# CHARACTERIZATION OF TRANSMISSION DATA FOR WEBSTER'S HORN EQUATION

Rakesh
410 Ewing Hall
University of Delaware
Newark, DE 19716
rakesh@math.udel.edu

**Abstract:** Consider the Initial Boundary Value problem

$$\begin{aligned} \eta^2(x)U_{tt} - (\eta^2(x)U_x)_x &= 0 \qquad \text{in} \quad 0 \leq x, \ t \in \mathbb{R} \\ U &= 0 \qquad \text{for } t < 0 \\ \eta(0)U_x(0,t) &= -\delta(t) \qquad \text{for } t \in \mathbb{R}\,. \end{aligned}$$

Suppose $\eta(x) > 0$, $\eta(x) = 1$ for $x \geq X$, and $\eta$ is in $H^1[0,X]$. We characterize the range of the map

$$\eta \mapsto U_t(X,.)$$

where $U_t(X,.)$ is restricted to the interval $[X, 3X]$. In addition, we obtain an upper bound on $\|\eta\|_{1,[0,X]}$ in terms of the function $U_t(X,.)$ which completes the scheme for recovering $\eta$ from $U_t(X,.)$ (restricted to $[X, 3X]$) described elsewhere.

# 1 Introduction

We are motivated by problems arising in seismology when the earth (modeled as the half space in three dimensions) is probed by a single source and the response measured on the surface of the earth (the boundary of the half space). From this response, measured on the surface, one wishes to determine the properties of the interior of the earth. Symes, in [10], has shown that if the source and all the receivers are on the surface of the half space then the inversion (if possible at all) is unstable because some discontinuities in the medium can not be sensed if the receivers and the source are in the configuration described above - one needs measurements almost all around the medium being probed.

Hopefully, a more tractable problem is one where receivers are placed all around the boundary of the region of inhomogeneity of the medium. One such problem is the case where the medium is the whole space in three dimensions, the medium is uniform outside a ball, a source is placed at the center of the ball, and the response is measured on the surface of the ball for a finite time period. From this response one wishes to determine the properties of the medium. The problem remains unresolved and we wanted to study one dimensional inverse problems similar to such problems. The one dimensional inverse problem we chose to study is the one in which the source and receiver are at different locations because that is the situation in the case described above - the one dimensional coincident source and receiver problem has been resolved and is discussed below briefly.

Wave propagation in layered mediums may be modeled by Webster's Horn equation which is the equation we study below. So consider the Initial Boundary Value problem

$$\begin{aligned} \mathcal{L}u \equiv \eta^2(x)u_{tt} - (\eta^2(x)u_x)_x &= 0 \qquad \text{in} \quad 0 \le x,\ t \in \mathbb{R} \\ u &= 0 \qquad \text{for } t < 0 \\ \eta(0)u_x(0,t) &= -f(t) \qquad \text{for } t \in \mathbb{R}\ . \end{aligned}$$

Suppose $\eta$ is in $H^1_{loc}[0,\infty)$ with $\eta(x) > 0$, and $\eta(x)$ constant for $x \ge X$, and $f(t)$ is a smooth function which is zero for $t << 0$. Then, as shown in [1], this problem has a unique solution, which is locally in $H^1$. Further, there it is shown that the distributional kernel of the map

$$f(t) \to u(x,t)$$

is $U(x, t-s)$ where $U(x,t)$ is zero for $t < x$, and in the region $t \ge x \ge 0$, $U(x,t)$ is the solution of the characteristic boundary value problem

$$\begin{aligned} \eta^2(x)U_{tt} - (\eta^2(x)U_x)_x &= 0 \qquad \text{in} \quad t \ge x \ge 0 \\ U(x,x) &= \frac{1}{\eta(x)} \qquad \text{in } x \ge 0 \\ U_x(0,t) &= 0 \qquad \text{in } t \ge 0\ . \end{aligned}$$

The trace of the solution $U$ on $t = t_0$ and on $x = x_0$ are in $H^1[0,t_0]$ and $H^1[x_0,T]$ respectively (for any $T > x_0$) and the maps

$$\begin{aligned} (0,\infty) &\to R & \qquad [0,X] &\to R \\ t &\mapsto \|U(.,t)\|_{H^1[0,t]} & x &\mapsto \|U(x,.)\|_{H^1[x,T]} \end{aligned}$$

$(T > X)$ are bounded. We will informally think of $U(x,t)$ as the solution of the boundary value problem

$$\mathcal{L}U \equiv \eta^2(x)U_{tt} - (\eta^2(x)U_x)_x = 0 \quad \text{in} \quad 0 \leq x, \ t \in \mathbb{R} \tag{1}$$

$$U = 0 \quad \text{for } t < 0 \tag{2}$$

$$\eta(0)U_x(0,t) = -\delta(t) \quad \text{for } t \in \mathbb{R} \ . \tag{3}$$

The inverse problem under consideration is the recovery of $\eta$ from a knowledge of $\eta(X)U_t(X,t)$ for $X \leq t \leq 3X$.

If $\eta$ is multiplied by a positive constant $\lambda$, then the new $U$ is $1/\lambda$ times the old $U$, leaving $\eta U_t$ unchanged. Hence one can hope to recover $\eta$ only up to a constant. So, we will consider all $\eta$ which are positive constant multiples of each other as essentially representing the same function. For any positive $\eta \in H^1_{loc}(0,\infty)$, which is constant for $x > X$, we define $\sigma(x) = \eta'(x)/\eta(x)$. Then $\sigma$ is in $L^2[0,X]$ ($\sigma$ is zero for $x > X$), and the map

$$\eta \rightarrow \sigma$$

is a homeomorphism (up to a constant multiple of $\eta$) between the class of $\eta$ under consideration and $L^2[0,X]$, with the inverse of this map being

$$\eta(x) = c\, exp(-\int_x^X \sigma(y)\, dy).$$

where $c$ is an arbitrary positive constant. Our goal is the analysis of the (forward) map

$$\begin{aligned} F \ : \ L^2[0,X] &\rightarrow L^2[X,3X] \\ \sigma &\rightarrow \eta(X)U_t(X,.) \end{aligned}$$

Here $L^2[a,b]$ refers to the class of real valued, square integrable functions on $[a,b]$.

In [6], the author and Sacks showed how $\sigma$ could be recovered from $F(\sigma)$, provided one knew an upper bound on the $L^2$ norm of $\sigma$ in terms of $F(\sigma)$. In this article we will characterize the range of $F$, obtain an upper bound on the $L^2$ norm of $\sigma$ in terms of $F(\sigma)$, and also provide a better organized proof (than in [6]) of the correctness of the algorithm for the recovery of $\sigma$ from $F(\sigma)$.

Because of applications to seismology, one dimensional inverse problems for the wave equation have received a lot of attention, but mostly directed towards inversion from the reflection data $\eta(0)U_t(0,t)$ (source and receiver at same location) or inversion from reflection and transmission data, instead of inversion from only transmission data $\eta(X)U_t(X,t)$ (source and receiver at different locations). Please refer to [2] and [3] for references related to inversion from reflection data.

Symes in [9] gave a characterization of the range of the map from the coefficient $\sigma$ to the reflection data $\eta(0)U_t(0,.)$ (restricted to $[0,2X]$) and later Sylvester et al in [8] obtained a characterization of the range of such a map for a related problem in the frequency domain. Browning in [1] established a relationship between the two problems and showed how Symes's result may be derived from the results in [8]. We recommend reading [1] for the interesting results there and the careful derivation of these results.

We obtain the characterization of the range of $F$ by deriving a relation between the transmission and reflection data and reducing the characterization problem to an application of Symes's characterization of the reflection data. The upper bound on the $L^2$ norm of $\sigma$ in terms of the $F(\sigma)$ may then be obtained by an appeal to results in [9] but we obtain a bound directly.

Claerbout in [4] studied the discrete version of the problem we consider, and some of our crucial ideas are just the continuous analogs of the arguments used by Claerbout in the discrete case. We strongly recommend reading [4].

One may also answer similar questions for the following problem where the source term takes the form $\eta(0)U_t(0,t) = -\delta(t)$ instead of $\eta(0)U_x(0,t) = -\delta(t)$. Consider the problem

$$\begin{aligned} \eta^2(x)V_{tt} - (\eta^2(x)V_x)_x &= 0 \qquad \text{in} \quad 0 \le x,\ t \in \mathbb{R} && (4)\\ V &= 0 \qquad \text{for } t < 0 && (5)\\ \eta(0)V_t(0,t) &= -\delta(t) \qquad \text{for } t \in \mathbb{R}\ . && (6) \end{aligned}$$

The goal is to recover $\sigma$ from $\eta(X)V_t(X,.)$ restricted to $[X, 3X]$.

This problem may be reduced to the previous inverse problem in the following manner. Since $\eta$ is constant for $x \ge X$, we have $(V_t+V_x)(X,t) = 0$. Hence the data $\eta(X)V_t(X,.)$ equals $-\eta(X)V_x(X,.)$. Introduce symbols $p(x,t) = \eta(x)V_t(x,t)$ and $q(x,t) = \eta(x)V_x(x,t)$. Then (4) may be replaced by the system of first order equations

$$p_t = q_x + \sigma q, \qquad q_t = p_x - \sigma p, \qquad x \ge 0,\ t \in R,$$

with the initial condition

$$p(.,t) = 0, \quad q(.,t) = 0, \qquad \text{for} \quad t < 0$$

and the boundary condition

$$p(0,t) = -\delta(t), \qquad t \in R$$

and the data is $-q(X,.)$ restricted to $[X, 3X]$.

If $(p,q)$ satisfies the above system then $(\tilde{p} = q, \tilde{q} = p)$ satisfies the above system of equations with the new $\tilde{\sigma} = -\sigma$ (which corresponds to replacing $\eta$ by $1/\eta$). Hence switching $U_x(0,t)$ to $U_t(0,t)$ corresponds to solving the old pde with $\eta$ replaced by $1/\eta$. So solving the previous inverse problem essentially solves the new inverse problem. The inversion problem for other types of (mixed) boundary conditions remains unsolved.

Finally, in the proofs given below, we have been a little careless, integrating distributions, and examining these distributions for certain values of $x$ or $t$. This was done to communicate the main ideas clearly and not clutter up the presentation. All the proofs could be rewritten with some effort by working in regions $t \ge |x|$ and $t \le |x|$, because the singularities are mostly concentrated on the lines $t = |x|$.

## 2 Transformation to Inverse Problem on a Symmetric Triangular Region

We reduce our inverse problem to an inverse problem on a symmetric triangular region. Let $J(x,t)$ be the solution of the following impulsive IVP

$$\mathcal{L}J = 0 \qquad \text{in} \quad 0 \leq x, \; t \in R \tag{7}$$

$$\eta(0)J(0,t) = H(t), \qquad J_x(0,t) = 0 \qquad \text{for } t \in R \, . \tag{8}$$

Applying the progressing wave expansion, and considering the appropriate Goursat problem, we will show later

**Proposition 1**

$$\begin{aligned} U(x,t) &= \frac{1}{\eta(x)} H(t-x) + \phi(x,t) \\ J(x,t) &= \frac{1}{2\eta(x)} (H(t-x) + H(t+x)) + \psi(x,t) \end{aligned}$$

*where the smoother remainders $\phi$ and $\psi$ have the following property - $\phi$ is supported in $t \geq x$, the derivatives of $\psi$ are supported in $|t| \leq x$, and the first order derivatives of $\phi$ and $\psi$ have traces on $x = constant$ and on $t = constant$ which are locally in $L^2$.*

One may easily verify that for any two functions $v(x,t)$ and $w(x,t)$, if we define

$$\begin{aligned} f(x,t,s) &= \eta^2(x) \left\{ v_t(x,t-s) w_x(x,s) - v_x(x,t-s) w_t(x,s) \right\} \\ g(x,t,s) &= \eta^2(x) \left\{ v_t(x,t-s) w_t(x,s) - v_x(x,t-s) w_x(x,s) \right\} \end{aligned}$$

then

$$\frac{\partial f}{\partial x} - \frac{\partial g}{\partial s} = (\mathcal{L}v)(x,t-s) w_t(x,s) - v_t(x,t-s)(\mathcal{L}w)(x,s). \tag{9}$$

Applying this to $v = U$ and $w = J$ and integrating with respect to $(x,s)$ over the region $[0,X] \times (-\infty, \infty)$ we obtain

$$\int_{-\infty}^{\infty} f(X,t,s) \, ds = \int_{-\infty}^{\infty} f(0,t,s) \, ds$$

where we have used the fact that $J_x(x,t)$ and $J_t(x,t)$ are compactly supported in $t$ for any fixed $x$.

Now since $\eta$ is constant for $x \geq X$, so $U_{tt} - U_{xx} = 0$ for $x \geq X$ and also $U = 0$ for $t < 0$. This may be shown to lead to $U_t(X,t) + U_x(X,t) = 0$ implying $U_x(X,t) = -U_t(X,t)$. Further, $\eta(0)J_t(0,t) = \delta(t)$, $\eta(0)J_x(0,t) = 0$, and $\eta(0)U_x(0,t) = -\delta(t)$. Hence substituting the replacements for $v$ and $w$ in the above integral identity we obtain

$$\eta^2(X)(J_t + J_x) * U_t(X,t) \;=\; -\,\eta^2(0) U_x * J_t(0,t) \;=\; \delta(t) \, . \tag{10}$$

(10) is an important relation - we will show that if $\eta(X)U_t(X,t)$ is known for $X \leq t \leq 3X$, then using (10), we may recover $(J_x + J_t)(X,t)$ for $-X \leq t \leq X$. Since $J$ satisfies the same PDE as $U$, this reduces our problem to the study of an inverse problem dealing with $J$, and there are advantages in this.

To recover $\eta(X)(J_x + J_t)(X,t)$ from the transmission data $\eta(X)u_t(X,t)$, $X \leq t \leq 3X$, we need to remove the singularities in (10), which will be done by determining the singular parts of $U_t$ and $J_x + J_t$. From Proposition 1, we have

$$\begin{aligned}(J_x + J_t)(X,t) &= \frac{1}{\eta(X)}\left(\delta(t+X) + j(t+X)\right) && (11)\\ U_t(X,t) &= \frac{1}{\eta(X)}\left(\delta(t-X) + e(t-X)\right) && (12)\end{aligned}$$

for locally square integrable functions $j(t)$ and $e(t)$ (wave escaping into $x > X$), with $j(t)$ supported in $[0, 2X]$ and $e(t)$ supported in $[0, \infty)$. So, if $\eta(X)u_t(X,t)$ is known for $X \leq t \leq 3X$, then from (12) $e(t)$ is known for $0 \leq t \leq 2X$.

Substituting (11) and (12) in (10), we obtain

$$\begin{aligned}\delta(t) &= \left(\delta(t-X) + e(t-X)\right) * \left(\delta(t+X) + j(t+X)\right)\\ &= \left(\delta(t) + e(t)\right) * \left(\delta(t) + j(t)\right). && (13)\end{aligned}$$

Using the support of $e(t)$ and $j(t)$, (13) may be rewritten as the Volterra equation

$$e(t) + j(t) + \int_0^t j(s)e(t-s)\, ds = 0, \qquad t \geq 0\,. \tag{14}$$

Since $e(t)$ is known on $0 \leq t \leq 2X$ and $e(t)$ is square integrable on [0,2X], we may solve the above Volterra equation and recover a unique square integrable $j(t)$ for $0 \leq t \leq 2X$. *Hence knowing $\eta(X)u_t(X,t)$ for $X \leq t \leq 3X$, we can recover $\eta(X)(J_x + J_t)(X,t)$ for $|t| < X$.*

From the proof of Proposition 1,

$$J(x,t) = \begin{cases} \frac{1}{\eta(0)} & \text{for } t > x\\ \frac{1}{2\eta(0)} + M(x,t) & \text{for } |t| < x\\ 0 & \text{for } t < -x \end{cases} \tag{15}$$

where $M(x,t)$ is the solution of the Goursat problem

$$\mathcal{L}M = 0 \qquad \text{for } |t| \leq x \tag{16}$$

$$M(x,x) = \frac{1}{2\eta(0)} - \frac{1}{2\eta(x)}, \;\; M(x,-x) = \frac{1}{2\eta(x)} - \frac{1}{2\eta(0)} \tag{17}$$

So $(J_x + J_t)(x,t) = (M_x + M_t)(x,t)$ in $|t| < x$ and our problem reduces to the following -

*given $\eta(X)(M_x + M_t)(X,t)$ for $|t| \leq X$ - recover $\sigma(x)$ .*

## 3 Estimating $\|\sigma\|_0$ by $F(\sigma)$

We now obtain an upper bound on the $L^2$ norm of $\sigma$ in terms of the transmission data $F(\sigma)$.

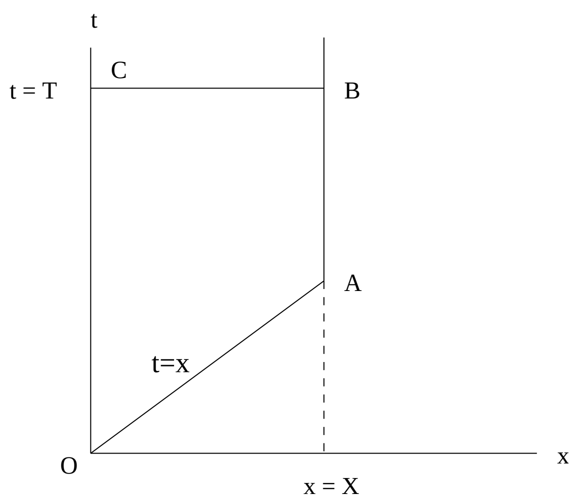
From Proposition 1, $U(x,x+) = 1/\eta(x)$, so in $t \geq x \geq 0$, $U$ is the solution of the Goursat problem

$$\begin{aligned} \eta^2(x)U_{tt} - (\eta^2(x)U_x)_x &= 0 \quad \text{in} \quad 0 \leq x \leq t & (18)\\ U(x,x) &= \frac{1}{\eta(x)} \quad \text{for } x \geq 0 & (19)\\ U_x(0,t) &= 0 \quad \text{for } t \geq 0 & (20) \end{aligned}$$

In addition $(U_t + U_x)(X,t) = 0$ for all $t$. Integrating the standard relation

$$0 = 2U_t(\eta^2 U_{tt} - (\eta^2 U_x)_x) = \left(\eta^2(U_t^2 + U_x^2)\right)_t - 2(\eta^2 U_x U_t)_x$$

over the trapezoidal region $OABC$ in the $x,t$ plane with vertices $O(0,0)$, $A(X,X)$, $B(X,T)$, and $C(0,T)$ (with $T > X$), we obtain




$$\begin{aligned} 0 \;=\; & \int_{BC} \eta^2(U_t^2 + U_x^2)(x,T) \;+\; 2\eta^2(0)\int_{OC} U_x(0,t)U_t(0,t) \\ & -\; 2\eta^2(X)\int_{AB} U_x(X,t)U_t(X,t) - \int_{OA} dx\; \eta^2(U_t^2 + U_x^2 + 2U_tU_x)(x,x) \end{aligned}$$

Now

$$\eta^2(U_t^2 + U_x^2 + 2U_tU_x)(x,x) = \eta^2(x)((U_t + U_x)(x,x))^2 = \eta^2(x)\left(\frac{d}{dx}U(x,x)\right)^2 .$$

So, using (20) on OC, (19) on OA, and $U_x + U_t = 0$ on $AB$, we obtain

$$\int_0^X \sigma(x)^2\, dx = E(T) + 2\eta^2(X)\int_{AB} U_t^2(X,t) \tag{21}$$

where

$$E(T) = \int_0^X \eta^2(x)(U_t^2 + U_x^2)(x, T)\, dx$$

is the energy at time $T$ in the interval $0 \leq x \leq X$.

Since $\sigma \in L^2(0, X)$, (21) implies that $\int_X^T U_t^2(X, t)\, dt$ (which is $\int_{AB} U_t^2(X, t)$) is bounded above by a finite quantity independent of $T$ implying $U_t(X, t) \in L^2(X, \infty)$. Letting $T$ approach infinity in (21), and using

**Proposition 2** $\lim_{T \to \infty} E(T) = 0$

which will be proved later, we obtain (using (12))

$$\int_0^X \sigma(x)^2\, dx = 2\eta^2(X) \int_X^\infty U_t^2(X, t)\, dt = 2 \int_0^\infty e^2(t)\, dt\ . \tag{22}$$

This estimates $\|\sigma\|_0$ in terms of the transmission data $e(t)$ for $t \in [0, \infty)$. However, this is not satisfactory since $F(\sigma)$ is the restriction of $e(t)$ to the interval $[0, 2X]$. However, the situation may be redeemed because the value of $e(t)$ on $[0, \infty)$ is completely determined by the value of $e(t)$ on $[0, 2X]$ as is shown below.

(14) may be rewritten as

$$j(t) + \int_0^t j(s) e(t - s)\, ds = -e(t), \quad t \geq 0$$

which is a Volterra equation for the unknown function $j$. Since $e(t)$ is in $L^2$, and known for $t \in [0, 2X]$, the Volterra equation may be solved for an $L^2$ function $j(t)$, $t \in [0, 2X]$, hence $j$ (which is supported in $[0, 2X]$) may be recovered from our data $F(\sigma)$.

Since $e$ and $j$ are in $L^2$, taking the Fourier Transform of (13) we obtain

$$(1 + \hat{e})(1 + \hat{j}) = 1 \tag{23}$$

giving

$$\hat{e} = -\frac{\hat{j}}{1 + \hat{j}}\ . \tag{24}$$

Note that this means that knowing $e(t)$ on $[0, 2X]$, we can recover $e(t)$ on $[0, \infty)$. But more importantly, (22) implies

$$\int_0^X \sigma^2(y)\, dy = 2 \int_0^\infty e^2(t)\, dt = \frac{1}{\pi} \int_{-\infty}^\infty |\hat{e}(\omega)|^2\, d\omega = \frac{1}{\pi} \int_{-\infty}^\infty \left| \frac{\hat{j}(\omega)}{1 + \hat{j}(\omega)} \right|^2 d\omega\ . \tag{25}$$

Indirectly, this gives the $L^2$ norm of $\sigma$ in terms of the data $F(\sigma)$.

## 4 Relation Between Reflection and Transmission Data

For a long time, for inversion of one dimensional hyperbolic problems, attention was mostly devoted to inversion from reflection data i.e. recovering the coefficients from $\eta(0)u_t(0,t)$, $0 \leq t \leq 2X$, where as we have been examining inversion from the transmission data $\eta(X)U_t(X,t)$, $X \leq t \leq 3X$. We have found an interesting relationship between these two data which is a modification and generalization of a result of Claerbout in [4] in the discrete case. This relationship will be crucial for us in characterizing the range of $F$ in the next section.

From Proposition 1, there is a function $r(t) \in L^2_{loc}[0,\infty)$, such that

$$\eta(0)U_t(0,t) = \delta(t) + r(t), \qquad t \in R. \tag{26}$$

We will establish a relationship between the reflected signal $r(t)$ and the transmitted signal $e(t)$ defined in (12).

In (9) take $v(x,t) = U(x,t)$ and $w(x,t) = U(x,-t)$. Then

$$\begin{aligned} f(x,t,s) &= \eta^2(x)\,\{U_t(x,t-s)U_x(x,-s) + U_x(x,t-s)U_t(x,-s)\} \\ g(x,t,s) &= -\,\eta^2(x)\,\{U_t(x,t-s)U_t(x,-s) + U_x(x,t-s)U_x(x,-s)\} \end{aligned}$$

and the right hand side of (9) will be zero. Integrating (9) over the region $x \in [0,X]$ and $s \in [-T,T]$, we obtain

$$\int_{-T}^{T} f(X,t,s)\,ds - \int_{-T}^{T} f(0,t,s)\,ds = \int_0^X g(x,t,T) - g(x,t,-T)\,dx\ . \tag{27}$$

Now, from the definition of $g(x,t,s)$ and using $2ab \leq a^2 + b^2$,

$$\begin{aligned} 2\int_0^X |g(x,t,-T)| &\leq \int_0^X \eta^2(x)(U_t^2 + U_x^2)(x,t+T)\,dx \ + \ \int_0^X \eta^2(x)(U_t^2+U_x^2)(x,T)\,dx \\ &= E(t+T) + E(T)\ . \end{aligned}$$

From Proposition 2, $E(T)$ approaches zero as $T$ approaches infinity. Further, $U(x,t)$ is zero for $t < 0$, hence for $|t| < T$, $g(x,t,T) = 0$. So letting $T$ approach infinity in (27), RHS approaches 0 (for a fixed $t$) and we have

$$\int_{-\infty}^{\infty} f(0,t,s)\,ds = \int_{-\infty}^{\infty} f(X,t,s)\,ds\ . \tag{28}$$

Combining $(U_x + U_t)(X,t) = 0$ with (12), we have

$$\begin{aligned} \text{RHS of (28)} &= -2\eta^2(X)\int_{-\infty}^{\infty} U_t(X,t-s)U_t(X,-s)\,ds \\ &= -2(\delta(t-X) + e(t-X)) * (\delta(-t-X) + e(-t-X)) \\ &= -2(\delta(t) + e(t)) * (\delta(-t) + e(-t)) \end{aligned}$$

For the LHS of (28), from (3) and (26), we have

$$\begin{aligned}\text{LHS of (28)} &= \int_{-\infty}^{\infty} \eta^2(0)\left\{U_t(0,t-s)U_x(0,-s)+U_x(0,t-s)U_t(0,-s)\right\}\,ds\\ &= \eta^2(0)U_t(0,t)*U_x(0,-t)+\eta^2(0)U_x(0,t)*U_t(0,-t)\\ &= -(\delta(t)+r(t))*\delta(-t)\;-\;(\delta(-t)+r(-t))*\delta(t)\\ &= -2\delta(t)-r(t)-r(-t)\end{aligned}$$

Hence (28) is equivalent to

$$2\delta(t)+r(t)+r(-t)=2(\delta(t)+e(t))*(\delta(-t)+e(-t)) \tag{29}$$

which establishes a relationship between the reflection data $r(t)$ and the transmission data $e(t)$. However, note that this relationship, at any time $t$, requires knowledge of $e(t)$ for all time $t \geq 0$ (because of the convolutions), and we are considering inverse problems where the data $e(t)$ is given only over $[0,2X]$.

Now (13) provides a convolution inverse of $\delta(t)+e(t)$, so convolving both sides of (29) with $\delta(t)+j(t)$, we obtain

$$(\delta(t)+j(t))*(2\delta(t)+r(t)+r(-t))=2(\delta(t)+e(-t)).$$

Since $e(t)$ is zero for $t<0$ we have

$$2j(t)+r(t)+r(-t)+j(t)*(r(t)+r(-t))=0, \qquad \text{for } t \geq 0.$$

Further, $j(t)$ is supported in $[0,2X]$. Hence we have the integral relation

$$2j(t)+(r(t)+r(-t))+\int_0^{2X} j(s)\,(r(t-s)+r(s-t))\,ds=0 \tag{30}$$

for $t \geq 0$.

This leads to an interesting observation. If the reflection data $r(t)$ is known over the interval $[0,2X]$ then $j(t)$ is obtained on $[0,2X]$ by solving an integral equation with a Toeplitz kernel. Hence the transmission data $e(X,t)$, $t \in [0,2X]$, may be recovered from (14).

Symes in [9] obtained a characterization of $r(t)$ which involved positivity of the operator with kernel $r(t-s)+r(s-t)$. This ties in with results on the existence and characterization of solutions of Toeplitz systems. We believe a careful examination of the known results on Toeplitz systems should lead to a characterization of $j(t)$ (so also of $J_x+J_t$) and hence to a characterization of the transmission data $e(t)$. However, we will characterize $j(t)$ using a related but different strategy.

## 5 Characterization Of The Range Of $F$

Our goal is to characterize the range of $F$ i.e. to classify the functions $e(t)$, $t \in [0,2X]$. We do not classify $e(t)$, $t \in [0,2X]$ explicitly; instead we classify the functions $j(t)$, $t \in [0,2X]$, and the characterization of $F$ follows from the one to one correspondence and relationship between $e$ and $j$ given by the Volterra equation (14) and the identity (13).

It is clear that we want $j$ to be in $L^2[0, 2X]$, but (25) suggests additional restrictions on $j$. It turns out that the conditions suggested by (25) are enough to classify $j$.

We will need a characterization of the Fourier Transform of functions in $L^2[0, \infty)$. We define the lower half plane

$$C_- = \{\ \zeta = \omega + i\theta\ :\ \omega, \theta \text{ real},\ \theta < 0\ \}$$

and $\overline{C_-}$ will be its closure. The Fourier Transform of a function $f$ in $L^2[0, \infty)$ is

$$\hat{f}(\zeta) = \int_0^\infty f(t) e^{-it\zeta}\, dt, \quad \zeta \in C_-\ .$$

Also, define $H_2^R(C_-)$ as the class of functions $F(\zeta)$ analytic in the lower half plane with

- $\overline{F(\omega + i\theta)} = F(-\omega + i\theta)$
- $\sup_{\theta<0}\ \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega + i\theta)|^2\, d\omega\ <\ \infty$

and endow this space with the norm which is the square root of the above supremum. Then the Paley Wiener theorem (see [5]) asserts that the map

$$\begin{array}{rcl} L^2[0,\infty) & \to & H_2^R(C_-) \\ f & \mapsto & \hat{f}(\zeta) \end{array}$$

is an isometric bijection. Further, $\hat{f}(\omega + i\theta)$ approaches $\hat{f}(\omega)$ in $L^2(-\infty, \infty)$ as $\theta \to 0^-$. This combined with (24) suggests

**Theorem** *Consider the forward map*

$$\begin{array}{rcl} G\ :\ L^2[0, X] & \to & L^2[0, 2X] \\ \sigma & \mapsto & \eta(X)(J_t + J_x)(X, t - X) - \delta(t) = j(t)\ . \end{array}$$

*Then $G$ is a continuous injection whose range is the open subset*

$$\mathcal{G} = \{\ j \in L^2[0, 2X]\ :\ 1 + \hat{j}(\zeta) \neq 0,\ \forall\ \zeta \in \overline{C_-}\ \}.$$

*Further*

$$\int_0^X \sigma^2(y)\, dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\hat{j}(\omega)}{1 + \hat{j}(\omega)} \right|^2 d\omega\ .$$

Actually, we believe that the map $G$ is a diffeomorphism onto its range and one should be able to prove this using "sideways" energy estimates as done in [11], or by using the diffeomorphism from $\sigma$ to $r$ proved in [11] and proving the smoothness of the the map taking $j$ to $r$ which is described below.

To prove this theorem we will need

**Proposition 3** *Suppose $j \in L^2[0,2X]$. Then $\hat{j}/(1+\hat{j})$ is in $H_2^R(C_-)$ if and only if $j \in \mathcal{G}$.*

**Proof of Proposition 3**
Note that $1+\hat{j}(\zeta)$ is a nonzero entire function so its zeros are isolated and countable in number, so $\hat{j}/(1+\hat{j})$ is not analytic only at a countable number of isolated complex numbers. Also, since $j$ is real valued so

$$\frac{\overline{\hat{j}(\omega+i\theta)}}{1+\overline{\hat{j}(\omega+i\theta)}} = \frac{\hat{j}(-\omega+i\theta)}{1+\hat{j}(-\omega+i\theta)} \ .$$

Further

$$\lim_{\substack{|\zeta|\to\infty \\ \zeta\in\overline{C_-}}} \hat{j}(\zeta) = 0. \tag{31}$$

This may be seen for $j \in C_0^\infty(0,2X)$ by an integration by parts (here $\zeta = \omega + i\theta$, $\theta \le 0$)

$$|\hat{j}(\zeta)| = \left| \int_0^{2X} j(t) e^{-it\zeta}\, dt \right| = \frac{1}{|\zeta|} \left| \int_0^{2X} j'(t) e^{-it\zeta}\, dt \right| \le \frac{1}{|\zeta|} \int_0^{2X} |j'(t)| e^{t\theta}\, dt \le \frac{1}{|\zeta|} \int_0^{2X} |j'(t)|\, dt$$

and this combined with the denseness of $C_0^\infty(0,2X)$ in $L^2[0,2X]$ and the inequality (note $\theta \le 0$)

$$|\hat{j}(\zeta)| \le \int_0^{2X} |j(t)| e^{t\theta}\, dt \le \sqrt{2X} \|j\| \ .$$

proves our claim for all $j$ in $L^2[0,2X]$.

Now suppose $\hat{j}/(1+\hat{j})$ is in $H_2^R(C_-)$. Then analyticity on $C_-$ of functions in $H_2^R(C_-)$ implies that $1+\hat{j}(\zeta) \neq 0$ for $\zeta \in C_-$. But we wish to prove this property for $\zeta \in \overline{C_-}$. Suppose $1+\hat{j}(\zeta)$ has a zero at some real number $a$. Then analyticity of $\hat{j}$ implies that

$$1+\hat{j}(\zeta) = (\zeta - a)^n \phi(\zeta), \quad \zeta \in C$$

with $n$ a positive integer, and $\phi$ analytic and non-zero in some rectangle

$$K = \{\ \zeta = \omega + i\theta \ : \ |w-a| \le \delta,\ |\theta| \le \delta\ \}$$

around $a$. Hence there is a positive lower bound $m$ for $|\phi(\zeta)|$ on $K$. Further, since $1+\hat{j}(a) = 0$, WLOG we may assume that $|\hat{j}(\zeta)| > 1/2$ if $\zeta \in K$. Then for $-\delta \le \theta < 0$

$$\int_{-\infty}^{\infty} \left| \frac{\hat{j}(\zeta)}{1+\hat{j}(\zeta)} \right|^2 d\omega \ \ge \ \int_{a-\delta}^{a+\delta} \frac{1}{4m^2 |w-a+i\theta|^{2n}}\, d\omega \ = \ \int_{-\delta}^{\delta} \frac{1}{4m^2(\omega^2+\theta^2)^n} d\omega$$

and the last integral approaches infinity as $\theta$ approaches 0 by the monotone convergence theorem - note $n \ge 1$. So $1+\hat{j}(\zeta)$ has no zeros in $\overline{C_-}$ and hence $j$ must be in $\mathcal{G}$.

Conversely, if $j \in \mathcal{G}$, then $\hat{j}/(1+\hat{j})$ is analytic in $C_-$ and continuous on $\overline{C_-}$. Further, from (31), and the fact that $1+\hat{j}(\zeta)$ is non-zero and continuous on $\overline{C_-}$ we conclude that $|1+\hat{j}|$ has a positive lower bound (say $m$) on $\overline{C_-}$. Hence

$$\left| \frac{\hat{j}(\zeta)}{1+\hat{j}(\zeta)} \right| \le \frac{|\hat{j}(\zeta)|}{m}, \qquad \zeta \in C_- \ .$$

This implies $\hat{j}/(1+\hat{j})$ is in $H_2^R(C_-)$ because $\hat{j}$ is in $H_2^R(C_-)$.

QED

**Proof of the Theorem**

Suppose $\sigma \in L^2[0,X]$. Then $j \in L^2[0,X]$ and in Section 3 we showed that $e \in L^2[0,\infty)$ and

$$\hat{e} = -\frac{\hat{j}}{1+\hat{j}} \ .$$

Hence the Proposition implies that $j \in \mathcal{G}$, proving that the range of $G$ is a subset of $\mathcal{G}$.

Now we show that the range of $G$ is actually $\mathcal{G}$. Suppose $j$ is in $\mathcal{G}$. Then the Proposition implies that $-\hat{j}/(1+\hat{j})$ is in $H_2^R(C_-)$ and hence by the Paley Wiener theorem there is a function (we call it) $e \in L^2[0,\infty)$ such that

$$\hat{e}(\zeta) = -\frac{\hat{j}(\zeta)}{1+\hat{j}(\zeta)}, \qquad \zeta \in \overline{C_-}.$$

Note that the right hand side is continuous on $\overline{C_-}$ and that this relation implies (13).

Now we construct $r$ guided by (29). Since $e(t)+e(-t)+e(t)*e(-t)$ is a real valued even function we may define a unique real valued function $r(t)$ supported in $[0,\infty)$ such that

$$r(t)+r(-t) = 2(e(t)+e(-t)+e(t)*e(-t)) \ .$$

Now $e$ is in $L^2[0,\infty)$, hence $e(t)*e(-t)$ is a bounded function and hence a tempered distribution. In addition, the Fourier Transform of $e(t)*e(-t)$ is $\hat{e}(\omega)\overline{\hat{e}(\omega)} = |\hat{e}(\omega|^2$. But $\hat{e}(\omega) = -\frac{\hat{j}(\omega)}{1+\hat{j}(\omega)}$ is a continuous function on $(-\infty,\infty)$ which approaches 0 as $|\omega|$ approaches infinity, hence $\hat{e}(\omega)$ is bounded. Hence $|\hat{e}(\omega)|^2$ is in $L^2(-\infty,\infty)$ because $\hat{e}(\omega)$ is in $L^2(-\infty,\infty)$. So $r(t)$ is in $L^2[0,\infty)$ and

$$2+\hat{r}(\omega)+\overline{\hat{r}(\omega)} = 2(1+\hat{e}(\omega)+\overline{\hat{e}(\omega)}+|\hat{e}(\omega)|^2) = 2|1+\hat{e}(\omega)|^2 = \frac{2}{|1+\hat{j}(\omega)|^2} \tag{32}$$

The function $r(t)$ generates a self adjoint operator

$$\begin{aligned} S \ : \ L^2[-X,X] \quad &\rightarrow \quad L^2[-X,X] \\ S(f)(t) \quad &= \quad 2f(t)+(r(t)+r(-t))*f(t) \\ &= \quad 2f(t)+\int_{-X}^{t} r(t-s)f(s)\,ds \ + \ \int_{t}^{X} r(s-t)f(s)\,ds \end{aligned}$$

Note that $S(f)$ is a function only on $[-X,X]$ but the RHS of the definition makes sense for all real $t$. Further, for $t \in [-X,X]$, the definition of $Sf(t)$ requires knowing $r(t)$ only over the interval $[0,2X]$. Also, note that $S(f)$ is in $L^2[-X,X]$, because using (32) and properties of Fourier Transforms

$$\begin{aligned} \|S(f)\|^2 \ &\leq \ \int_{-\infty}^{\infty} |2f(t)+(r(t)+r(-t))*f(t)|^2\,dt \\ &= \ \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{|\hat{f}(\omega)|^2}{|1+\hat{j}(\omega)|^4}\,d\omega \ \leq \ C\|f\|^2 \end{aligned}$$

because $|1+\hat{j}(\omega)|$ is bounded below by a positive number.

Further, $S$ is positive definite, because if $f \in L^2[-X,X]$ then

$$\begin{aligned}(Sf,f) &= \int_{-X}^{X} S(f)(t)f(t)\,dt = \int_{-\infty}^{\infty} (2f(t) + (r(t)+r(-t)) * f(t))f(t)\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\hat{f}(\omega)|^2}{|1+\hat{j}(\omega)|^2}\,dw \;\geq\; C\|f\|^2\end{aligned}$$

for some positive number $C$ because $|1+\hat{j}(\omega)|$ is bounded above on $(-\infty,\infty)$.

Hence, by the characterization theorem of Symes (Theorem 1 in [9] or Theorem 4.2 in [1]), there is a unique $\sigma \in L^2[0,X]$ which generates this $r(t)$ through (26). To complete the proof of the claim that $\mathcal{G}$ is the range of $G$ we need to show that the $j$ generated from this $\sigma$ through (11) and the $j$ we started out with are identical.

From our construction of $e$ and $r$ from $j$, in the proof of this theorem, it is clear that (13) and (29) hold. Further, these relations also hold for any $e, j, r$ defined through (26) - (11). From (13) and (29) we conclude that

$$(\delta(t)+j(t)) * (2\delta + r(t) + r(-t)) = 2\delta(t) + 2e(-t)\;.$$

which simplifies to

$$2j(t) + (r(t)+r(-t)) * j(t) = 2e(-t) - r(t) - r(-t)$$

or

$$2j(t-X) + (r(t)+r(-t)) * j(t-X) = 2e(-t-X) - r(t-X) - r(-t-X).$$

Remembering the supports of $r, j, e$ the above relation leads to

$$S(j(t-X))(t) = -r(t-X), \qquad t \in [-X,X]\;. \tag{33}$$

So, the $j$ that we started out with and the $j$ constructed from the $\sigma$ which was constructed from the $r$, both obey (33). But $S$ is injective because it is positive definite, hence (33) has at most one solution. Hence the two $j(t-X)$ coincide if $t-X \in [-X,X]$, so the two $j$ coincide since they are supported in $[0,2X]$. This completes the proof of the claim that the range of $G$ is $\mathcal{G}$.

$\mathcal{G}$ is open because if $\phi \in L^2[0,2X]$ with $\|\phi\| \leq \epsilon$, then

$$\begin{aligned}|\hat{\phi}(\zeta)| &\leq \int_0^{2X} |\phi(t)|e^{t\theta}\,dt, \qquad \zeta = \omega + i\theta,\;\; \theta \leq 0 \\ &\leq \int_0^{2X} |\phi(t)|\,dt \;\leq\; \sqrt{2X}\epsilon\end{aligned}$$

and $|1+\hat{j}(\zeta)|$ is bounded below by a positive number for all $\zeta \in \overline{C_-}$, for a given $j \in L^2[0,2X]$. So $1+\hat{j}(\zeta) + \hat{(\phi)}(\zeta) \neq 0$ on $\overline{C_-}$ provided $\epsilon$ is small enough.

The continuity of $G$ may be proved using arguments used to prove the continuous dependence of solutions on the coefficients. The injectivity of $G$ was proved indirectly in [6] where we gave an inversion procedure (with proof) provided an upper bound on $\|\sigma\|_0$ was known.

QED

## 6 Coefficient Recovery From Data

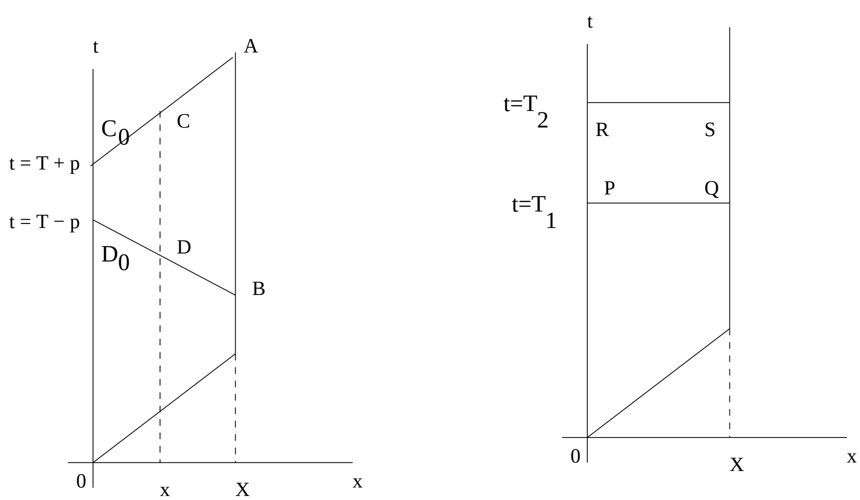
We now describe how $\eta$ may be recovered from the data $\eta(X)(M_x + M_t)(X,t)$ on $|t| \leq X$. This was described in [6] with the assumption that we have an upper bound on $\|\sigma\|_0$ in terms of the data (which we have proved here) but we restate it here for completeness and because it is short.

From (16) and (17), we observe that $M(x,t)$ is odd in $t$. Hence $M_t(x,t)$ is even in $t$ and $M_x(x,t)$ is odd in $t$. So, given $\eta(X)(M_x + M_t)(X,t)$ for $|t| \leq X$, we can recover its odd and even parts $\eta(X)M_x(X,t)$ and $\eta(X)M_t(X,t)$. Hence, again appealing to the ideas in [7] we can recover $\eta$ up to a constant.

## 7 Energy Decay

**Proof of Proposition 2**
The literature contains proofs of such results but they seem to require $\eta$ to be smoother than $H^1$ so we have chosen to prove the result independently. From (1) we have

$$\begin{aligned} 0 &= 2U_x\left(\eta^2 U_{tt} - (\eta^2 U_x)_x\right) \\ &= 2(\eta^2 U_t U_x)_t - (\eta^2(U_t^2 + U_x^2))_x + 2\eta\eta'(U_t^2 - U_x^2). \end{aligned}$$



Integrating this over the trapezoidal region ABCD formed by the lines $x = x$, $x = X$, $t = x+T+p$ and $t = -x+T-p$, for some small positive number $p$, we obtain

$$\int_{CD} \eta^2(U_t^2 + U_x^2) - \int_{AB} \eta^2(U_t^2 + U_x^2) + \int_{AC} \eta^2(U_x + U_t)^2 \, dx + \int_{BD} \eta^2(U_t - U_x)^2 + \int\int_{ABCD} 2\eta\eta'(U_t^2 - U_x^2) = 0 \, .$$

Let us define the sideways energy

$$H(x) = \int_{-x+T-p}^{x+T+p} \eta^2(x)(U_t^2 + U_x^2)(x,t) \, dt = \int_{CD} \eta^2(U_t^2 + U_x^2) \tag{34}$$

then using the non-negativity of the integrals on AC and BD, the previous equation implies

$$H(x) \le H(X) + \int\int_{ABCD} 2\eta\eta'(U_t^2 - U_x^2) \le H(X) + 2\int_x^X |\sigma(y)| H(y) \, dy \, .$$

for all $x \in [0, X]$. Here $\sigma = \eta'/\eta$.

So Gronwall's inequality implies

$$H(x) \le H(X) e^{2\int_0^X |\sigma|} \qquad \text{for all } x \in [0, X].$$

Integrating $H(x)$ over the interval $[0, X]$ we obtain

$$\int\int_{ABC_0D_0} \eta^2(U_t^2 + U_x^2) \le K H(X) \tag{35}$$

for some constant $K$, where $K$ depends only on the $L^2$ norm of $\sigma$ and $X$ and $p$. Note that $K$ does not depend on $T$ because the area of $ABC_0D_0$ does not depend on $T$.

We also have the standard energy identity obtained from

$$0 = 2U_t(\eta^2 U_{tt} - (\eta^2 U_x)_x) = \left(\eta^2(U_t^2 + U_x^2)\right)_t - 2(\eta^2 U_x U_t)_x$$

by integrating the above relation over the rectangular region PQRS formed by the lines $t = T_1$, $t = T_2$, $x = 0$ and $x = X$, $(0 < T_1 < T_2)$ , we have

$$E(T_2) - E(T_1) - 2\int_{QS} \eta^2 U_x U_t + 2\int_{PR} \eta^2 U_x U_t = 0$$

where

$$E(t) = \int_0^X \eta^2(x)(U_x^2 + U_t^2)(x,t) \, dx \, .$$

But on $QS$ we have $U_t = -U_x$ and on $PR$ we have $U_x = 0$ which leads to

$$E(T_2) + 2\int_{QS} \eta^2 U_x^2 = E(T_1) \, .$$

Hence

$$E(T_2) \leq E(T_1)$$

for any $T_2 > T_1 > X$. So $E(t)$ is a decreasing function if $t > X$.

Now, from (35), and the decreasing nature of $E(t)$, we have

$$\begin{aligned}
pE(T) &\leq \int_{T-p}^{T} E(t)\, dt \;\leq\; \int\int_{ABC_0D_0} \eta^2(U_t^2 + U_x^2) \\
&\leq KH(X) \;=\; K\int_{T-X-p}^{T+X+P} \eta(X)^2(U_t^2 + U_x^2)(X,t)\, dt \\
&= 2K\int_{T-X-p}^{T+X+P} \eta(X)^2 U_t^2(X,t)\, dt \qquad \text{(because } U_t = -U_x \text{ on } x = X\text{)}.
\end{aligned}$$

But since $U_t(X,t)$ is in $L^2[X,\infty)$ (see (21)), the integral on the right approaches zero as $T$ approaches infinity. Hence $E(T)$ approaches zero as $T$ approaches infinity.

## 8 Expansions of $U$ and $J$

**Proof of Proposition 1**

We first prove the result for $\phi$. From the introduction

$$U(x,t) = \begin{cases} V(x,t) & t \geq x \\ 0 & t < x \end{cases}$$

where $V(x,t)$ is the solution of the Goursat Problem

$$\begin{aligned}
\eta^2(x)V_{tt} - (\eta^2(x)V_x)_x &= 0 \qquad \text{in} \quad t \geq x \geq 0 \\
V(x,x) &= \frac{1}{\eta(x)} \qquad \text{in } x \geq 0 \\
V_x(0,t) &= 0 \qquad \text{in } t \geq 0\ .
\end{aligned}$$

Using standard techniques one may show that $V$ is locally in $H^1$ and $V_x$ and $V_t$ have traces on $t = \textit{constant}$ and $x = \textit{constant}$ which are locally in $L^2$. So

$$U(x,t) = \frac{1}{\eta(x)}H(t-x) + \phi(x,t)$$

where

$$\phi = \begin{cases} V(x,t) - 1/\eta(x) & t \geq x \\ 0 & t < x \end{cases}$$

Then because the trace of $V(x,t) - 1/\eta(x)$ is zero on $t = x$, one may show that $\phi$ is locally in $H^1$ (including near $t = x$). Hence the $x$ and $t$ derivative of $\phi$ are zero extensions of the $x$ and $t$ derivative of $V - 1/\eta$ proving the claim about $\phi$ in the proposition.

From the progressing wave expansion one may show that $J$ has the form

$$J(x,t) = \frac{1}{2\eta(x)}(H(t-x) + H(t+x)) + \psi(x,t)$$

where $\psi$ has no jump across $t = x$ or $t = -x$. Hence

$$J(x,t) = \begin{cases} \frac{1}{\eta(x)} + \psi(x,t) & \text{for } t > x \\ \frac{1}{2\eta(x)} + \psi(x,t) & \text{for } |t| < x \\ \psi(x,t) & \text{for } t < -x \end{cases} .$$

Now from the speed of propagation argument applied to (7) and (8), we know that $\psi(x,t) = 0$ for $t + x \leq 0$. In the region $t > x > 0$, $J(x,t)$ solves the IVP (7) and (8). Noting that the coefficients of $\mathcal{L}$ are independent of $t$, we may show that $J$ is independent of $t$ in the region $t > x > 0$ and solves the ODE

$$-(\eta^2(x)J'(x))' = 0, \quad \eta(0)J(0) = 1, \quad J_x(0) = 0 .$$

Hence $J(x,t) = 1/\eta(0)$ in the region $t > x > 0$, implying that $\psi(x,t) = 1/\eta(0) - 1/\eta(x)$ in this region. Hence

$$J(x,t) = \begin{cases} \frac{1}{\eta(0)} & \text{for } t > x \\ \frac{1}{2\eta(0)} + M(x,t) & \text{for } |t| < x \\ 0 & \text{for } t < -x \end{cases}$$

where $M(x,t)$ is the solution of the Goursat problem

$$\mathcal{L}M = 0 \qquad \text{for } |t| \leq x$$

$$M(x,x) = \frac{1}{2\eta(0)} - \frac{1}{2\eta(x)}, \;\; M(x,-x) = \frac{1}{2\eta(x)} - \frac{1}{2\eta(0)}$$

Again using standard techniques one may show that $M$ is in $H^1$ and $M_x$ and $M_t$ have traces on $t = \textit{constant}$ and $x = \textit{constant}$ which are in $L^2$. Hence

$$\psi(x,t) = \begin{cases} \frac{1}{\eta(0)} - \frac{1}{\eta(x)} & \text{for } t > x \\ \frac{1}{2\eta(0)} - \frac{1}{2\eta(x)} + M(x,t) & \text{for } |t| < x \\ 0 & \text{for } t < -x \end{cases}$$

So $\psi$ is locally in $H^1$ in the regions $t > x$, $|t| < x$, and $t < -x$. Also, $\psi$ has no jumps across $t = |x|$, hence one may show that $\psi$ is actually locally in $H^1$ over the whole $x, t$ space, and that the $x$ and $t$ derivative are extensions of the $x$ and $t$ derivative of $M - 1/(2\eta)$. This proves our claim about $\psi$.

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