An inverse problems for a hyperbolic PDE in one space dimension

Rakesh

June 13, 2009

1 Introduction

The following notes are based on the work in [Sy83], [Sy86], and [Ra01] and we recommend supplementing them by reading [Br00] and [SWG96]. There are other approaches to this problem (see the references in [Bu80]) and the Boundary Control Method introduced by Belishev - I may post his notes from a workshop in Italy in 2009 on my web page.

For a smooth function $\sigma(x)$ on $[0, \infty)$ consider the IBVP

$$\mathcal{L}U := U_{tt} - U_{xx} - 2\sigma U_x = 0, \qquad (x,t) \in [0,\infty) \times \mathbb{R} \tag{1}$$

$$U_x(0,t) = -\delta(t), \qquad t \in \mathbb{R} \tag{2}$$

$$U(x,t) = 0 \qquad t < 0. \tag{3}$$

Given $\sigma(.)$, this is a well posed problem. The inverse problem consists of recovering $\sigma(.)$ given the additional data $U_t(0,t)$. We will reformulate the problem in a manner so that we do not deal with $\delta(t)$ and so we can work in the standard function spaces.

Because of the speed of propagation one knows that $U(x,t) = 0$ for $t < x$, so we should be able to express $U(x,t)$ as

$$U(x,t) = u(x,t)H(t-x) \tag{4}$$

for a smooth function $u(x,t)$ (which may be defined for all (x,t) - even when $t < x$). Then

$$U_x = -u\delta(t-x) + u_x H(t-x), \quad U_{xx} = u\delta'(t-x) - 2u_x\delta(t-x) + u_{xx}H(t-x),$$
$$U_{tt} = u\delta'(t-x) + 2u_t\delta(t-x) + u_{tt}H(t-x)$$

hence

$$0 = \mathcal{L}U = 2(u_t + u_x + \sigma u)(x,t)\delta(t-x) + (\mathcal{L}u)H(t-x).$$

This forces $\mathcal{L}u = 0$ in the region $t \geq x \geq 0$ and the transport equation $(u_t + u_x + \sigma u)(x,x) = 0$. Further, the BC (2) imply

$$-\delta(t) = U_x(0,t) = -u(0,t)\delta(t) + u_x(0,t)H(t)$$

which forces $u(0,0) = 1$ and $u_x(0,t) = 0$ for $t \geq 0$. So using the transport equation, we have $u(x,x)$ is the solution of the ODE

$$\frac{d}{dx}(u(x,x)) + \sigma(x)u(x,x) = 0, \quad u(0,0) = 1$$

whose solution is $u(x,x) = e^{-\int_0^x \sigma(y)\,dy}$. So $U(x,t)$ is given by (4) where $u(x,t)$ is the solution of the characteristic boundary value problem

$$u_{tt} - u_{xx} - 2\sigma u_x = 0, \qquad 0 \le x \le t, \tag{5}$$

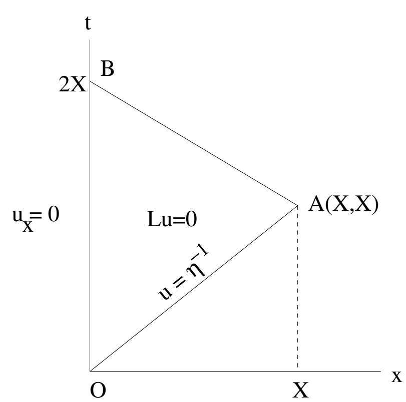
$$u_x(0,t) = 0, \qquad t \ge 0 \tag{6}$$

$$u(x,x) = \eta(x)^{-1}, \qquad x \ge 0 \tag{7}$$

where, for convenience, we have introduced the function $\eta(x) := e^{\int_0^x \sigma(y)\,dy}$, closely related to $\sigma(x)$. Further note that

$$U_t(0,t) = u(0,t)\delta(t) + u_t(0,t)H(t) = u(0,0)\delta(t) + u_t(0,t)H(t) = \delta(t) + u_t(0,t)H(t).$$

Throughout these notes we will assert that certain IBVP are well posed without giving the proofs. Since the operator we consider is a first order perturbation of the one dimensional wave equation, whatever IBVP is well posed for the one dimensional wave equation will most likely be well posed for a first order perturbation. The proof for the wave equation can be obtained by exact formulas and then the proof for the perturbation, for smooth coefficients, can be done using a Volterra equation approach where the Volterra equation is obtained by using the formulas for the exact formulas for the one dimensional wave equation. Then results can be proved for less regular coefficients by obtaining energy estimates and taking limits.



Now, suppose $X > 0$ and $\sigma(x) \in L^2[0,X]$, then the characteristic boundary value problem (5) - (7) may be shown to be a well posed problem in the region $0 \le x \le X$, $x \le t \le 2X - x$, with a unique solution $u(x,t)$ which is H^1 on this region and has H^1 traces on any line and u_x and u_t have L^2 traces on any non-characteristic line. Hence we may define the forward map

$$\mathcal{F} : L^2[0,X] \to L^2[0,2X] \tag{8}$$

$$\sigma(.) \mapsto u_t(0,.) \tag{9}$$

and the inverse problem under consideration is really the study of the map $\mathcal{F}$ and its inverse. In particular we wish to answer atleast the following questions.

1. Is $\mathcal{F}$ injective?
2. What is the range of $\mathcal{F}$?
3. If $\mathcal{F}$ is injective then how to construct the inverse of $\mathcal{F}$?
4. Is $\mathcal{F}^{-1}$ continuous i.e. will the reconstruction be stable?

Note that $\mathcal{F}$ is non-linear because if for $i = 1, 2$, u_i is the solution of (5)-(7) for $\sigma = \sigma_i$, then $u_1 + u_2$ is *not* the solution corresponding to $\sigma_1 + \sigma_2$.

We will answer all these questions. The second question is equivalent to asking which functions $r(.) \in L^2[0, 2X]$ can arise as the response $r(.) = u_t(0, .)$ of the medium corresponding to some $\sigma \in L^2[0, X]$. For any $r(.) \in L^2[0, 2X]$ define the operator

$$R : L^2[0, 2X] \to L^2[0, 2X]$$

$$(R\phi)(t) = (\delta(t) + r(t)H(t)) * \phi(t) = \phi(t) + \int_0^t r(t - s)\, \phi(s)\, ds.$$

Then we have the following theorem.

Theorem 1.

1. *$\mathcal{F}$ is injective;*
2. *The range of $\mathcal{F}$ consists of all functions $r(.) \in L^2[0, 2X]$ for which $R + R^* \geq cI$ for some positive c (dependent on r);*
3. *The range of $\mathcal{F}$ is an open subset of $L^2[0, 2X]$ and $\mathcal{F}^{-1}$ is locally Lipschitz continuous on its domain.*

Let λ_{min} denote the smallest element in the spectrum of the self-adjoint operator $R + R^*$; then

$$\lambda_{min} = \inf_{\phi \in L^2[0,2X],\ \|\phi\|=1} \langle (R + R^*)\phi, \phi \rangle = \inf_{\phi \in L^2[0,2X],\ \|\phi\|=1} 2\langle R\phi, \phi \rangle$$

and the condition on $R + R^*$ in the second item of Theorem 1 is equivalent to the statement that $\lambda_{min} > 0$. The proof of Theorem 1 will include a reconstruction algorithm for the construction of the inverse of $\mathcal{F}$.

2 The necessity of the characterization condition

We observe that if $\sigma \in L^2[0, X]$ then $\eta \in H^1[0, X]$, $\eta'/\eta = \sigma$, and η is positive and bounded away from zero because η is continuous. Also, for any function $F(x, t)$ we have

$$\eta^2 \mathcal{L} F = \eta^2 (F_{tt} - F_{xx} - 2\sigma F_x) = (\eta^2 F)_{tt} - (\eta^2 F_x)_x.$$

From this we can derive the useful identity

$$2\eta^2 \, \mathcal{L}F \, F_t = 2(\eta^2 F_t)_t F_t - 2(\eta^2 F_x)_x F_t = (\eta^2 F_t^2)_t - 2(\eta^2 F_x F_t)_x + 2\eta^2 F_x F_{xt}$$
$$= (\eta^2 (F_t^2 + F_x^2))_t - 2(\eta^2 F_x F_t)_x. \quad (10)$$

We first show that the operator R corresponding to $r(.) = u_t(0,.)$ satsfies the positivity condition in Theorem 1. For this r, we note that

$$(R\phi)(t) = (\delta(t) + u_t(0,t)) * \phi(t) = U_t(0,t) * \phi(t) = V_t(0,t)$$

where $V(x,t) = U(x,t) * \phi(t)$ is the solution of an IBVP similar to (1)-(3) except the BC is $-\phi(t)$ instead of $-\delta(t)$, that is

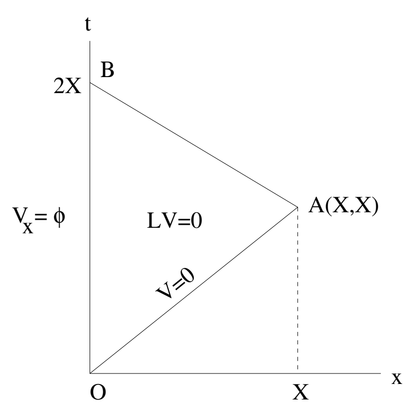
$$V_{tt} - V_{xx} - 2\sigma V_x = 0, \qquad (x,t) \in [0,\infty) \times \mathbb{R}, \quad (11)$$
$$V_x(0,t) = -\phi(t), \qquad t \in \mathbb{R}, \quad (12)$$
$$V(x,t) = 0, \qquad t < 0. \quad (13)$$

Here we have extended $\phi(t)$ as a zero function for t outside $[0, 2X]$.

From the speed of propagation we know that $V(x,t)$ is supported in the region $t \geq x$. Further $V(x,t)$ is the solution of the characteristic boundary value problem

$$V_{tt} - V_{xx} - 2\sigma V_x = 0, \qquad 0 \leq x \leq X, \quad x \leq t \leq 2X - x, \quad (14)$$
$$V_x(0,t) = -\phi(t), \qquad 0 \leq t \leq 2X, \quad (15)$$
$$V(x,x) = 0, \qquad 0 \leq x \leq X. \quad (16)$$

Note the zero condition in (16) which is a consequence of the fact that ϕ is the L^2 limit of a sequence of smooth functions on $[0, 2X]$ which are zero in a neighborhood of $x = 0$, and the speed of propagation is 1; compare (16) with (7).



Since (14)-(16) is well posed with the usual regularity properties, we may define the escaping operator

$$
\begin{aligned}
E \,:\, L^2[0,2X] &\to L^2_\eta[0,X] \\
(E\phi)(x) &= \frac{d}{dx} V(x, 2X-x).
\end{aligned}
$$

Here $L^2_\eta[0,X]$ stands for the space of all L^2 functions on $[0,X]$ except with the equivalent norm $\|f\|^2_\eta = \int_0^X \eta(x)^2 f(x)^2\,dx$..

Using (10) with V replacing F, integrating over the region OAB and using Green's theorem, we obtain

$$
\begin{aligned}
0 &= \iint_{OAB} 2\,\eta^2 \mathcal{L}V\, V_t \\
&= 2\int_{OB} \eta^2 V_x V_t\,dt + \int_{BA} \eta^2 (V_t^2 + V_x^2 - 2V_tV_x)(x, 2X-x)\,dx - \int_{OA} \eta^2 (V_t^2 + V_x^2 + 2V_tV_x)(x,x)\,dx \\
&= 2\int_{OB} \eta^2 V_x V_t\,dt + \int_{BA} \eta^2 \left(\frac{d}{dx} V(x,2X-x)\right)^2 dx - \int_{OA} \eta^2 \left(\frac{d}{dx} V(x,x)\right)^2 dx \\
&= -2\int_0^{2X} \phi(t)(R\phi)(t)\,dt + \int_0^X \eta(x)^2 (E\phi)(x)^2\,dx
\end{aligned}
$$

implying

$$2\langle R\phi, \phi\rangle = \|E\phi\|^2_\eta \tag{17}$$

which gives us

$$R + R^* = E^*E. \tag{18}$$

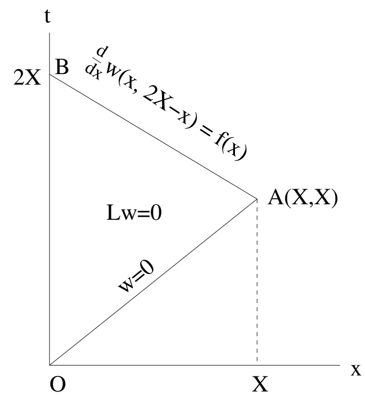
So $R + R^* \geq 0$; however we make the stronger claim that $R + R^* > 0$ for some $\epsilon > 0$. This follows quickly because from (17)

$$\langle (R+R^*)\phi, \phi\rangle = \|E\phi\|^2_\eta \geq \frac{1}{\|E^{-1}\|^2}\|\phi\|^2$$

and the fact that E has a bounded inverse with $E^{-1}f = w_x(0,t)$ where $w(x,t)$ is the solution of the Goursat problem

$$w_{tt} - w_{xx} - 2\sigma w_x = 0, \qquad 0 \leq x \leq X, \quad x \leq t \leq 2X - x, \tag{19}$$

$$w(x,x) = 0, \quad \frac{d}{dx}(w(x, 2X-x)) = f(x), \qquad 0 \leq x \leq X. \tag{20}$$



This completes the proof of the necessity of the positivity condition. For future use we note that $\|E^{-1}\|^2$ is the reciprocal of the smallest eigenvalue of E^*E which by (18) is $\lambda_{\min}$ - the smallest eigenvalue of $R+R^*$, that is

$$\|E^{-1}\|^2 = \lambda_{min}^{-1}. \tag{21}$$

3 Estimating σ in terms of R

One important step in the inversion is an upper bound on $\|\sigma\|_{L^2[0,X]}$ in terms of the data $u_t(0,.)$ or equivalently in terms of the operator R associated to $u_t(0,.)$. We will prove

Proposition 1.

$$\|\sigma\|_{L^2[0,X]} \le \lambda_{min}^{-1/2} \|u_t(0,.)\|_{L^2[0,X]}. \tag{22}$$

Proof Using (10) with F replaced by u, integrating over the region OAB and noting (5)-(7) we have

$$\begin{aligned}
0 &= \iint_{OAB} 2\,\eta^2 \mathcal{L}u\, u_t \\
&= 2\int_{OB} \eta^2 u_x u_t \,dt + \int_{BA} \eta^2 (u_t^2 + u_x^2 - 2u_t u_x)(x, 2X-x)\,dx - \int_{OA} \eta^2 (u_t^2 + u_x^2 + 2u_t u_x)(x,x)\,dx \\
&= \int_{BA} \eta^2 \left(\frac{d}{dx} u(x, 2X-x)\right)^2 dx - \int_{OA} \eta^2 \left(\frac{d}{dx} u(x,x)\right)^2 dx \\
&= \int_{BA} \eta^2 \left(\frac{d}{dx} u(x, 2X-x)\right)^2 dx - \int_0^X \sigma(x)^2\,dx.
\end{aligned}$$

Hence

$$\int_0^X \sigma(x)^2\,dx = \int_{BA} \eta^2 \left(\frac{d}{dx} u(x, 2X-x)\right)^2 dx. \tag{23}$$

We will now estimate the RHS of (23) in terms of the R associated to $u_t(0,.)$.

Let $w(x,t)$ be the solution of (19)-(20) with $f = \frac{d}{dx}(u(x, 2X-x))$. One may verify the identity

$$\eta^2 w_t \mathcal{L}u + \eta^2 u_t \mathcal{L}w = (\eta^2(u_t w_t + u_x w_x))_t - (\eta^2(u_x w_t + u_t w_x))_x.$$

Integrating this over OAB and noting (5)-(7) and (19)-(20) we obtain

$$\begin{aligned} 0 &= \int_{OB} \eta^2(u_x w_t + u_t w_x) + \int_{BA} \eta^2(u_x - u_t)(w_x - w_t)\, dx - \int_{OA} (u_x + u_t)(w_x + w_t)\, dx \\ &= \int_{OB} u_t w_x\, dt + \int_{BA} \eta^2 \left(\frac{d}{dx} u(x, 2X-x)\right)^2 dx. \end{aligned}$$

So (here $f = \frac{d}{dx}u(x, 2X-x)$) we have

$$\|f\|_\eta^2 = -\int_0^{2X} u_t(0,t) w_x(0,t)\, dt = -\langle u_t(0,.), E^{-1} f\rangle \leq \|u_t(0,.)\|\, \|E^{-1}\|\, \|f\|_\eta$$

which results in $\|f\|_\eta \leq \|u_t(0,.)\|\, \|E^{-1}\| = \|u_t(0,.)\| \lambda_{min}^{-1/2}$ which proves our claim when combined with (23).

4 Inversion

To complete the proof of Theorem 1 we need to show that for each $r(.) \in L^2[0, 2X]$ for which $R + R^* > 0$, there is a unique $\sigma \in L^2[0, X]$ so that if $w(x,t)$ is the solution of the sideways IVP

$$\begin{aligned} &w_{tt} - w_{xx} - 2\sigma w_x = 0, \qquad 0 \leq x \leq X,\ x \leq t \leq 2X - x, \\ &w_x(0,t) = 0, \quad w(0,t) = 1 + \int_0^t r(s)\, ds, \qquad 0 \leq t \leq 2X. \end{aligned} \tag{24}$$

then $w(x,x) = \eta(x)^{-1} = e^{-\int_0^x \sigma(y)\, dy}$. Note that the second condition in (24) is equivalent to the conditions $w_t(0,t) = r(t)$ and $w(0,0) = 1$.

4.1 The local step

The construction of $\sigma(.)$ will be done piece by piece, repeating the same procedure - first over the interval $[0, \delta]$, then over $[\delta, 2\delta]$ and so on. Each step is really an application of the following proposition for some choice of f, g, Y, η_* where σ has already been constructed over $[0, Y]$ and $f = w_x(Y,.)$, $g = w_t(Y,.)$, $\eta^* = \eta(Y) = e^{\int_0^Y \sigma(y)\, dy} = w(Y,Y)^{-1}$. Note that under these assumptions the condition $w(x,x) = \eta(x)^{-1}$, for $x \geq Y$, may be rewritten as

$$w(x,x) = \eta(x)^{-1} = e^{-\int_0^x \sigma(y)\, dy} = e^{-\int_0^Y \sigma(y)\, dy - \int_Y^x \sigma(y)\, dy} = \eta_*^{-1} e^{-\int_Y^x \sigma(y)\, dy}.$$

Proposition 2. *Suppose $0 < Y < X$, $f, g \in L^2[Y, 2X - Y]$ and η_* a positive constants. Then there is a $\delta > 0$ and a unique $\sigma \in L^2[Y, Y + \delta]$ so that if $w(x,t)$ is the solution of the sideways IVP*

$$w_{tt} - w_{xx} - 2\sigma w_x = 0, \qquad Y \leq x \leq Y + \delta,\ x \leq t \leq 2X - x, \tag{25}$$

$$w_x(Y,t) = f(t), \quad w(Y,t) = \eta_*^{-1} \int_{2Y}^{t} g(s)\, ds, \qquad Y \leq t \leq 2X - Y. \tag{26}$$

then $w(x,x) = \eta_*^{-1} e^{-\int_Y^x \sigma(y)\, dy}$ for $Y \leq x \leq Y + \delta$. Further δ is independent of Y and dependent only on $\eta_*, \|f\|, \|g\|$.

Proof of Proposition 2 For any $Z \in (Y, X]$, define the complete metric space

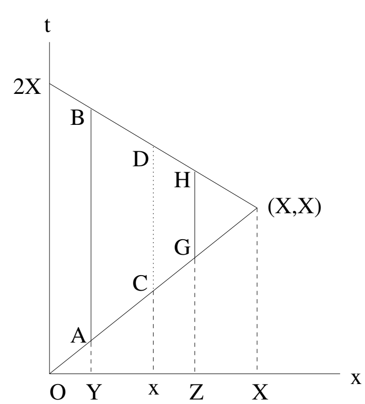
$$\mathcal{S}_{Z,\eta_*,f,g} = \{\sigma \in L^2[Y, Z] \,:\, \|\sigma\|^2 \leq 16\eta_*^2(\|f\|^2 + \|g\|^2)\,\}.$$

and define the map

$$M \,:\, \mathcal{S}_{Z,\eta_*,f,g} \to \mathcal{S}_{Z,\eta_*,f,g}$$
$$(M\sigma)(x) = -\frac{w(x,x)'}{w(x,x)}.$$

Then noting that $\sigma = \eta'/\eta = -\frac{d}{dx}(\eta^{-1})/\eta^{-1}$, we will have proved the proposition if we can show that M has a unique fixed point for some $Z \in (Y, X]$. At the moment, it is not even clear whether M is well defined because $w(x,x)$ may be zero for some $x \in [Y, Z]$ and even if $w(x,x)$ is non-zero it is not clear that $M\sigma$ is in $\mathcal{S}_{Z,\eta_*,f,g}$. However, by taking Z close enough to Y we will be able to show that M is well defined. Next, we will prove the existence of a unique fixed point by showing that M is a contraction map, provided Z is close enough to Y.

To prove these two claims we need a sideways energy identity for the wave equation, similar to the standard energy identity, except the roles of x and t are reversed. For any function $p(x,t)$ we have $2(\mathcal{L}p + 2\sigma p_x)p_x = 2(p_{tt} - p_{xx})p_x = (2p_x p_t)_t - (p_t^2 + p_x^2)_x$.



Define $J(x) := \int_x^{2X-x}(p_t^2 + p_x^2)(x,t)\,dt$; integrating the above identity over the region $ACDB$ we obtain

$$J(Y) - J(x) - \int_{AC}(p_x + p_t)^2\,dx - \int_{BD}(p_x - p_t)^2\,dx = 2\int_{ACBD}\mathcal{L}p\,p_x + \sigma p_x^2. \tag{27}$$

Hence (27) implies that (here we deviate from what is typically done when deriving energy identities)

$$\begin{aligned} J(x) + \int_{AC}(p_x+p_t)^2\,dx &\le J(Y) + 2\left(\int_{ACBD}|\mathcal{L}p|^2\right)^{1/2}\left(\int_Y^x J(y)\right)^{1/2} + 2\int_Y^x|\sigma(y)|J(y)\,dy \\ &\le J(Y) + 2\sqrt{Z-Y}\sqrt{J_*}\left(\int_{AGHB}|\mathcal{L}p|^2\right)^{1/2} + 2\int_Y^x|\sigma(y)|J(y)\,dy, \qquad \forall x\in[Y,Z] \end{aligned}$$

where J_* is the supremum of $J(y)$ for $y\in[Y,Z]$ (which can be shown to be finite). Hence by Gronwall's inequality, for all $x\in[Y,Z]$ we have

$$\begin{aligned} J(x) + \int_{AC}(p_x+p_t)^2\,dx &\le \left(J(Y) + 2\sqrt{Z-Y}\sqrt{J_*}\left(\int_{AGHB}|\mathcal{L}p|^2\right)^{1/2}\right)e^{2\int_Y^x|\sigma(y)|\,dy} \\ &\le \left(J(Y) + 2\sqrt{Z-Y}\sqrt{J_*}\left(\int_{AGHB}|\mathcal{L}p|^2\right)^{1/2}\right)e^{2\sqrt{x-Y}\left(\int_Y^x|\sigma(y)|^2\,dy\right)^{1/2}}. \end{aligned}$$

Hence taking Z close enough to Y (depending only on $\|\sigma\|$), we have

$$\begin{aligned} J_* + \int_Y^Z(p(x,x)')^2dx &\le 2J(Y) + 4\sqrt{Z-Y}\sqrt{J_*}\left(\int_{AGHB}|\mathcal{L}p|^2\right)^{1/2} \\ &\le 2J(Y) + \frac{1}{2}J_* + 4|Z-Y|\int_{AGHB}|\mathcal{L}p|^2. \end{aligned}$$

Hence for Z close enough to Y (depending only on $\|\sigma\|$), for all $x\in[Y,Z]$, we have

$$\int_x^{2X-x}(p_x^2+p_t^2)(x,t)\,dt + \int_Y^Z(p(x,x)')^2dx \le 4\int_Y^{2X-Y}(p_x^2+p_t^2)(Y,t)dt + 8|Z-Y|\int_{AGHB}|\mathcal{L}p|^2. \tag{28}$$

Applying (28) with w replacing p, we obtain

$$\int_{GH}(w_x^2+w_t^2)(Z,t)\,dt + \int_{AC}(w(x,x)')^2\,dx \le 4(\|f\|^2+\|g\|^2). \tag{29}$$

One consequence of this is that for Z close to Y (determined only by $\eta_*, \|f\|, \|g\|$)

$$w(x,x) = w(Y,Y) + \int_Y^x w(y,y)'\,dy \ge \eta_*^{-1} - (Y-x)^{1/2}\sqrt{\int_{AC}(w(y,y)')^2\,dy} \ge \frac{1}{2\eta_*} \tag{30}$$

So one can divide by $w(x,x)$ in the definition of M. Also, again from (29)

$$\|M\sigma\|^2 = \int_Y^Z(w(x,x)')^2/w(x,x)^2\,dx \le 4\eta_*^2\int_Y^Z(w(x,x)')^2 \le 16\eta_*^2(\|f\|^2+\|g\|^2)$$

and hence M is well defined.

It remains to prove that M is a contraction. Suppose $\sigma_1, \sigma_2 \in \mathcal{S}_{Z,\eta_*,f,g}$ and w_1, w_2 are solutions of (25), (26) for σ_1, σ_2 respectively. Let $p = w_2 - w_1$; then $\mathcal{L}_2 p = (\sigma_2 - \sigma_1) w_{1x}$ and $p_x(0,.) = 0$, $p_t(0,.) = 0$. So applying (28) we obtain

$$\begin{aligned}\int_Y^Z (p(x,x)')^2 dx &\leq 8|Z-Y| \int_{AGHB} |\mathcal{L}_2 p|^2 = 8|Z-Y| \int_{AGHB} |\sigma_2 - \sigma_1|^2 \, |w_{1x}|^2 \\ &= 8|Z-Y| \int_Y^Z |(\sigma_2-\sigma_1)(x)|^2 \int_x^{2X-x} |w_{1x}(x,t)|^2 \, dt \\ &\preccurlyeq |Z-Y| \, \|\sigma_1 - \sigma_2\|^2 \qquad (31)\end{aligned}$$

with the constant dependent only on $\eta_*, \|f\|, \|g\|$ and $\|\sigma_1\|, \|\sigma_2\|$, that is only on $\eta_*, \|f\|, \|g\|$. Further, since $p(Y,Y) = 0$, we have

$$|p(x,x)| = \left| \int_Y^x p(y,y)' \, dy \right| \leq \sqrt{x-Y} \left(\int_Y^Z (p(y,y)')^2 \, dy \right)^{1/2} \preccurlyeq \sqrt{Z-Y} \, \|\sigma_1 - \sigma_2\|, \qquad \forall x \in [Y,Z].$$

Now

$$(M\sigma_1 - M\sigma_2)(x) = \frac{w_2(x,x)'}{w_2(x,x)} - \frac{w_1(x,x)'}{w_1(x,x)} = \frac{(w_2-w_1)(x,x)'}{w_1(x,x)} - \frac{(w_2-w_1)(x,x) w_1(x,x)'}{w_1(x,x) w_2(x,x)}.$$

Hence

$$|(M\sigma_1 - M\sigma_2)(x,x)| \leq 2\eta_* |p(x,x)'| + 4\eta_*^2 |w_1(x,x)'| \, |p(x,x)|$$

so

$$\|M\sigma_1 - M\sigma_2\|^2 \preccurlyeq \int_Y^Z |p(x,x)'|^2 \, dx + \max_{x\in[Y,Z]} |p(x,x)|^2 \preccurlyeq |Z-Y| \|\sigma_1 - \sigma_2\|^2$$

with the constant dependent only on $\eta_*, \|f\|, \|g\|$. So M is a contraction if Z is close enough to Y.

4.2 Global Inversion

Let R be the operator associated to r on $[0,2X]$ and λ_{min} the smallest eigenvalue of $R + R^*$, which is positive by hypothesis. We observe that is $\tilde{R}$ is the operator associated to r restricted to $[0,2Y]$ for some $Y < X$ then the $\tilde{\lambda}_{min}$, the smallest eigenvalue of $\tilde{R} + \tilde{R}^*$, is larger then the smallest eigenvalue of $R + R^*$ because for any ϕ supported in $[0,2Y]$ we have $\langle \tilde{R}\phi, \phi \rangle = \langle R\phi, \phi \rangle$, hence

$$\tilde{\lambda}_{min} = 2 \inf_{\phi \in L^2[0,2Y]} \langle \tilde{R}\phi, \phi \rangle \geq 2 \inf_{\phi \in L^2[0,2X]} \langle R\phi, \phi \rangle = \lambda_{min}. \qquad (32)$$

Suppose we have recovered σ up to $[0,Y]$ for some $Y < X$. Then the r restricted to $[0,2Y]$ is the response of the medium to σ to on $[0,Y]$. Hence by Proposition 1 we have

$$\|\sigma\|_{L^2[0,Y]} \leq \|r\|_{L^2[0,2Y]} \, \tilde{\lambda}_{min}^{-1} \leq \|r\|_{L^2[0,2X]} \, \lambda_{min}^{-1}.$$

So we have a bound on $\|\sigma\|_{L^2[0,Y]}$ independent of Y. Since $\eta(x) = e^{\int_0^x \sigma(y)\,dy}$, the bound on $\|\sigma\|$ allows us to find positive η_* and η^* so that $\eta_* \leq \eta(x) \leq \eta^*$ for all $x \in [0, Y]$. Note that η_* and η^* can be chosen independent of Y. Finally, taking $p = w$ where w is the solution of (25)-(26) not over the x interval $[Y, Z]$ but over the interval $[0, Y]$ and using arguments simpler than but similar to those used to obtain (28) (we do not need smallness of $|Z - Y|$) we obtain

$$\int_Y^{2Y} (w_x^2 + w_t^2)(Y,t)\, dt \preccurlyeq \int_0^{2X} (w_x^2 + w_t^2)(0,t)\, dt = \int_0^{2X} |r(t)|^2\, dt$$

with the constant depending only on $\|\sigma\|$ and Y. Hence we have an upper bound on $\|w_t(Y,.)\|$, $\|w_x(.,Y)\|$ dependent only on $\|r\|$ and R and independent of Y. Finally, $w(Y,Y) = \eta(Y)^{-1}$ which lies in $[1/\eta^*, 1/\eta_*]$ which is independent of Y.

So if we start the local step with $Y = 0$, $f = 0$, $g = r$ and $\eta_* = 1$ then we can determine σ over the interval $[0, \delta]$. Then repeat the local step with $Y = \delta$, $f = w_x(Y,.)$, $g = w_t(Y,.)$ and $w(Y,Y) = \eta(Y)^{-1}$ and we can determine σ over the interval $[\delta, 2\delta]$; note that the δ is a global constant dependent only on r and R because of the bounds obtained earlier in this subsection. so we can continue this until $Y + \delta \geq X$.

References

[Br00] Brian L Browning. Time And Frequency Domain Scattering For The One Dimensional Wave Equation, Inverse Problems 16, 1377-1403 (2000).

[Bu80] R Burridge. The Gelfand-Levitan, the Marchenko, and the Gopinath-Sondhi integral equations of inverse scattering theory, regarded in the context of inverse impulse-response problems, Wave Motion 2, 305-323 (1980).

[Ra01] Rakesh. A One Dimensional Inverse Problem For A Hyperbolic System With Complex Coefficients, Inverse Problems, 17, 1401-1417 (2001).

[SWG96] Sylvester, D P Winebrenner, and F Gylys-Colwell. Layer stripping for the Helmholtz equation, SIAM J. Appl. Math., 56, 736-754 (1996).

[Sy86] Symes, William W.On the relation between coefficient and boundary values for solutions of Webster's horn equation. SIAM J. Math. Anal. 17 (1986), no. 6, 1400–1420.

[Sy83] Symes, William W. Impedance profile inversion via the first transport equation. J. Math. Anal. Appl. 94 (1983), no. 2, 435–453.