Recovering a function from its spherical mean values in two and three dimensions

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1 Introduction

The problem of determining a function from a subset of its spherical means has a rich history in pure and applied mathematics. Its connection with photoacoustic imaging has already been made abundantly clear in earlier chapters in this volume: when the sound speed ν_s in a medium is constant the pressure at time t and point $\mathbf{r}$ is expressed in terms of spherical means of the pressure, and its time derivative, at any earlier time t'. We begin this chapter with a review of the connections between spherical means and the several equations which arise in the time or frequency domains in the analysis of photoacoustic imaging and show how the inverse problem of photoacoustic imaging can be formulated as the problem of recovering a function from some collection of its spherical means. In §2 we discuss the uniqueness problem of when a function is determined by a collection of its spherical means, and when the initial value of a wave equation is determined by the value of the solution on a subset of the spatial domain over a given time interval. In §3 we specialize to the question of actually recovering (or approximately recovering) a function supported in a region D in space (or in the plane) from its spherical means with centers on the boundary of D for fairly general regions D. In connection with this topic, it is convenient to also discuss some related work on the characterization of the range of the spherical mean transform, and of related wave equations. (Several other chapters in this volume have some discussion of the range: see the chapter by S. Patch and the chapter by M. Agranovsky, P. Kuchment, and L. Kunyansky.) In §4, we specialize to the case when the family of spherical means are centered on sets with simple geometry, and present some of the filtered back-projection formulas that have been found. This section is largely independent of the preceding two, and so the reader interested in explicit formulas can jump directly from the introduction. We shall confine the discussion of this chapter to two and three dimensions, since these are relevant to photoacoustic tomography. Many of the results do have higher dimensional generalizations, and the interested reader may pursue these in the original articles.

Some notations: Partial derivatives will be written out fully as in $\frac{\partial u}{\partial r_i}$, as ∂_{r_i}, or in subscript form as u_{r_i}. For the spatial Laplacian, we will use both the engineers convention ∇^2 and the

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mathematicians convention of Δ. With $\mathbf{r}$ a point in Euclidean n-space, this means

$$\nabla^2 u(\mathbf{r}) = \Delta u(\mathbf{r}) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial r_i^2}. \tag{1}$$

A region $D \subset \mathbf{R}^n$ will be an open set with smooth boundary.

As seen in Chapter ??, the pressure function p in photoacoustic tomography in a volume filled by an acoustically homogeneous liquidlike medium can be modeled by the inhomogeneous wave equation:

$$p_{tt}(\mathbf{r},t) - \nu_s^2 \nabla^2 p(\mathbf{r},t) = \Gamma \partial_t H(\mathbf{r},t), \tag{2}$$

where ν_s is the wave speed, H is the heating function, and $\Gamma = \frac{\beta \nu_s^2}{C_p}$ where β is the thermal coefficient of volume expansion and C_p is the specific heat capacity at constant pressure. It is usually assumed that H has the form $H(\mathbf{r},t) = A(\mathbf{r}) I_e(t)$ where I_e is a temporal illumination function and A is a spatial absorption function, and that $p = 0$ for $t < 0$. The inverse problem is to recover the absorption function A.

Thermoacoustic tomography also leads to a similar inverse problem in dimension two. We just sketch the reduction here, and refer to the chapter by G. Paltauf for further details. In the case of constant wave speed, we integrate (2) over the family of lines parallel to a given direction θ (i.e. take the x-ray transform in direction θ) to obtain a wave equation in two dimensions. Indeed, if $\mathbf{r}^\perp$ is taken as coordinate on a hyperplane orthogonal to θ, then by the rotation invariance of the Laplacian,

$$\nabla^2 = (\nabla^\perp)^2 + (\theta \cdot \nabla)^2,$$

then well-known results for the x-ray transform show that

$$\int [(\nabla^\perp)^2 + (\theta \cdot \nabla)^2] p(\mathbf{r}^\perp + s\theta, t)\, ds = (\nabla^\perp)^2 \int p(\mathbf{r}^\perp + s\theta, t)\, ds.$$

The x-ray transform of the right hand side of (2) is just the x-ray transform of (a constant multiple of) the spatial component of the heating function times the temporal illumination term. This reduction of dimension is relevant to photacoustic tomography with integrating line detectors (see the chapter by Paltauf), since an integrating line detector can be modeled as a device performing the x-ray transform of the pressure function. In this situation, the inverse problem of recovering the spatial absoption functions breaks into two stages. The first stage is to recover the x-ray transform of the absorption, in some family of directions, from the values of the two-D wave equation at those points $\mathbf{r}'$ corresponding to locations of the integrating line detectors, while the second stage is to invert the x-ray transform.

When the wave speed ν_s is constant, there is a close connection between the solution of the wave equation and spherical means. We now remind the reader of the solution operator of the following single datum initial value problem for the wave equation, in dimensions two and three:

$$u_{tt}(\mathbf{r},t) - \nu_s^2 \nabla^2 u(\mathbf{r},t) = 0 \tag{3}$$

$$u(\mathbf{r},0) = 0 \qquad u_t(\mathbf{r},0) = g(\mathbf{r}), \tag{4}$$

under the assumption that ν_s is constant. If, for the moment, we denote the solution operator by $\mathcal{W}$, so that $u = \mathcal{W}\, g$, then the solution of the non-homogenous wave equation with full

Cauchy data

$$v_{tt}(\mathbf{r},t) - \nu_s^2 \nabla^2 u(\mathbf{r},t) = F(\mathbf{r},t) \tag{5}$$
$$v(\mathbf{r},0) = f(\mathbf{r}) \qquad v_t(\mathbf{r},0) = g(\mathbf{r})$$

is given by

$$v(\mathbf{r},t) = \partial_t(\mathcal{W}\, f)(\mathbf{r},t) + \mathcal{W}\, g(\mathbf{r},t) + \int_0^t \mathcal{W}\, F(\cdot,s)(\mathbf{r},t-s)\, ds.$$

The last term above is the solution of the non-homogeneous problem with zero initial data, as in (2). We follow the presentation in [1, chap. V].

In three space dimensions, for $t > 0$ the solution of (3), (4) is given by

$$u(\mathbf{r},t) = \frac{1}{4\pi} t \int_{S^2} g(\mathbf{r} + \nu_s t\omega)\, dS(\omega), \tag{6}$$

where ω is a point on the unit sphere S^2 and $dS(\omega)$ is the area element on the unit sphere. In dimension two, the solution takes the form

$$u(\mathbf{r},t) = \frac{1}{2\pi\nu_s} \int_{|\mathbf{r}'|<\nu_s t} \frac{g(\mathbf{r}+\mathbf{r}')}{\sqrt{\nu_s^2 t^2 - |\mathbf{r}'|^2}}\, d^2 r'. \tag{7}$$

The integrals can be expressed in terms of the (normalized) spherical mean operator, which is defined for continuous functions with compact support on $\mathbf{R}^n$ by

$$(\mathcal{M}\, f)(\mathbf{r},\sigma) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(\mathbf{r} + \sigma\omega)\, dS(\omega), \tag{8}$$

where S^{n-1} is the unit sphere in $\mathbf{R}^n$, $|S^{n-1}|$ its area (length) and $dS(\omega)$ the area element ($n \geq 3$) or arc length element ($n = 2$). Here $\mathbf{r}$ is the center of the sphere over which f is averaged, and $\sigma \geq 0$ is the radius. We note that $\mathcal{M}\, f$ has a natural extension to a function on $\mathbf{R}^n \times \mathbf{R}$ even with respect to σ. Incorporating the definition of the spherical mean operator, we may rewrite the solution (6) to (3), (4) as

$$u(\mathbf{r},t) = t(\mathcal{M}\, g)(\mathbf{r}, \nu_s t), \qquad n = 3 \tag{9}$$

and (7) as

$$u(\mathbf{r},t) = \int_0^t \frac{\tau}{\sqrt{t^2 - \tau^2}} (\mathcal{M}\, g)(\mathbf{r}, \nu_s \tau)\, d\tau, \qquad n = 2. \tag{10}$$

In view of (2) we also need the solution of the inhomogeneous problem

$$\frac{\partial^2}{\partial t^2} u(\mathbf{r},t) - \nu_s^2 \nabla^2 u(\mathbf{r},t) = F(\mathbf{r},t) \tag{11}$$
$$u(\mathbf{r},t) = 0 \qquad t < 0.$$

In dimension three it is given explicitly [1, eq. V.1.40] by

$$u(\mathbf{r},t) = \int_0^t (t-\tau)(\mathcal{M}\, F)(\mathbf{r}, \nu_s(t-\tau), \tau)\, d\tau = \int_{\mathbf{R}^4} \frac{\delta(\nu_s(t-t') - |\mathbf{r}-\mathbf{r}'|)}{4\pi\nu_s |\mathbf{r}-\mathbf{r}'|} F(\mathbf{r}',t')\, dt'\, d^3 r'. \tag{12}$$

Applying (12) to the model (2), where the forcing function has the form $F(\mathbf{r},t) = \Gamma A(\mathbf{r})\partial_t I(t)$, we see that

$$p(\mathbf{r},t) = \int_0^t (t-\tau)(\Gamma\,\mathcal{M}\,A)(\mathbf{r},\nu_s(t-\tau))I'(\tau)\,d\tau. \tag{13}$$

To recover the spherical means of the absorption function from the pressure in (13) requires a deconvolution, but if the temporal illumination function factor of the heating function in (2) is idealized to a delta impulse, then (13) simplifies and the solution is given for $t > 0$ by

$$p(\mathbf{r},t) = \partial_t(t(\mathcal{M}(\Gamma A))(\mathbf{r},\nu_s t)). \tag{14}$$

This is the same as (for $t > 0$) the solution of the initial value problem

$$\begin{gathered} p_{tt}(\mathbf{r},t) - \nu_s^2\nabla^2 p(\mathbf{r},t) = 0 \\ p(\mathbf{r},0) = p_0(\mathbf{r}) = \Gamma A(\mathbf{r}) \qquad p_t(\mathbf{r},0) = 0. \end{gathered}$$

While it is important to maintain units in the physical analysis, it is a nuisance when dealing with the relation between the wave equation and spherical means. Thus in the sequel, at least when dealing with the constant wave speed model, we shall assume that the wave speed is unity. This may be effected by rescaling time by the magnitude of the wave speed. We will also assume that the temporal illumination function is a delta impulse, or that deconvolution of the temporal impulse has already been performed, so that we are working with a pure initial value problem. Finally, we shall simply represent the initial values f or g, incorporating all physical constants in the initial data.

Since many papers on the inversion problem have worked with the wave equation in the frequency domain, we remind the reader of this approach. We will confine the discussion to the initial value problem (3), (4), in dimension $n = 3$, assuming $\nu_s = 1$. While the solution has a natural extension to $t \in \mathbf{R}$ as an odd function of t, we consider the solution where u is zero for $t < 0$. Then the Fourier transform

$$\tilde{u}(\mathbf{r},k) = \int_{\mathbf{R}} u(\mathbf{r},t)e^{ikt}\,dt = \int_0^\infty u(\mathbf{r},t)e^{ikt}\,dt, \tag{15}$$

satisfies the Helmholtz equation

$$\nabla^2\tilde{u}(\mathbf{r},k) + k^2\tilde{u}(\mathbf{r},k) = -g(\mathbf{r}). \tag{16}$$

Using (9), the solution can be expressed through the spherical means

$$\tilde{u}(\mathbf{r},k) = -\int_0^\infty t(\mathcal{M}\,g)(\mathbf{r},t)e^{ikt}\,dt = -\int_{\mathbf{R}^3} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}g(\mathbf{r}')\,d^3r'. \tag{17}$$

Here the factor

$$G_k(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{18}$$

is a fundamental solution (free space Green's function) of the Helmholtz equation called the outgoing fundamental solution. For $k > 0$ it satisfies the radiation condition,

$$\lim_{|\mathbf{r}|\to\infty}(\mathbf{r}\cdot\nabla_{\mathbf{r}}G - ik|\mathbf{r}|G) = 0. \tag{19}$$

The complex conjugate of G_k is called the incoming fundamental solution.

At several points later on, we shall need the composition of the δ distribution with a differentiable function. If $q(\mathbf{r})$ is a differentiable function whose gradient is non-zero everywhere on $Z = \{\mathbf{r} : q(\mathbf{r}) = 0\}$, the generalized function $\delta(q)$ is defined by

$$\int_{\mathbf{R}^n} \phi(\mathbf{r})\delta(q(\mathbf{r}))\, d^n r = \int_Z \frac{\phi(\mathbf{r}_0)}{|\nabla q\,|(\mathbf{r}_0)}\, dS(r_0), \tag{20}$$

where $dS(r_0)$ is the area element on Z. For the calculus of such generalized functions we refer the reader to [2].

2 Uniqueness

We study which subsets S of $\mathbf{R}^n$, $n = 2, 3$, have the property that knowing $(\mathcal{M}\, f)(\mathbf{r}, t)$ on $S \times [0, \infty)$ will be enough to uniquely determine f for all smooth, compactly supported functions f on $\mathbf{R}^n$. Such subsets S of $\mathbf{R}^n$ are called *sets of injectivity* for $\mathcal{M}$. We state a few results regarding injectivity sets - see [3] for a more thorough discussion.

Injectivity sets for the mean value operator, in two dimensions, were completely characterized in Theorem B in [4] and we state the result below. For any positive integer N define Σ_N to be the Coxeter system of N lines (in the complex plane)

$$\Sigma_N := \{re^{i\pi k/N} \ : \ k = 1, \cdots, N, \ r \in \mathbf{R}\}.$$

Theorem 1. *A subset $S \subseteq \mathbf{R}^2$ is a set of injectivity for $\mathcal{M}$ on $\mathbf{R}^2$ if and only if S is not contained in any set of the form $Q(\Sigma_N) \cup Y$ for some N, for some rigid motion Q and some finite set Y.*

So, the only subsets S of $\mathbf{R}^2$ which fail to be injectivity sets are those for which all the points in S, except for a finite number, lie on an even number of equiangular rays emanating from some point in $\mathbf{R}^2$. When S is one of the Coxeter system of lines, corresponding to a positive integer N, then one may construct a smooth compactly supported function f on $\mathbf{R}^2$ which is anti-symmetric (odd) about all the lines in Σ_N and hence $(\mathcal{M}\, f)(\mathbf{r}, \sigma) = 0$ for all $\mathbf{r} \in S$, $\sigma \in \mathbf{R}$. In fact take $f(\mathbf{r}) = \phi(|\mathbf{r}|)\psi(\theta)$ where ϕ is compactly supported smooth function on $(0, \infty)$ and $\psi(\theta)$ is a periodic odd function on $[-\pi, \pi]$ of period π/N.

In the three dimensional case, injectivity sets have not yet been characterized. We cite a few results which are often sufficient in applications. From [5], if D is a non-empty bounded open set with smooth boundary S, then S is a set of uniqueness. If S is a piece of a surface which does not bound a domain, then uniqueness can sometimes be established in classes of functions with further restrictions on the support. One such result is [3, Thm. 19]: we state it only for $n = 3$ though it is true in all dimensions.

Theorem 2. *Suppose S is a smooth surface in $\mathbf{R}^3$ and f a compactly supported continuous function on $\mathbf{R}^3$ such that $(\mathcal{M}\, f)(\mathbf{r}, \sigma) = 0$ for all $(\mathbf{r}, \sigma) \in S \times \mathbf{R}$. If there is a point $\mathbf{r}_0 \in S$ so that the support of f lies on one side of the tangent plane to S at $\mathbf{r}_0$ and touches the tangent plane at most at $\mathbf{r}_0$, then $f \equiv 0$.*

An easy consequence of Theorem 2 is that a function, supported in a convex domain D, is uniquely determined by its spherical means values over spheres centered on any (non-empty) relatively open subset of the boundary of D. In the three dimensional case, if the data is available only on sets S which intersect the support of f then additional restrictions on f may be necessary to uniquely determine f from $(\mathcal{M} f)(\mathbf{r}, \sigma)$ with $(\mathbf{r}, \sigma) \in S \times \mathbf{R}$. Some results may be derived using the following result, which is a consequence of Theorem 1.1 in [6]; also see Theorem 1 in [7]. In [6] and [7], the result is stated for solutions of the wave equation but because of the relationship between mean values and solutions of the wave equation given in (6), (7), there are corresponding results for $\mathcal{M}$. We state this for $n = 3$ but it holds in all dimensions.

Theorem 3. *Suppose f is a continuous function on $\mathbf{R}^3$ and $\mathbf{r}_0 \in \mathbf{R}^3$ so that $\left(\partial_{\mathbf{r}}^{\alpha}(\mathcal{M} f)\right)(\mathbf{r}_0, \sigma) = 0$ for all $\sigma \in [0, L)$ and all multi-indices α then $f \equiv 0$ on $|\mathbf{r} - \mathbf{r}_0| < L$.*

As a sample application, we show how this implies the following well known result in [8]. Suppose f is a continuous function on $\mathbf{R}^3$ and is even in the r_1 variable. If $(\mathcal{M} f)(\mathbf{r}, \sigma) = 0$ for all $\sigma \in [0, L)$ and all $\mathbf{r}$ which lie in the intersection of the plane $r_1 = 0$ and some ball $|\mathbf{r}| < \epsilon$ then f is zero on the ball $|\mathbf{r}| < L$. This is so because the hypothesis implies that all r_2, r_3 derivatives of $(\mathcal{M} f)(\mathbf{r}, \sigma)$ are zero on $\{0\} \times [0, L)$; further the first order r_1 derivative of $(\mathcal{M} f)(\mathbf{r}, \sigma)$ is also zero on $0 \times [0, L]$ because f is even in r_1; finally all higher order $\mathbf{r}$ derivatives are zero because, from (6) $\sigma(\mathcal{M} f)(\mathbf{r}, \sigma)$ satisfies (3) (with σ playing the role of t and $\nu_s = 1$), so higher order r_1 derivatives may be expressed in terms of the r_2, r_3 and σ derivatives.

Because of the relationships (9), (10), the uniqueness for $\mathcal{M}$ follows from uniqueness results for the following problem for solutions of (3), (4) for $n = 2$, $n = 3$. Suppose D is a bounded region in $\mathbf{R}^n$, S its boundary, and g is supported in D; is g uniquely determined from $u|_{S \times [0,T]}$ if T is large enough? The case when ν_s is a constant is already covered by Theorems 1 and 2, but there is also interest in this uniqueness question when ν_s is not constant, and the following may be proved.

Theorem 4. *Suppose D is a bounded region in $\mathbf{R}^n$ with a smooth boundary S and ν_s a smooth, positive function on $\mathbf{R}^n$. Let g be a smooth function supported in D and u the solution of the IVP (3), (4). If $u = 0$ on $S \times [0, T]$ and $T >$ diameter(D) then $g = 0$.*

Here the diameter of D refers to the maximum distance between any two points in D where the distance between points is measured using the length element which is ν_s^{-1} times the Euclidean length element.

We give the short proof next. From the hypothesis, u is the solution of the well posed exterior IBVP problem

$$u_{tt}(\mathbf{r}, t) - \nu_s^2 \nabla^2 u(r, t) = 0 \qquad (\mathbf{r}, t) \in (\mathbf{R}^n \setminus D) \times [0, T] \tag{21}$$

$$u(\mathbf{r}, t) = 0 \qquad (\mathbf{r}, t) \in S \times [0, T] \tag{22}$$

$$u(\mathbf{r}, 0) = 0, \quad u_t(\mathbf{r}, 0) = 0 \qquad \mathbf{r} \in \mathbf{R}^n \setminus D. \tag{23}$$

Hence $u = 0$ on $(\mathbf{R}^n \setminus D) \times [0, T]$, and since u is odd in t, we conclude that $u = 0$ on $(\mathbf{R}^n \setminus D) \times [-T, T]$. Choose a $p \in \mathbf{R}^n \setminus \overline{D}$ so that the distance of p from any point in D is less than T. Now u is zero in a neighborhood of the line segment $\{p\} \times [-T, T]$; hence by the unique continuation result Theorem 3 in [9] and a standard argument (see [10]) we have $u(\mathbf{r}, t) = 0$ when $\text{dist}(\mathbf{r}, p) + |t| < T$. In particular $u_t(\mathbf{r}, 0) = 0$ for all $\mathbf{r} \in D$.

3 General inversion of $\mathcal{M}$

Suppose D is a region in $\mathbf{R}^n$ for $n = 2, 3$ and let S be its boundary. In this section we discuss the recovery of a smooth function f, supported in D, from a knowledge of $\mathcal{M} f$ on $S \times [0, L]$ for some $L > 0$. Define the diameter of D to be the maximum distance between any two points in D - we denote it by $\text{diam}(D)$. We note that if f is supported in D then $(\mathcal{M} f)(\mathbf{r}, \sigma) = 0$ for all $\mathbf{r} \in S$ if $|\sigma| \geq \text{diam}(D)$. So if $L \geq \text{diameter}(D)$ then we know $(\mathcal{M} f)(\mathbf{r}, \sigma)$ for all $\mathbf{r} \in S$ and all $\sigma \in \mathbf{R}$. Also, if $L < \text{diam}(D)/2$ then for certain D (for example, a ball) there will be an open subset U of D which is disjoint from all spheres of radius less than L and centered on the boundary. For such D and non-zero f supported in U, $(\mathcal{M} f)(\mathbf{r}, \sigma) = 0$ for all $(\mathbf{r}, \sigma) \in S \times [0, L]$, so that there is no hope of recovering f completely from a knowledge of $\mathcal{M} f$ on $S \times [0, L]$.

Define $F(\mathbf{r}, \sigma) = (\mathcal{M} f)(\mathbf{r}, \sigma)$ for all $(\mathbf{r}, \sigma) \in \mathbf{R}^n \times \mathbf{R}$; note that F is a smooth function. Then, as shown on page 699 of [8] we have

$$F_{\sigma\sigma} + \frac{n-1}{\sigma} F_\sigma - \Delta_{\mathbf{r}} F = 0 \qquad \text{on} \;\; \mathbf{R}^n \times \mathbf{R} \tag{24}$$

$$F(\mathbf{r}, \sigma = 0) = f(\mathbf{r}), \quad F_\sigma(\mathbf{r}, \sigma = 0) = 0, \qquad \mathbf{r} \in \mathbf{R}^n. \tag{25}$$

First consider the case when $L \geq \text{diam}(D)$. Then $F(\mathbf{r}, L) = 0,\; F_\sigma(\mathbf{r}, L) = 0$ for all $\mathbf{r} \in \overline{D}$ and hence F is the solution of the backward initial boundary value problem (IBVP)

$$F_{\sigma\sigma} + \frac{n-1}{\sigma} F_\sigma - \Delta_{\mathbf{r}} F = 0 \qquad \text{on} \;\; \overline{D} \times (0, L];$$

$$F(\mathbf{r}, L) = 0, \quad F_\sigma(\mathbf{r}, L) = 0 \qquad \text{for all } \mathbf{r} \in \overline{D};$$

$$F(\mathbf{r}, \sigma) = (\mathcal{M} f)(\mathbf{r}, \sigma) \qquad \text{for all } (\mathbf{r}, \sigma) \in S \times (0, L].$$

This suggests the following inversion scheme. Given a smooth function $a(\mathbf{r}, \sigma)$ on $S \times (0, L)$, let $G(\mathbf{r}, \sigma)$ on $\overline{D} \times (0, L]$ be the solution of the backward IBVP

$$G_{\sigma\sigma} + \frac{n-1}{\sigma} G_\sigma - \Delta_{\mathbf{r}} G = 0 \qquad \text{on} \;\; \overline{D} \times (0, L]; \tag{26}$$

$$G(\mathbf{r}, L) = 0, \quad G_\sigma(\mathbf{r}, L) = 0 \qquad \text{for all } \mathbf{r} \in \overline{D}; \tag{27}$$

$$G(\mathbf{r}, \sigma) = a(\mathbf{r}, \sigma) \qquad \text{for all } (\mathbf{r}, \sigma) \in S \times (0, L]. \tag{28}$$

This is a well posed problem and one may also solve this numerically. If $a = \mathcal{M} f$ then $G = F$ and hence, knowing $\mathcal{M} f$ on $S \times (0, L)$ we may recover f from $\mathcal{M} f$ by first solving the backward IBVP (26) - (28) and then $f(\mathbf{r}) = \lim_{\sigma \to 0^+} G(\mathbf{r}, \sigma)$ for all $\mathbf{r} \in D$.

Implementations of the above inversion scheme may encounter difficulties, because solving (26) - (28) near $\sigma = 0$ may pose problems because $\sigma = 0$ is a singularity for the coefficient of the operator in (26). As noted in the final remarks in [11], if a actually lies in the range of $\mathcal{M}$ then the singular coefficient is no problem, but the computation will be unstable with respect to errors outside the range, including errors generated by a numerical implementation. In the odd dimensional case, $n = 3$, there is a better alternative based on the relation (9). If

$$u(\mathbf{r}, \sigma) := \sigma \, (\mathcal{M} f)(\mathbf{r}, \sigma), \qquad (\mathbf{r}, \sigma) \in \mathbf{R}^n \times \mathbf{R} \tag{29}$$

then $u(\mathbf{r},\sigma)$ is an odd (in σ) solution of the initial value problem

$$u_{\sigma\sigma} - \Delta_{\mathbf{r}} u = 0 \qquad (\mathbf{r},\sigma) \in \mathbf{R}^3 \times \mathbf{R} \tag{30}$$

$$u(\mathbf{r},0) = 0, \quad u_\sigma(\mathbf{r},0) = f(\mathbf{r}), \qquad \mathbf{r} \in R^3. \tag{31}$$

Further, $u(\mathbf{r},\sigma) = 0$, $u_\sigma(\mathbf{r},\sigma) = 0$ for $\sigma \geq L$, for all $\mathbf{r} \in \overline{D}$ (note $L \geq \operatorname{diam}(D)$). This suggests another inversion scheme for $n = 3$. Given a smooth function $b(\mathbf{r},\sigma)$ on $S \times [0,L]$, let $v(\mathbf{r},\sigma)$ be the solution of the backward IBVP

$$v_{\sigma\sigma} - \Delta_{\mathbf{r}} v = 0, \qquad (\mathbf{r},\sigma) \in \overline{D} \times [0,L], \tag{32}$$

$$v(\mathbf{r},L) = 0, \quad v_\sigma(\mathbf{r},L) = 0, \qquad \mathbf{r} \in \overline{D}, \tag{33}$$

$$v(\mathbf{r},\sigma) = b(\mathbf{r},\sigma) \qquad \text{for all } (\mathbf{r},\sigma) \in S \times [0,L]. \tag{34}$$

This is a well posed problem, see e.g. [12], and may be solved numerically. Further, if $b(\mathbf{r},\sigma) = \sigma\,(\mathcal{M}\,f)(\mathbf{r},\sigma)$ on $S \times [0,L]$ for some smooth function f on $\mathbf{R}^3$ with support in D then $v(\mathbf{r},\sigma) = u(\mathbf{r},\sigma)$ on $\overline{D} \times [0,L]$ and hence $f(\mathbf{r}) = v_\sigma(\mathbf{r},0)$ for all $\mathbf{r} \in \overline{D}$ (we also have $v(\mathbf{r},0) = 0$ on $\overline{D}$ but that is not relevant). It is important to note that the above scheme is stable in the sense that if the data is corrupt but the corruption is small, then because the IBVP being solved in the inversion scheme is well posed, the corruption in the function recovered will also be small, even if the corrupt data is out of the range of $\mathcal{M}$.

Various forms of such backward propagation methods have been studied (often under the name *time-reversal*) both to derive reconstruction formulas and for numerical reconstruction methods. We mention [13, 14, 15, 16].

We now consider the case when $\operatorname{diam}(D)/2 \leq L < \operatorname{diam}(D)$. The above inversion scheme is not available because we have less data than it requires. However, for $n = 3$, another inversion scheme is available, based again on the fact that u defined by (29) is a solution of the IVP (30), (31). Given the function $b(\mathbf{r},\sigma) = \sigma(\mathcal{M}\,f)(\mathbf{r},\sigma)$ for $(\mathbf{r},\sigma) \in S \times [0,L]$, taking the odd (in σ) extension of b, we observe that $b = u$ on $S \times [-L,L]$. Further, since f is supported in D we have $u(\mathbf{r},0) = 0$ and $u_\sigma(\mathbf{r},0) = 0$ for $\mathbf{r} \in (\mathbf{R}^3 \setminus D)$. So let $w(\mathbf{r},\sigma)$ be the solution of the exterior IBVP

$$w_{\sigma\sigma} - \Delta_{\mathbf{r}} u = 0, \qquad (\mathbf{r},\sigma) \in (\mathbf{R}^3 \setminus D) \times [-L,L], \tag{35}$$

$$w(\mathbf{r},0) = 0, \quad w_\sigma(\mathbf{r},0) = 0, \qquad \mathbf{r} \in \mathbf{R}^3 \setminus D, \tag{36}$$

$$w(\mathbf{r},\sigma) = b(\mathbf{r},\sigma) \qquad \text{for all } (\mathbf{r},\sigma) \in S \times [-L,L] \tag{37}$$

This is a well posed problem and may be solved numerically; in particular we obtain the normal derivative $\partial_\nu w$ on $S \times [-L,L]$. But $w = u$ on $(\mathbf{R}^3 \setminus D) \times [-L,L]$ hence we have recovered $\partial_\nu u$ on $S \times [-L,L]$; note we were given u on $S \times [-L,L]$ to begin with. The Kirchoff formula (see [17]) for solutions of the wave equation in three space dimensions expresses $u(\mathbf{r},\sigma)$ in terms of the values of u and $\partial_\nu u$ on the intersection of the cylinder $S \times \mathbf{R}$ with the downward light cone with vertex $(\mathbf{r},\sigma)$, see Figure 1 for an illustration.

From Kirchoff's formula we have

$$u(\mathbf{r},\sigma) = \frac{1}{4\pi}\int_S \frac{[\partial_\nu u](\mathbf{r}',\sigma)}{|\mathbf{r}'-\mathbf{r}|} + \left(\frac{[u](\mathbf{r}',\sigma)}{|\mathbf{r}'-\mathbf{r}|^2} + \frac{[u_\sigma](\mathbf{r}',\sigma)}{|\mathbf{r}'-\mathbf{r}|}\right)\partial_\nu(|\mathbf{r}'-\mathbf{r}|)\,dS(r')$$

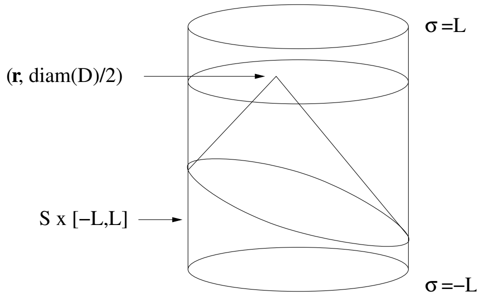

Figure 1: The downward light cone and its intersection with the boundary of the cylinder

where $[v](\mathbf{r}', \sigma) := v(\mathbf{r}', \sigma - |\mathbf{r}' - \mathbf{r}|)$. Hence, we can reconstruct $u(\mathbf{r}, \sigma)$ and $u_\sigma(\mathbf{r}, \sigma)$ for all $\mathbf{r} \in \overline{D}$ and $\sigma = \text{diam}(D)/2$ because $L \geq \text{diam}(D)/2$ and we know u and $\partial_\nu u$ on $S \times [-L, L]$. Now let $v(\mathbf{r}, \sigma)$ be the solution of the backward IBVP

$$
\begin{aligned}
v_{\sigma\sigma} - \Delta_{\mathbf{r}} v = 0, \qquad & (\mathbf{r}, \sigma) \in \overline{D} \times [0, \text{diam}(D)/2], \\
v(\mathbf{r}, \text{diam}(D)/2) = \text{known}, \quad & v_\sigma(\mathbf{r}, \text{diam}(D)/2) = \text{known}, \qquad \mathbf{r} \in \overline{D}, \\
v(\mathbf{r}, \sigma) = b(\mathbf{r}, \sigma) \qquad & \text{for all } (\mathbf{r}, \sigma) \in S \times [0, \text{diam}(D)/2].
\end{aligned}
$$

This well posed problem may be solved numerically and we can recover $v_t(\mathbf{r}, 0) = u_t(\mathbf{r}, 0) = f(\mathbf{r})$ for all $\mathbf{r} \in \overline{D}$.

These inversion schemes also help characterize the range of $\mathcal{M}$ in the special case when D is a ball and $L \geq \text{diam}(D)$ - see Theorems 10 and 11 in [11]. By this we mean the problem to specify necessary and sufficient conditions that a function $a(.,.)$ on $S \times (0, L]$ be equal to $\mathcal{M} f$, for some function f on $\mathbf{R}^n$ with support in D. In [11] there are range characterizations valid in all dimensions, with one of the equivalent conditions in terms of the solution of (26)-(28). We present this in the theorem below, when $n = 2$. For odd dimensions, [18] gave characterizations of the range of the solution operator of (3) with one non-zero initial datum supported in D. One of these characterizations is in terms of the solution of (32)-(34). Since the measured data of thermoacoustic tomography is the boundary trace of the solution of the wave equation, and because the inversion scheme based on (32)-(34) is perhaps better than the one based on (26)-(28), we present this version in the theorem below, for $n = 3$.

Theorem 5 (Range characterization)**.** *Suppose D is a ball in $\mathbf{R}^n$ of diameter L, $n = 2, 3$, S is the boundary of D, and $a(\mathbf{r}, \sigma)$ is a smooth function on $S \times (0, L]$ for which $\partial_\sigma^k a(., L) = 0$ for $k = 0, 1, 2, \cdots$.*

- *When $n = 3$, define $b(\mathbf{r}, \sigma) := \sigma a(\mathbf{r}, \sigma)$ and let v be the solution of the IBVP (32) - (34). Then $a = \mathcal{M} f$ on $S \times (0, L]$ for some smooth function f on $\mathbf{R}^3$ with support in D if and only if $v(\mathbf{r}, 0) = 0$ for all $\mathbf{r} \in \overline{D}$.*
- *When $n = 2$, let $G(\mathbf{r}, \sigma)$ be the solution of the backward IBVP (26) - (28). Then $a = \mathcal{M} f$ on $S \times (0, L]$ for some smooth function f on $\mathbf{R}^2$ with support in D if and only if $\lim_{\sigma \to 0^+} G_\sigma(\mathbf{r}, \sigma) = 0$ for all $\mathbf{r} \in D$ and*

$$
\int_0^L \int_0^{2\pi} \sigma^{2k+1} \, \phi_m(\theta) \, a\big(\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta, \sigma\big) \, d\theta \, d\sigma = 0
$$

for all $m = 0, 1, 2\cdots$ and all non-negative integers k strictly less than $m/2$; here $\phi_m(\theta)$ stands for $\cos(m\theta)$ and $\sin(m\theta)$.

While the above numerical schemes should be effective, we describe an approximate inversion scheme given by Beylkin in [19] (see also [20]) which was motivated by applications to geophysics. Suppose $\overline{D}$ is a strictly convex region in $\mathbf{R}^n$ with a smooth boundary S. Restricting the centers of the spheres to S, $\mathcal{M}$ is a Fourier Integral Operator mapping smooth functions on $\mathbf{R}^n$ with support in D to smooth functions on $S \times (0, \infty)$. Further, if $\mathcal{M}^*$ is the L^2 adjoint of $\mathcal{M}$ then $\mathcal{M}^* \mathcal{M}$ is an elliptic pseudodifferential operator and we can construct a parametrix $\mathcal{P}$ for $\mathcal{M}^* \mathcal{M}$ so that $\mathcal{P}\,\mathcal{M}^*\,\mathcal{M} = I + \mathcal{K}$ where I is the identity and K is a compact operator. Thus $\mathcal{P}\,\mathcal{M}^*$ is an approximate inverse for $\mathcal{M}$. It is also possible to incorporate a (smooth) cut-off factor when measurements can not be made over the full boundary S, and this was already present in [19]. Beylkin modified this approach to construct a different approximate inverse for $\mathcal{M}$.

For any smooth function f on $\mathbf{R}^n$ with support in D, and any $(\mathbf{r}', \sigma) \in S \times (0, \infty)$

$$(\mathcal{M}\, f)(\mathbf{r}', \sigma) = \frac{1}{\sigma^{n-1}|S^{n-1}|} \int_{|\mathbf{r}-\mathbf{r}'|=\sigma} f(\mathbf{r})\, dS(r) = \frac{1}{\sigma^{n-1}|S^{n-1}|} \int_{\mathbf{R}^n} f(\mathbf{r})\, \delta(\sigma - |\mathbf{r} - \mathbf{r}'|)\, d^n r.$$

Comparing this with (3.5) in [19], our problem corresponds to the problem studied in [19] when $\hat{\phi}(x, \xi) = |x - \xi|/2$, $\tilde{\phi}(x, \eta) = |x - \eta|/2$, $\xi = \eta$, his $a(x, \xi, \eta) = (|S^{n-1}|\,|x - \xi|^{n-1})^{-1}$, his $\mathcal{R} = \mathcal{M}$, and $u(t, \xi, \eta) = \partial_t^{n-1}(\mathcal{M}\, f)(\xi, t)$.

Since $\overline{D}$ is strictly convex, to every pair $(\mathbf{r}, \theta)$ with $\mathbf{r} \in D$, $\theta \in \mathbf{R}^n$, $|\theta| = 1$, we can associate a positive number $\sigma(\theta, \mathbf{r})$ representing the distance from $\mathbf{r}$ to S in the direction θ; so the ray from $\mathbf{r}$, in the direction θ hits S at the point $\mathbf{r} + \sigma(\theta, \mathbf{r})\theta$. For smooth functions $a(\mathbf{r}', \sigma)$ on $S \times [0, \infty)$ which vanish to infinite order at $\sigma = 0$ and are zero for large σ, define the approximate inverse $\mathcal{B}$ as follows

$$\text{(n=3)} \qquad (\mathcal{B}\, a)(\mathbf{r}) = \frac{-1}{2^3 \cdot \pi} \int_{|\theta|=1} \sigma^2\, a_{\sigma\sigma}(\mathbf{r}', \sigma)\, d\theta, \qquad \mathbf{r} \in D. \tag{38}$$

$$\text{(n=2)} \qquad (\mathcal{B}\, a)(\mathbf{r}) = \frac{-1}{2^2 \cdot \pi} \int_{|\theta|=1} \sigma \int_{-\infty}^{\infty} \frac{a_\sigma(\mathbf{r}', \sigma')}{\sigma - \sigma'}\, d\sigma'\, d\theta, \qquad \mathbf{r} \in D. \tag{39}$$

where $\sigma = \sigma(\theta, \mathbf{r})$ and $\mathbf{r}' = \mathbf{r} + \sigma(\theta, \mathbf{r})\theta$, and $a(\mathbf{r}', \sigma)$ is assumed to be zero for $\sigma < 0$.

Theorem 6 (Beylkin)**.** *Suppose $\overline{D}$ is a strictly convex region in $\mathbf{R}^n$, $n = 2, 3$, and $\sigma(\theta, \mathbf{r})$ and $\mathcal{B}$ are defined as above. Then, for any smooth function f on $\mathbf{R}^n$ with support in D we have*

$$(\mathcal{B}\,\mathcal{M}\, f)(\mathbf{r}) = f(\mathbf{r}) + (\mathcal{K} f)(\mathbf{r}), \qquad \mathbf{r} \in D$$

for some compact operator $\mathcal{K}$.

The operator $\mathcal{K}$ is also a smoothing operator so the singularities of f and the singularities of $\mathcal{B}\,\mathcal{M}\, f$ are the same and hence the singularities of f may be recovered from $\mathcal{M}\, f$ by the use of the operator $\mathcal{B}$. The definition of $\mathcal{B}$ may be given in terms of an integral on S instead of an integral on the unit sphere, using the change of variables $\theta \mapsto \mathbf{r}' = \mathbf{r} + \sigma(\theta, \mathbf{r})\theta$. If S is the level surface $\phi(\mathbf{r}') = c$, with $\nabla\phi$ pointing out of D, then for any function $h(\mathbf{r}')$ on S we have

$$\int_S h(\mathbf{r}')\, \frac{(\mathbf{r}' - \mathbf{r}) \cdot \nabla\phi(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|^n\, |\nabla\phi(\mathbf{r}')|}\, dS(r') = \int_{|\theta|=1} h(\mathbf{r} + \sigma(\theta, \mathbf{r})\theta)\, d\theta.$$

The inner integral in the formula for $\mathcal{B}$, for $n = 2$, is called the Hilbert transform of a and more about it may be found in [21].

When ν_s is constant, it is essentially equivalent to recover g from $\mathcal{M}\, g$ or from $u|_{S\times[0,L]}$, where u is the solution of (3), (4). When ν_s is not constant, there are as yet no exact inversion schemes for recovery of the initial data from $u|_{S\times[0,L]}$ similar to the schemes discussed above. This is because for a general sound speed there is no simple analytic expression for the solution of the initial value problem, and because $u(\mathbf{r}, t)$ need not be zero for large t when $\mathbf{r}$ is restricted to D. It is also clear that some hypothesis is necessary to guarantee that the effect of an initial disturbance in any part of the domain eventually propagates to the boundary. A technical condition which suffices for this is that the domain is *non-trapping.* The recent work of Agranovsky and Kuchment in [16] does give a series type inversion provided that u is observed on the boundary for all time, the domain is non-trapping, and all Dirichlet eigenvalues and eigenfunctions of $-\nu_s^2\nabla^2$ are known. Another approach to recovery of the initial data which only requires observation for finite time is given in [22]. It is based on the method of quasireversibility. However, this approach requires knowing both the value of u on $S \times [0, L]$ and the value of the normal derivative of u on $S \times [0, L]$, when L is sufficiently large (as well as a requirement on the spatial variation of the wave speed.)

4 Explicit inversion formulas

Explicit (exact) inversion formulas from spherical means are known for some simple geometries. By this we mean that the given data consists of all spherical means over spheres whose centers lie on some simple geometric object, and that there may be a further hypothesis relating the support of the unknown function and the object. The first case, known to us, of such an explicit inversion formula was given in [23]. Their geometric object was a hyperplane in odd dimensional space, with the additional hypothesis that the function be supported on one side of the hyperplane (or even with respect to the hyperplane). For the hyperplane case, a formula expressing the reconstruction kernel in terms of an unevaluated integral was given by Norton and Linzer, [24], in their study of ultrasonic reflection tomography. This paper has been an inspiration to many subsequent works, including [19], and papers by Wang and his collaborators, see e.g. [25]. Other works involving the the hyperplane geometry can be found in [26], [27], [28]. Another geometry where an inversion formula is known is that of an infinite cylinder in three dimensional space, under the assumption that the function to be recovered is supported inside the cylinder. The original formula in the frequency domain is in [24]; a substantially better result can be found in [14]. We do not dwell further on either of these geometries, but take up the case of spherical (resp. circular) geometry in three (resp. two) dimensions. Here it is assumed that the unknown function is supported inside a ball (resp. disk) and that the spherical (resp. circular) means are known for all spheres (resp. circles) centered on the boundary of the ball. Explicit inversion formulas based on infinite series expansions were found first: [29] in two dimensions; [24] in three dimensions. The series expansion of the latter paper was reworked in [30]. Very recently there have been new developments in the series approach. Kuyansky, see [31], [32], shows that for the inner product of the unknown function with the Dirichlet eigenfunctions can be computed from boundary data in domains where the eigenfunctions of the Dirichlet Laplacian are explicitly known. Furthermore, in some cases when the FFT can be brought into play, for example, the rectangle, this gives rapidly computable series expansions. The second new development, [16] already mentioned above, is a series expansion in the case of variable wave speed, subject to a decay hypothesis on the energy of the waves inside the region (implied by a geometric condition called *non-trapping*) and assuming known all the eigenfunctions and eigenvalues of the operator $-\nu_s^2\nabla^2$ subject to Dirichlet boundary conditions. Further details on both are in the cited papers and the chapter of Agranovsky, Kuchment and Kunyansky in this volume. We now concentrate on formulas of filtered back-projection type.

4.1 The ball in 3D

We suppose now that the unknown function is supported in a ball B_{R_0} of radius R_0, which we assume to be centered at the origin, and that the spherical mean transform is known for all spheres centered on the boundary sphere S_{R_0} of the ball. Several exact inversion formulas involving filtration and back-projection are known.

First let us comment on the hypothesis that measurements are made on the full sphere. This hypothesis may be tenable in a laboratory setting, but would be very unlikely to hold in a clinical setting. The development of effective reconstruction techniques in the case of partial measurements is an unresolved problem. Some discussion can be found in the chapter by Y. Xu and L. H. Wang and in the chapter by S. Patch.

The first inversion formulas were presented in [33]: slightly different proofs can be found in [34]. Another formula, see (47) below, was given in [14], with an alternate derivation provided in [31]. We will present another proof of (47) later in this section.

Theorem 7. *Let f be a smooth function supported in the closure of B_{R_0}. Then for $\mathbf{r} \in B_{R_0}$, the following inversion formulas hold:*

$$f(\mathbf{r}) = -\frac{1}{2\pi R_0}\int_{S_{R_0}} \frac{\partial_\sigma^2(\sigma^2(\mathcal{M}\,f)(\mathbf{r}_0,\sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\,dS(r_0), \tag{40}$$

$$f(\mathbf{r}) = -\frac{1}{2\pi R_0}\int_{S_{R_0}} \frac{(\partial_\sigma\sigma\partial_\sigma\sigma(\mathcal{M}\,f)(\mathbf{r}_0,\sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\,dS(r_0), \tag{41}$$

$$f(\mathbf{r}) = -\frac{1}{2\pi R_0}\nabla^2\int_{S_{R_0}} |\mathbf{r}-\mathbf{r}_0|(\mathcal{M}\,f)(\mathbf{r}_0,|\mathbf{r}-\mathbf{r}_0|)\,dS(r_0). \tag{42}$$

In the language of computed tomography, in the first two formulas, filtration is performed first and then back-projection, while the third depends on filtration of the back-projection.

Commuting one derivative with σ in (40) and then subtracting (41) yields a consistency condition: for every smooth f supported in the closure of B_{R_0}, then

$$0 = \int_{S_{R_0}} \frac{\partial_\sigma(\sigma\,\mathcal{M}\,f(\mathbf{r}_0,\sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\,dS(r_0), \tag{43}$$

on B_{R_0}. The condition (43), together with natural support and smoothness conditions, is also sufficient that a candidate function G on $S_{R_0} \times [0, 2R_0]$ be the spherical means of a smooth function supported in the closed ball: see the paragraph after Theorem 3 in [18].

The proof of (42) is based on the explicit computation of the integral on the right as an integral operator applied to f. Applying (20) with $q = |\mathbf{r}' - \mathbf{r}_0|^2 - |\mathbf{r} - \mathbf{r}_0|^2$ gives

$$4\pi|\mathbf{r} - \mathbf{r}_0|\, \mathcal{M}\, f(\mathbf{r}_0, |\mathbf{r} - \mathbf{r}_0|) = 2 \int_{\mathbf{R}^3} f(\mathbf{r}')\delta(|\mathbf{r}' - \mathbf{r}_0|^2 - |\mathbf{r} - \mathbf{r}_0|^2)\, d^3r'.$$

Inserting this in the integral in (42) and interchanging order,

$$\int_{S_{R_0}} |\mathbf{r} - \mathbf{r}_0|(\mathcal{M}\, f)(\mathbf{r}_0, |\mathbf{r} - \mathbf{r}_0|)\, dS(r_0) = \frac{1}{2\pi} \int_{\mathbf{R}^3} f(\mathbf{r}') \int_{S_{R_0}} \delta(|\mathbf{r}' - \mathbf{r}_0|^2 - |\mathbf{r} - \mathbf{r}_0|^2)\, dS(r_0)\, d^3r'.$$

The evaluation of the inner integral uses some algebraic manipulations, properties of the delta distribution, and an appropriate set of polar coordinates on the sphere, see [33] for the details. The result is that

$$\int_{S_{R_0}} \delta(|\mathbf{r}' - \mathbf{r}_0|^2 - |\mathbf{r} - \mathbf{r}_0|^2)\, dS(r_0) = \frac{\pi R_0}{|\mathbf{r}' - \mathbf{r}|}. \tag{44}$$

Thus the right hand side of (42) is

$$-\frac{1}{4\pi}\nabla^2 \int_{\mathbf{R}^3} f(\mathbf{r}')\frac{1}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

which is equal to $f(\mathbf{r})$, since $\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$ is the fundamental solution of the Laplacian on $\mathbf{R}^3$.

The formula (40 follows from (42) via an inner product relation, as explained in ([33]). Namely, if u_i, $i = 1, 2$ are solutions of (3), (4), for $\nu_s = 1$ and $g = f_i$, $i = 1, 2$ respectively, where the f_i are supported in B_{R_0}, then (42) implies

$$\int_{B_{R_0}} f_1(\mathbf{r}) f_2(\mathbf{r})\, d^3r = -\frac{2}{R_0} \int_0^\infty \int_{S_{R_0}} t\, u_1(\mathbf{r}_0, t) u_{2tt}(\mathbf{r}_0, t)\, dS(r_0)\, dt. \tag{45}$$

Interchanging the roles of u_1 and u_2 in (45), adding the two equations, integrating by parts with respect to t, and using that $u_1 u_2 = 0$ when $t = 0$ yields a companion identity in a more symmetric form:

$$\int_{B_{R_0}} f_1(\mathbf{r}) f_2(\mathbf{r})\, d^3r = \frac{2}{R_0} \int_0^\infty \int_{S_{R_0}} t\, u_{1t}(\mathbf{r}_0, t) u_{2t}(\mathbf{r}_0, t)\, dS(r_0)\, dt. \tag{46}$$

This in turn is equivalent to the second inversion formula (41).

Another inversion formula for the spherical geometry is among those found by Xu and Wang under the name *universal back-projection algorithm.* (See their chapter in this volume.) In the spherical case, $n = 3$, it may be written

Theorem 8 (Xu and Wang)**.** *Let f be a smooth function supported in the ball of radius R_0. Then for $\mathbf{r} \in B_{R_0}$*

$$f(\mathbf{r}) = \frac{1}{2\pi R_0} \nabla \cdot \int_{S_{R_0}} \mathbf{r}_0 \frac{\partial_\sigma(\sigma(\mathcal{M}\, f)(\mathbf{r}_0, \sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|}\, dS(r_0). \tag{47}$$

Remark. The formula of Xu and Wang applies in several geometries, but does not appear to have as direct consequence either of the formulas (41) or (40) which give filtered back-projection reconstruction without a weight depending on the reconstruction point. Compare formula (20) of [14].

Their proof is in the frequency domain and uses the outgoing, (18), and incoming fundamental solutions. We give a derivation based on Theorem 7 and (43). By (42),

$$
\begin{aligned}
-2\pi R_0 f(\mathbf{r}) &= \nabla^2 \int_{S_{R_0}} |\mathbf{r}-\mathbf{r}_0| (\mathcal{M} f)(\mathbf{r}_0, |\mathbf{r}-\mathbf{r}_0|)\, dS(r_0) \\
&= \nabla \cdot \int_{S_{R_0}} \nabla_{\mathbf{r}} (|\mathbf{r}-\mathbf{r}_0|\, \mathcal{M} f(\mathbf{r}_0, |\mathbf{r}-\mathbf{r}_0|))\, dS(r_0) \\
&= \nabla \cdot \int_{S_{R_0}} \partial_\sigma (\sigma (\mathcal{M} f)(\mathbf{r}_0, \sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}\, dS(r_0) \\
&= \nabla \cdot \left(\mathbf{r} \int_{S_{R_0}} \frac{\partial_\sigma (\sigma (\mathcal{M} f)(\mathbf{r}_0, \sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\, dS(r_0) \right) \\
&\qquad - \nabla \cdot \left(\int_{S_{R_0}} \mathbf{r}_0 \frac{\partial_\sigma (\sigma (\mathcal{M} f)(\mathbf{r}_0, \sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\, dS(r_0) \right) \\
&= -\nabla \cdot \int_{S_{R_0}} \mathbf{r}_0 \frac{\partial_\sigma (\sigma (\mathcal{M} f)(\mathbf{r}_0, \sigma))|_{\sigma=|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\, dS(r_0),
\end{aligned}
$$

since the first term on the right in the penultimate line is zero by (43).

4.2 The disk in 2D

Recently, there have been two discoveries of filtered back-projection type reconstruction formulas for the inversion of the circular mean transform with centers on a circle. As remarked in the introduction, these have relevance for photoacoustic tomography with integrating line detectors. The first has appeared in [31], and is a 2D analog of the result of Xu and Wang mentioned in the previous section. Indeed, Kunyansky has found a formula for each dimension greater than one. We will state his result here, but refer the reader to the chapter of Agranovsky, Kuchment,and Kunyansky for further details.

Theorem 9 (Kunyansky)**.** *Let f be smooth and supported in the closure of D_{R_0}, the disk of radius R_0 centered at the origin. Then for $\mathbf{r} \in D_{R_0}$*

$$
f(\mathbf{r}) = \frac{1}{8\pi R_0} \nabla_{\mathbf{r}} \cdot \int_{S_{R_0}} \mathbf{r}_0 V(\mathbf{r}_0, |\mathbf{r}-\mathbf{r}_0|)\, ds(r_0), \tag{48}
$$

where

$$
V(\mathbf{r}_0, \sigma) = \int_0^\infty \int_0^{2R_0} (N_0(\lambda\sigma) J_0(\lambda s) - J_0(\lambda\sigma) N_0(\lambda s))\, s\, \mathcal{M} f(\mathbf{r}_0, s)\, ds\, \lambda\, d\lambda.
$$

In this expression, J_0 is the Bessel function of order zero, and N_0 is the Neumann function of order zero.

It should be remarked that formal interchange of order of integration leads to a divergent inner integral. However, in the chapter of Agranovsky, Kuchment, and Kunyansky it is shown how this may be regularized. The result is the same inversion formula as given below in (56).

The second discovery was reported in [35] and gives inversion formulas from spherical means and from the trace of the solution of the wave equation in all even dimensions. The inversion formulas are analogous to the odd dimensional case reported in Theorem 7 above. We give statements only for dimension two.

Theorem 10. *Let f be a smooth function supported in the closure of D_{R_0}, the disk of radius R_0 centered at the origin. Then for $\mathbf{r} \in D_{R_0}$*

$$f(\mathbf{r}) = \frac{1}{2\pi R_0} \nabla^2 \int_{S_{R_0}} \int_0^{2R_0} \log|\sigma^2 - |\mathbf{r} - \mathbf{r}_0|^2|\, \sigma (\mathcal{M} f)(\mathbf{r}_0, \sigma)\, d\sigma\, ds(r_0), \tag{49}$$

$$f(\mathbf{r}) = \frac{1}{2\pi R_0} \int_{S_{R_0}} \int_0^{2R_0} \log|\sigma^2 - |\mathbf{r} - \mathbf{r}_0|^2|\, (\partial_\sigma \sigma \partial_\sigma (\mathcal{M} f))(\mathbf{r}_0, \sigma)\, d\sigma\, ds(r_0). \tag{50}$$

Another version, using the odd extension F_o of $F = \mathcal{M} f$ reads as follows.

Theorem 11. *Let f be a smooth function supported in the closure of D_{R_0}, the disk of radius R_0 centered at the origin, and let F_o be the extension of $\mathcal{M} f(\mathbf{r}_0, \sigma)$ to $\sigma < 0$ as an odd function of σ. (Note F_o is still smooth for $\mathbf{r}_0 \in S_{R_0}$ by the support hypothesis on f.) Then for $\mathbf{r} \in D_{R_0}$*

$$f(\mathbf{r}) = \frac{1}{2\pi R_0} \int_{S_{R_0}} \int_{-2R_0}^{2R_0} \frac{(\sigma \partial_\sigma F_o)(\mathbf{r}_0, \sigma)}{|\mathbf{r} - \mathbf{r}_0| - \sigma}\, d\sigma\, ds(r_0), \tag{51}$$

$$f(\mathbf{r}) = \frac{1}{2\pi R_0} \int_{S_{R_0}} |\mathbf{r} - \mathbf{r}_0| \int_{-2R_0}^{2R_0} \frac{(\partial_\sigma F_o)(\mathbf{r}_0, \sigma)}{|\mathbf{r} - \mathbf{r}_0| - \sigma}\, d\sigma\, ds(r_0). \tag{52}$$

where the inner integrals are to interpreted in principal value sense.

These have forms very similar to the standard filtered back-projection formulas for the Radon transform, see [36], for which there exists extensive numerical experience. Questions such as optimal implementation of the filtration step or sampling conditions with respect to angular and radial variables have not yet been addressed, but the analysis may benefit from the corresponding results for the Radon transform.

The proof of Theorem 10 is based on the following identity, which is similar to the identity (44).

Proposition. *Let D_{R_0} be the disk of radius R_0, and let S_{R_0} be the boundary circle. Then for $\mathbf{r}$, $\mathbf{r}' \in D_{R_0}$ with $\mathbf{r} \neq \mathbf{r}'$,*

$$\int_{S_{R_0}} \log\Big||\mathbf{r} - \mathbf{r}_0|^2 - |\mathbf{r}' - \mathbf{r}_0|^2\Big|\, ds(r_0) = 2\pi R_0 \log|\mathbf{r} - \mathbf{r}_0| + 2\pi R_0 \log R_0. \tag{53}$$

In view of (10), the expression of the solution of (3), (4) in terms of circular means is more complicated in two dimensions than in three. However, the inverse of the Abel operator appearing in (10) is well known, and shows that to determine the circular mean on S_{R_0} at radius σ only requires the solution of (3), (4) on S_{R_0} for $0 \le \tau \le \sigma$. Combining this with the inversion formula (49) gives the following.

Theorem 12. *If g is smooth and supported in the closure of B_{R_0}, and u is the solution of (3), (4), then for $\mathbf{r} \in B_{R_0}$,*

$$g(\mathbf{r}) = \frac{1}{R_0\pi^2}\nabla^2 \int_{S_{R_0}} \int_0^{2R_0} u_t(\mathbf{r}_0, t) K(t, |\mathbf{r} - \mathbf{r}_0|)\, dt\, ds(r_0), \tag{54}$$

where

$$K(t, s) = \int_t^{2R_0} \frac{r}{\sqrt{r^2 - t^2}} \log|r^2 - s^2|\, dr.$$

Finally, we note that in the two dimensional case there is a consistency condition similar to (43).

Proposition. *Let f be smooth and supported in the closure of D_{R_0}. Then for $\mathbf{r} \in D_{R_0}$,*

$$0 = \int_{S_{R_0}} \int_0^{2R_0} \frac{\sigma}{\sigma^2 - |\mathbf{r} - \mathbf{r}_0|^2} (\mathcal{M} f)(\mathbf{r}_0, \sigma)\, d\sigma\, ds(r_0). \tag{55}$$

The proof uses the identity $0 = \int_{S_{R_0}} \int_0^\infty u_t(\mathbf{r}_0, t) v(\mathbf{r}_0, t)\, dt\, ds(r_0)$ for solutions u, v of (3), (4) with initial values f and g respectively (established in the proof of Theorem 1.6 in [35]) and a modification of the proof of the $n = 2$ case of Theorem 1.5 of that paper.

Combining (55) with formula (49) gives another inversion formula of divergence type for circular means centered on a circle. As mentioned above, this result also appears in the chapter of Agranovsky, Kuchment, and Kunyansky.

Theorem 13. *Let f be smooth and supported in the closure of D_{R_0}. Then for $\mathbf{r} \in D_{R_0}$,*

$$f(\mathbf{r}) = \frac{1}{\pi R_0}\nabla \cdot \int_{S_{R_0}} \mathbf{r}_0 \int_0^{2R_0} \frac{\sigma}{\sigma^2 - |\mathbf{r} - \mathbf{r}_0|^2} (\mathcal{M} f)(\mathbf{r}_0, \sigma)\, d\sigma\, ds(r_0). \tag{56}$$

The proof is quite similar to the proof given for Theorem 8 above.

5 Summary

We have presented some of the recent mathematical work on uniqueness and recovery of the initial value g of (3), (4) from observations of the solution on the boundary of a domain containing the support of the initial data. In the case of constant wave speed, this can also be interpreted in terms of spherical means. When the domain is a ball, there are convenient formulas of filtered back-projection type in all dimensions. Such formulas are not yet established for other closed regions, though in some cases good series expansions can be found. Notable open problems remain in the situation where measurements can be made on only part of the boundary, and in the case when wave speed is variable.

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