \frac{s(e + sy)}{y \cdot (e + sy)}\, dr \\ &= a(|e + y|) b(e + y)\, \frac{e + y}{y \cdot (e + y)} - \int_{|e+s^*y|}^{|e+y|} a(r)\, b(e + sy)\, \frac{d}{dr}\left(\frac{s(e + sy)}{y \cdot (e + sy)}\right)\, dr\end{aligned}$$

Now

$$\frac{d}{dr}\left(\frac{s(e+sy)}{y\cdot(e+sy)}\right)=\frac{d}{ds}\left(\frac{s(e+sy)}{y\cdot(e+sy)}\right)\frac{ds}{dr}=\frac{(e+2sy)(y\cdot(e+sy))-s(e+sy)\,|y|^2}{(y\cdot(e+sy))^2}\,\frac{r}{y\cdot(e+sy)}$$
$$=\frac{(e+2sy)(y\cdot e)+s^2y\,|y|^2}{(y\cdot(e+sy))^2}\,\frac{|e+sy|}{y\cdot(e+sy)},$$

hence

$$\left|\int_{s^*}^1 a'(|e+sy|)b(e+sy)\frac{s(e+sy)}{|e+sy|}\,ds\right|\preccurlyeq\frac{|a(|e+y|)|}{\sigma}+\frac{1}{\sigma^3}\int_{|e+y|}^1|a(r)|\,dr.$$

So we have shown that for all $y\in B_{\delta_1}(0)$ we have

$$\left|\int_0^1 a'(|e+sy|)b(e+sy)\frac{s(e+sy)}{|e+sy|}\,ds\right|\preccurlyeq|a(|e+y|)|+\int_{|e+y|}^1|a(r)|\,dr.$$

Hence, for $t\in[0,\delta_1]$, using (30), we have

$$\int_{|y|\le t}\left|\int_0^1 a'(|e+sy|)b(e+sy)\frac{s(e+sy)}{|e+sy|}\,ds\right|^2dy\preccurlyeq\int_{|y|\le t}|a(|e+y|)|^2\,dy+\int_{|y|\le t}\int_{|e+y|}^1|a(r)|^2\,dr\,dy$$
$$\preccurlyeq\int_{1-t}^1|a(r)|^2\,dr=\int_0^t|a(1-r)|^2\,dr$$

which estimates the third integral in (41). Estimating the second integral in (41) is easier because it just requires the change of variables used above - no integration by parts is needed. With this and noting $\delta_1<1$, we obtain

$$|a(1-t)|^2\preccurlyeq\int_0^t|a(1-r)|^2\,dr,\qquad\text{for all}\ \ t\in[0,\delta_1].$$

QED

4 Proofs of Propositions

We give the proofs of Propositions 1 and 2.

4.1 Proof of Proposition 1

Define

$$E_1(t,T)=\int_{|y|\le\min(t,T-t)}(w_t^2+|\nabla w|^2)(y,t)\,dy,\qquad E(t,T)=E_1(t,T)+\int_{|y|\le\min(t,T-t)}w(y,t)^2\,dy.$$

We have the identity

$$\begin{aligned}
w_t\,\Box w &= w_t w_{tt} - \Delta w\, w_t = \frac{1}{2}(w_t^2)_t - \nabla\cdot(w_t\nabla w) + \nabla w\cdot\nabla w_t \\
&= \frac{1}{2}(w_t^2)_t - \nabla\cdot(w_t\nabla w) + \frac{1}{2}\partial_t(|\nabla w|^2) = \frac{1}{2}(w_t^2+|\nabla w|^2)_t - \nabla\cdot(w_t\nabla w).
\end{aligned}$$

Integrating this over the region $D_{\tau,T}$ and using the divergence theorem, for $\tau \geq T/2$ we have

$$\begin{aligned}
2\int_{D_{\tau,T}} (\Box w)(y,t)\,w_t(y,t)\,dy\,dt &= E_1(\tau,T) + \int_{\tau\leq|y|\leq T/2} (w_t^2+|\nabla w|^2 - 2w_t\hat{y}\cdot\nabla w)(y,T-|y|)\,dy \\
&\quad - \int_{|y|\leq T/2} (w_t^2+|\nabla w|^2 + 2w_t\hat{y}\cdot\nabla w)(y,|y|)\,dy \\
&= E_1(\tau,T) + \int_{\tau\leq|y|\leq T/2} |(\hat{y}w_t - \nabla w)(y,T-|y|)|^2\,dy - \int_{|y|\leq T/2} |(\hat{y}w_t+\nabla w)(y,|y|)|^2\,dy \\
&= E_1(\tau,T) + \int_{\tau\leq|y|\leq T/2} |\nabla(w(y,T-|y|))|^2\,dy - \int_{|y|\leq T/2} |\nabla(w(y,|y|))|^2\,dy.
\end{aligned}$$

Hence, by the Cauchy-schwartz inequality

$$\begin{aligned}
E_1(\tau,T)+&\int_{\tau\leq|y|\leq T/2} |\nabla(w(y,T-|y|))|^2\,dy \\
&\preccurlyeq \int_{|y|\leq T/2} |\nabla(w(y,|y|))|^2\,dy + \int_{D_{\tau,T}} |(\Box w)(y,t)|^2\,dy\,dt + \int_{D_{\tau,T}} w_t^2(y,t)\,dy\,dt. \qquad (43)
\end{aligned}$$

Also, for any (y,t) with $t \geq |y|$ we have

$$w^2(y,t) = w^2(y,|y|) + 2\int_{|y|}^t w(y,s)w_t(y,s)\,ds \leq w^2(y,|y|) + \int_{|y|}^t w^2(y,s)+w_t^2(y,s)\,ds$$

Hence

$$\begin{aligned}
\int_{|y|\leq T-\tau} |w(y,\tau)|^2\,dy &\leq \int_{|y|\leq T/2} w^2(y,|y|)\,dy + \int_0^\tau E(t)\,dt, \\
\int_{T-\tau\leq|y|\leq T/2} |w(y,T-|y|)|^2 &\leq \int_{|y|\leq T/2} |w(y,|y|)|^2\,dy + \int_0^\tau E(t)\,dt.
\end{aligned}$$

Adding these to (43), we obtain, for $\tau \geq T/2$ that

$$\begin{aligned}
E(\tau) + &\int_{T-\tau\leq|y|\leq T/2} |\nabla(w(y,T-|y|))|^2 + |w(y,T-y)|^2\,dy \\
&\preccurlyeq \int_{|y|\leq T/2} |\nabla(w(y,|y|))|^2 + |w(y,|y|)|^2\,dy + \int_{D_{\tau,T}} |(\Box w)(y,t)|^2\,dy\,dt + \int_0^\tau E(t)\,dt. \qquad (44)
\end{aligned}$$

When $\tau \in [0,T/2]$ a similar but simpler argument gives us a similar relation except the second term on the LHS will be absent. Then the usual argument for hyperbolic operators gives us (a) and (b) - note that the LHS of (b) is the second term on the LHS of (44) when we take $\tau = T$.

Proof of (34) From (13) we have

$$\begin{aligned}
|y|^2\Box &= |y|^2\partial_t^2 - |y|^2\Delta = |y|^2\partial_t^2 - (y\cdot\nabla)(y\cdot\nabla) - y\cdot\nabla - \Omega \\
&= (|y|\partial_t - y\cdot\nabla)(|y|\partial_t + y\cdot\nabla) + y\cdot\nabla(|y|)\partial_t - y\cdot\nabla - \Omega \\
&= (|y|\partial_t - y\cdot\nabla)(|y|\partial_t + y\cdot\nabla) + (|y|\partial_t - y\cdot\nabla) - \Omega \\
&= (|y|\partial_t - y\cdot\nabla)(|y|\partial_t + y\cdot\nabla + 1) - \Omega
\end{aligned} \tag{45}$$

Let $h(y) = (|y|w_t + y\cdot\nabla w + w)(y, \tau - |y|)$. Noting that for any function $\psi(y,t)$

$$(y\cdot\nabla)(\psi(y,\tau-|y|)) = (y\cdot(\nabla\psi) - |y|\psi_t)\,(y,\tau-|y|) = -(|y|\psi_t - y\cdot\nabla\psi)(y,\tau-|y|)$$

and using (45), we have

$$\begin{aligned}
|y|^2(\Box w)(y,\tau-|y|) &= \left((|y|\partial_t - y\cdot\nabla)(|y|w_t + y\cdot\nabla w + w)\right)(y,\tau-|y|) - (\Omega w)(y,\tau-|y|) \\
&= -(y\cdot\nabla)((|y|w_t + y\cdot\nabla w + w)(y,\tau-|y|)) - (\Omega w)(y,\tau-|y|) \\
&= -(y\cdot\nabla)(h(y)) - (\Omega w)(y,\tau-|y|).
\end{aligned}$$

Hence

$$y\cdot((\nabla h)(y)) \;=\; -|y|^2(\Box u)(y,\tau-|y|) - (\Omega w)(y,\tau-|y|).$$

This may be written as

$$s\frac{d}{ds}(h(sy)) = \text{RHS evaluated at } sy\ .$$

Dividing by s and integrating this over the interval $[1, \tau/(2|y|)]$ we obtain

$$\begin{aligned}
h(\tau\hat{y}/2) - h(y) &= \int_1^{\tau/(2|y|)} s|y|^2(\Box w)(sy,\tau-s|y|) - \frac{1}{s}(\Omega w)(sy,\tau-s|y|)\,ds \\
&= \int_{|y|}^{\tau/2} r(\Box w)(r\hat{y},\tau-r) - \frac{1}{r}(\Omega w)(r\hat{y},\tau-r)\,dr.
\end{aligned}$$

Then (34) follows if we note that

$$\begin{aligned}
h(\tau\hat{y}/2) &= \frac{\tau}{2}w_t(\tau\hat{y}/2,\tau/2) + \frac{\tau\hat{y}}{2}\cdot\nabla w(\tau\hat{y}/2,\tau/2) + w(\tau\hat{y}/2,\tau/2) \\
&= z\cdot\nabla(w(z,|z|))|_{z=\tau\hat{y}/2} + w(\tau\hat{y}/2,\tau/2) = (z\cdot\nabla + 1)(f(z))|_{z=\tau\hat{y}/2}.
\end{aligned}$$

4.2 Proof of Proposition 2

The set $(z-e).z = 0$ is the sphere $|z - e/2| = 1/2$. This sphere cuts $|z| = 1-\rho$ in a circle at a generic point p with $|p-e| = \sqrt{1-(1-\rho)^2}$, see figure 4. Since $\sqrt{1-(1-\rho)^2} > \rho$ if $\rho\in(0,1)$, there is a real number $\overline{\rho}$ strictly between ρ and $\sqrt{1-(1-\rho)^2}$.

If $z \in B_{\overline{\rho}}(e) \cap B_{1-\rho}(0)$ then z is in the interior of the sphere $(z-e) \cdot z = 0$ because

$$(z-e) \cdot z = \frac{|z-e|^2 + |z|^2 - 1}{2} = \frac{\overline{\rho}^2 + (1-\rho)^2 - 1}{2} < 0$$

by the definition of $\overline{\rho}$. If x^* is the point of intersection of $[e, z]$ with the boundary of $B_{1-\rho}(0)$ then $[e, x^*)$ lies outside $B_{1-\rho}(0)$ and $[x^*, z]$ lies in $B_{1-\rho}(0)$. Also, for any $x \in [x^*, z]$, $x = e + s(z-e)$ for some $s \in (0, 1]$; in fact $s \geq \rho/\bar{\rho}$ because $|z-e| \leq \bar{\rho}$ and $|x-e| \geq |e| - |x| \geq 1 - (1-\rho) = \rho$. Hence for $x \in [x^*, z]$ we have

$$(z-e) \cdot x = (z-e) \cdot ((1-s)e + sz) = (1-s)(z-e) \cdot e + s(z-e) \cdot z \leq \frac{\rho}{\overline{\rho}}(z-e) \cdot z$$

because both $(z-e) \cdot z$ and $(z-e) \cdot e$ are negative. Since $(z-e) \cdot z < 0$ for all $z \in B_{\overline{\rho}}(e) \cap B_{1-\rho}(0)$, the maximum of the continuous function $(z-e) \cdot z\rho/\overline{\rho}$ on this compact set will be a negative number which we call $-\sigma$.

Next we show that for $z \in B_{\overline{\rho}}(e) \setminus B_{1-\rho}(0)$, the segment $[e, z]$ is outside $B_{1-\rho}(0)$. If z is outside the downward cone with vertex e, axis along the e axis, and through the generic points p; then $[e, z]$ is outside $B_{1-\rho}(0)$ because this downward cone is tangential to $B_{1-\rho}(0)$ since $(p-e).p = 0$. So assume that z is inside this cone, that is

$$\begin{aligned}
0 &\leq \frac{e \cdot (e-z)}{|e-z|} - \frac{e \cdot (e-p)}{|e-p|} = \frac{(e-z) \cdot (e-z) + z \cdot (e-z)}{|e-z|} - \frac{(e-p) \cdot (e-p)}{|e-p|} \\
&= |e-z| + \frac{z \cdot (e-z)}{|e-z|} - |e-p| \leq \overline{\rho} - \sqrt{1-(1-\rho)^2} + \frac{z \cdot (e-z)}{|e-z|} \\
&< \frac{z \cdot (e-z)}{|e-z|}.
\end{aligned}$$

Hence $z \cdot (e-z) > 0$ and this guarantees that the point on $[e, z]$ closest to the origin is z, and hence $[e, z]$ is outside $B_{1-\rho}(0)$.

QED

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