Distributions

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These notes are based on the exposition in [GeSh68] and [Ho03]. Let Ω be an open set in $\mathbb{R}^n$ and $C_0^\infty(\Omega)$ the vector space of all compactly supported functions on Ω which are infinitely differentiable.

Definition 1. *A distribution on Ω is a linear map $u \;: C_0^\infty(\Omega) \to \mathbb{R}$ such that for every compact subset K in Ω there is a constant C_K and a positive integer N so that*

$$|\langle u, \phi \rangle| \leq C_K \sup_{x \in K} \sum_{|\alpha| \leq N} |(\partial^\alpha \phi)(x)|$$

for all $\phi \in C_0^\infty(\Omega)$ with support in K.

If f is a locally integrable function on Ω then f is a distribution defined by

$$\langle f, \phi \rangle = \int_\Omega f(x)\, \phi(x)\, dx, \qquad \forall \phi \in C_0^\infty(\Omega).$$

The Dirac delta function $\delta(x)$ is a distribution defined by

$$\langle \delta(x), \phi(x) \rangle = \phi(0), \qquad \forall \phi \in C_0^\infty(\Omega).$$

For any smooth function u on Ω, by the integration by parts formula, we have

$$\int_\Omega (\partial_i u)(x)\, \phi(x)\, dx = -\int_\Omega u(x)\, (\partial_i \phi)(x)\, dx, \qquad \forall \phi \in C_0^\infty(\Omega).$$

This suggests a natural definition of the derivative of a distribution.

Definition 2. *If u is a distribution on Ω then $\partial_i u$ is a distribution defined by*

$$\langle \partial_i u, \phi \rangle = -\langle u, \partial_i \phi \rangle.$$

We want to give meaning to objects such as $\delta(|x|^2 - 1)$ as a distribution on $\mathbb{R}^n$. Suppose f is a real valued C^1 function on $\mathbb{R}^n$ and S the zero set of f, that is $S = \{x \in \mathbb{R}^n \;:\; f(x) = 0\}$; further assume that $(\nabla f)(x) \neq 0$ at every point of S. We may define $\delta(f)$ using pullbacks as in Hormander's Vol 1???? but we want to give a computational definition.

Definition 3. *If f is a real valued function C^1 function on $\mathbb{R}^n$ with $(\nabla f)(x) \neq 0$ for every $x \in S$, the zero set $f(x) = 0$. Then we define the distribution $\delta(f)$ as*

$$\langle \delta(f), \phi \rangle = \int_S \frac{\phi}{|\nabla f|}\, dS.$$

Let us see why this is the natural definition of $\delta(\phi)$. We will write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. If p is any point on S then $(\nabla f)(p) \neq 0$, say $(\partial_n f)(p) \neq 0$; then by the inverse function theorem applied to the map

$$x \mapsto (x', f(x)), \tag{1}$$

there are neighborhoods U of p and V of $(p', 0)$ and a C^1 inverse $y \mapsto (y', g(y))$ of (1) so that $g(f(x)) = x_n$, $f(g(y)) = y_n$ for $x \in U$, $y \in V$. We assume that these relations hold for all $x, y \in \mathbb{R}^n$ - the actual situation can be handled by a partition of unity argument.

The surface $S : f(x) = 0$ is mapped to the region $y_n = 0$; further S is also given by the equation $x_n = h(x')$ where we define $h(x') := g(x', 0)$. Consider the following formal computation with a change of variables $x \mapsto y = (x', f(x))$ (note $y' = x'$ and $dy = |(\partial_n f)(x)|\, dx$);

$$\begin{aligned}\langle \delta(f), \phi \rangle &= \int_{\mathbb{R}^n} \delta(f(x))\, \phi(x)\, dx = \int_{\mathbb{R}^n} \delta(y_n)\, \frac{\phi(x)}{|(\partial_n f)(x)|}\, dy = \int_{\mathbb{R}^{n-1}, y_n = 0} \frac{\phi(x)}{|(\partial_n f)(x)|}\, dy' \\ &= \int_{\mathbb{R}^{n-1}} \frac{\phi(x', h(x'))}{|(\partial_n f)(x', h(x'))|}\, dx' = \int_S \frac{1}{\sqrt{1 + |\nabla_{x'} h|^2}}\, \frac{\phi}{|\partial_n f|}\, dS.\end{aligned}$$

Now $f(x', h(x')) = 0$ so $\nabla_{x'} f + \partial_n f\, \nabla_{x'} h = 0$ so

$$1 + |\nabla_{x'} h|^2 = \frac{(\partial_n f)^2 + |\nabla_{x'} f|^2}{(\partial_n f)^2} = \frac{|\nabla f|^2}{|\partial_n f|^2}.$$

Hence formally

$$\langle \delta(f), \phi \rangle = \int_S \frac{\phi}{|\nabla f|}\, dS.$$

One useful distribution is the fundamental solution of the wave equation in three space dimensions, that is a distribution $G(x, t)$ so that

$$\begin{aligned}(\Box G)(x, t) := (\partial_t^2 G - \Delta_x G)(x, t) = \delta(x, t, \qquad (x, t) \in \mathbb{R}^3 \times \mathbb{R} \\ G(x, t) = 0, \qquad \text{for } t < 0.\end{aligned}$$

It takes considerable effort to show that

$$G(x, t) = \frac{1}{4\pi}\, \frac{\delta(t - |x|)}{|x|}$$

that is for any $\phi(x, t) \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ if we define the distribution G as

$$\langle G, \phi \rangle = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\phi(x, |x|)}{|x|}\, dx \tag{2}$$

then $\langle \Box G, \phi \rangle = \phi(0, 0)$. Note that from (2) it is clear that $\langle G, \phi \rangle = 0$ for any ϕ supported in $t < 0$.

References

[GeSh68] I M Gelfand and G E Shilov. Generalized Functions, Volume 2: Spaces of Fundamental and Generalized Functions, Academic Press (1968).

[Ho03] Lars Hormander. The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis (Classics in Mathematics), Springer Verlag (2003).