Carleman estimates and a formally determined inverse problem for a hyperbolic PDE

Rakesh

June 13, 2009

1 Introduction

Throughout the notes $A \preccurlyeq B$ will mean that A is less than or equal to a constant multiple of B. Our goal is the study of the following inverse problem. Suppose $T > 0$, D is a bounded open subset of $\mathbb{R}^n$ with a smooth boundary, k, q are smooth functions on $\overline{D}$, and f a smooth function on $\partial D \times [0,T]$. Consider the IBVP

$$\mathcal{L}u := u_{tt} - \Delta_x u - q(x)u = 0, \qquad (x,t) \in D \times [0,T] \tag{1}$$

$$u(x,0) = k(x),\ u_t(x,0) = 0, \qquad x \in \overline{D}, \tag{2}$$

$$\partial_\nu u(x,t) = f(x,t), \qquad (x,t) \in \partial D \times [0,T]. \tag{3}$$

Here $\partial_\nu u$ is the exterior normal derivative of u and the f, k are such that boundary condition and the initial conditions match to a sufficiently high order so that we are guaranteed that u will be C^3. The inverse problem under consideration is the recovery of $q(.)$ from $u|_{\partial D \times [0,T]}$.

Under appropriate conditions on k and T, Bukhgeim and Klibanov in [BuKl81] proved uniqueness for this problem. These ideas were modified by Imanuvilov and Yamamoto in [ImYa01] to prove stability for this inverse problem and our presentation draws freely from that article. The main result they proved and which we will reprove is the following.

Theorem 1. *Suppose k and q_i, $i = 1,2$ are smooth functions on $\overline{D}$, f a smooth function on $\partial D \times [0,T]$ satisfying the matching conditions and u_i, $i = 1,2$ are the solutions of the IBVP (1)-(3). If $|k(x)| > 0$ for all $x \in \overline{D}$ and there is an $a \in \mathbb{R}^n$ so that $\sup_{x \in \overline{D}} |x - a| < T$ (radius of D is smaller than T), then*

$$\|q_1 - q_2\|_{L^2(D)} \preccurlyeq \|\partial_t(u_1 - u_2)\|_{H^1(\partial D \times [0,T])} \tag{4}$$

with the constant depending only on k, a, D, T, f and the C^2 norms of q_i on $\overline{D}$.

To understand what will be required in the proof, let us try to prove just the uniqueness in Theorem 1. Let $q = q_1 - q_2$ and $u = u_1 - u_2$; then

$$u_{tt} - \Delta_x u - q_1(x)u = qu_2, \qquad (x,t) \in D \times [0,T]$$

$$u(x,0) = 0,\ u_t(x,0) = 0, \qquad x \in \overline{D},$$

$$\partial_\nu u(x,t) = 0, \qquad (x,t) \in \partial D \times [0,T].$$

In addition we are given $u = 0$ on $\partial D \times [0,T]$ and we have to show that $q = 0$ on D. So the initial data is zero, the Cauchy data on the boundary is zero, but the PDE is inhomogeneous and the

RHS is a product of $q(x)$ and a function of x, t; we have to show that $q = 0$. None of the standard estimates for the wave equation will work because all of them estimate the solution in the interior and one boundary value in terms of the initial data, the RHS and the other boundary data. The first trick is to have $q(x)$ appear in a place other than the RHS of the pde. This is done by taking $v = u_t$ and noting that the initial conditions for v are

$$v(x,0) = u_t(x,0) = 0, \quad v_t(x,0) = u_{tt}(x,0) = (\Delta u + q u_1)(x,0) = q(x) u_1(x,0) = q(x) k(x).$$

Hence v is the solution of

$$\begin{aligned} v_{tt} - \Delta_x v - q_1(x) v &= q u_{2t}, && (x,t) \in D \times [0,T] \\ v(x,0) = 0, \; v_t(x,0) &= q(x) k(x), && x \in \overline{D}, \\ \partial_\nu v(x,t) &= 0, && (x,t) \in \partial D \times [0,T]. \end{aligned}$$

In addition we have $v = 0$ on $\partial D \times [0,T]$ and we have to show that $q = 0$ on D. Noting that $|k(x)| > 0$, if we could bound the initial data by the boundary Cauchy data $v, \partial_\nu v$ and the RHS of the pde, and if we could make the contribution from the RHS of the PDE small compared to the initial data, then we will have proved uniqueness. Standard energy estimates for the wave equation do not do this. Carleman estimates, which bound the solution in the interior solely by time-like Cauchy data and the RHS will be the right tool. Also, the contribution from the RHS term will be made small using Carleman estimates because they are weighted estimates. In [BuKl81], Bukhgeim and Klibanov introduced the use of Carleman estimates for tackling inverse problems and we spend some time discussing and deriving these estimates.

2 Carleman and timelike estimates for the wave operator

The presentation is based on Chapter VIII of [Ho76] and an exposition on Daniel Tataru's web page [TaWeb] of the results by Tataru and others on Carleman estimates. Both of these sources have substantially more general results than what is covered in our notes. Tataru has other material on this topic on his web page. Chapter 5 of [Kr09] has a good discussion about the origins of the idea of pseudo-convexity - a property crucial for Carleman estimates.

2.1 Carleman estimates

Suppose $P(x, D)$ is a second order operator on $\mathbb{R}^n$ with real coefficients, where $D_j = \partial_j$. Let $p(x, D)$ be the highest order terms; then the function $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is called the principal symbol of $P(x, D)$. Consider the system of ODEs on $\mathbb{R}^n \times \mathbb{R}^n$ (Hamilton's equation associated to $p(x, \xi)$).

$$\frac{dx_i}{ds} = \frac{\partial p}{\partial \xi_i}(x, \xi), \quad \frac{d\xi_i}{ds} = -\frac{\partial p}{\partial x_i}(x, \xi), \quad i = 1, \cdots, n. \tag{5}$$

A solution of (5), over some interval in s, is a curve in $\mathbb{R}^n \times \mathbb{R}^n$, and is called a bicharacteristic. One may show that if $p(x,\xi) = 0$ at some point on a bicharacteristic then $p(x,\xi) = 0$ at all points on this bicharacteristic. Bicharacteristics on which $p(x,\xi)$ is zero are called null bicharacteristics and the null bicharacteristics play an important role in tracking singularities and the transmission of "energy" of solutions of hyperbolic PDEs. The curves in $\mathbb{R}^n$ obtained by projecting null bicharacteristics under the projection $(x,\xi) \mapsto x$ are called the rays associated to $P(x,D)$.

Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with a smooth boundary and $\psi(x)$ and $\phi(x)$ are smooth functions on $\overline{\Omega}$ with $\nabla\psi \neq 0$ and $\nabla\psi \neq 0$ at every point in $\overline{\Omega}$. Below $p^{(j)}$, p_k will mean $\partial_{\xi_j} p$, $\partial_{x_k} p$ respectively and repeated indices will mean a summation.

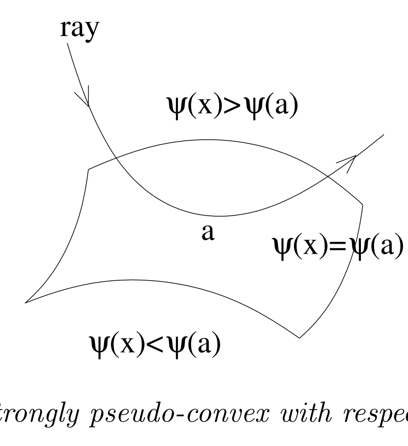
Definition 1. *ψ is said to be pseudo-convex with respect to $P(x,D)$ on $\overline{\Omega}$ if*

$$\psi_{jk}(x)\, p^{(j)}(x,\xi)\, p^{(k)}(x,\xi) + \left(p_k^{(j)}(x,\xi)\, p^{(k)}(x,\xi) - p_k(x,\xi) p^{(j,k)}(x,\xi)\right) \psi_j(x) \, > \, 0 \tag{6}$$

for all $x \in \overline{\Omega}$ and all non-zero $\xi \in \mathbb{R}^n$ satisfying

$$p(x,\xi) = 0, \quad \psi_j(x) p^{(j)}(x,\xi) = 0. \tag{7}$$

Pseudoconvexity has an interesting geometrical interpretation; one can check that ψ is pseudoconvex with respect to $P(x,D)$ iff $\frac{d^2}{ds^2}\psi(x(s)) > 0$ at every critical point of $\psi(x(s))$ for any ray $x(s)$ of $P(x,D)$, so $\psi(x(s))$ has a local minimum at that critical point. This condition means that either a ray of $P(x,D)$ is transverse to the contours of ψ or if the ray is tangential to a contour $\psi(x) = \psi(a)$ at some point a then, near a, the ray lies in the region $\psi(x) > \psi(a)$.



Definition 2. *ψ is said to be strongly pseudo-convex with respect to $P(x,D)$ on $\overline{\Omega}$ if ψ is pseudo-convex and*

$$\psi_{jk}(x)\, p^{(j)}(x,\zeta)\, \overline{p^{(k)}(x,\zeta)} + \sigma^{-1} Im \left(p_k(x,\zeta)\, \overline{p^{(k)}(x,\zeta)}\right) \, > \, 0 \tag{8}$$

for all $x \in \overline{\Omega}$ and all $\zeta = \xi + i\sigma\nabla\psi(x)$, $\xi \in \mathbb{R}^n$, $\sigma \neq 0$, satisfying

$$p(x,\zeta) = 0, \quad \psi_j(x) p^{(j)}(x,\zeta) = 0. \tag{9}$$

Because $P(x, D)$ has order 2, one may show that pseudo-convexity implies strong pseudo-convexity. For obtaining Carleman estimates, one needs a condition stronger than strong pseudoconvexity, which we call the "special condition" - it has no standard name.

Definition 3. *ϕ satisfies a* special condition *on $\overline{\Omega}$ for the operator $P(x, D)$ if ϕ is pseudo-convex on $\overline{\Omega}$ and (8) holds for all $x \in \overline{\Omega}$ and all $\zeta = \xi + \sigma \nabla \phi(x)$ with $\xi \in \mathbb{R}^n$, $\sigma \neq 0$, satisfying*

$$p(x, \zeta) = 0. \tag{10}$$

Note that (9) implies (10) so *special condition* implies strong pseudo-convexity but the converse is not true. However, we have ([Ho76, Theorem 8.6.3) the following proposition.

Proposition 1. *Suppose ψ is strongly pseudoconvex with respect to $P(x, D)$ on $\overline{\Omega}$, then for large enough λ, $\phi = e^{\lambda\psi}$ satisfies the* special condition.

So the way to construct a function which satisfies the *special condition* with respect to a second order operator is to construct a function ψ which is pseudoconvex. Then ψ is also strongly pseudo-convex and hence $\phi = e^{\lambda\psi}$ satisfies the special condition for large enough λ. When using Carleman estimates the contours of ψ will be crucial. However $\phi = e^{\lambda\psi}$ so the contours of ϕ are the contours of ψ hence we can use ϕ - for which we will have Carleman estimates.

For the rest of this subsection we will use x_0 instead of t and $x \in \mathbb{R}^{n+1}$ will be split as $x = (x_0, x')$ with x_0 real and $x' \in \mathbb{R}^n$. We find a family of functions which are pseudoconvex with respect to the wave operator $\Box := \partial_0^2 - \partial_j^2 = -\epsilon_j \partial_j^2$ where $\epsilon_j = -1$ if $j = 0$ and 1 if $j \geq 1$; the principal symbol of $\Box$ is $-\epsilon_j \xi_j^2$. The pseudo-convexity conditions (6) and (7) reduce to

$$\epsilon_j \epsilon_k \psi_{jk}(x) \xi_j \xi_k > 0, \qquad \forall \xi \neq 0 \text{ with } \epsilon_j \xi_j^2 = 0,\ \epsilon_j \psi_j \xi_j = 0. \tag{11}$$

The *special condition* for ϕ is pseudoconvexity (that is (11) with ϕ replacing ψ) and $\epsilon_j \epsilon_k \phi_{jk}(\xi_j + i\sigma\phi_j)(\xi_k - i\sigma\phi_k) > 0$ for all ξ, σ with $\sigma \neq 0$, $\epsilon_j(\xi_j + i\sigma\phi_j)^2 = 0$, that is

$$\epsilon_j \epsilon_k \phi_{jk} \xi_j \xi_k + \sigma^2 \epsilon_j \epsilon_k \phi_{jk} \phi_j \phi_k > 0, \qquad \forall\ (\xi, \sigma) \text{ with } \sigma \neq 0,\ \epsilon_j \xi_j^2 - \sigma^2 \epsilon_j \phi_j^2 = 0,\ \epsilon_j \phi_j \xi_j = 0. \tag{12}$$

So the *special condition* for ϕ is equivalent to (replacing (11), (12) by)

$$\epsilon_j \epsilon_k \phi_{jk} \xi_j \xi_k + \sigma^2 \epsilon_j \epsilon_k \phi_{jk} \phi_j \phi_k > 0 \tag{13}$$

for all $(\xi, \sigma) \neq 0$, satisfying

$$\epsilon_j \xi_j^2 - \sigma^2 \epsilon_j \phi_j^2 = 0, \qquad \epsilon_j \phi_j \xi_j = 0. \tag{14}$$

We seek a pseudoconvex function of the form $\psi(x) = a_0 x_0^2 + a|x'|^2$ for $\Box$. Then the pseudo-convexity condition 11) is that for all (x, ξ) with $\xi \neq 0$, $\epsilon_j \xi_j^2 = 0$, $a_0 \epsilon_0 x_0 \xi_0 + a \epsilon_j x_j \xi_j = 0$ one has

$$0 < a_0 \xi_0^2 + a|\xi'|^2 = (a_0 + a)|\xi'|^2.$$

This will be true if $a_0 + a > 0$ because $\xi \neq 0$ and $\xi_0^2 = |\xi'^2|$. So $\psi(x) := -\beta x_0^2 + |x'|^2$ and $\psi(x) := x_0^2 - \beta|x'|^2$ are pseudoconvex if $0 < \beta < 1$ over a region where $\nabla\psi$ is not zero. For the time-like Cauchy problem the function $\psi(x,t) := -\beta t^2 + |x-p|^2$ with $0 < \beta < 1$ and arbitrary $p \in \mathbb{R}^n$ is the pseudo-convex function that is used. We can use the p because $\Box$ is translation invariant in x.

We now state Carleman estimates for the wave operator $\mathcal{L} = \Box + a_0(x,t)\partial_t + a_i(x,t)\partial_i + b(x,t)$.

Theorem 2. *Suppose Ω is a region in $\mathbb{R}^n \times \mathbb{R}$, the coefficients of $\mathcal{L}$ are bounded on $\overline{\Omega}$, and ϕ satisfies the special property with respect to $\Box$, then for large enough σ*

$$\sigma \int_\Omega e^{2\sigma\phi}(|\partial u|^2 + \sigma^2 u^2) \preccurlyeq \int_\Omega e^{2\sigma\phi}|\mathcal{L}u|^2 + \sigma \int_{\partial\Omega} e^{2\sigma\phi}(|\partial u|^2 + \sigma^2 u^2), \qquad \forall u \in C^2(\overline{\Omega}) \tag{15}$$

with the constant independent of σ and u. Here $\partial u = (u_t, \partial_x u)$.

Please see [Ta94] for more general Carleman estimates. We postpone the proof of Theorem 2 to Section 4.

2.2 An estimate for the timelike Cauchy problem

We prove the following estimate for the timelike Cauchy problem for the wave operator $\mathcal{L} = \Box + a_0(x,t)\partial_t + a_i(x,t)\partial_i + b(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$.

Theorem 3. *Suppose D is a bounded open subset of $\mathbb{R}^n$ with a smooth boundary and there is an $a \in \mathbb{R}^n$ so that $|x - a| < T$ for all $x \in \overline{D}$. If the coefficients of $\mathcal{L}$ are bounded on $\overline{\Omega}$, then*

$$\int_{-T}^{T}\int_D |\partial u|^2 + u^2 \preccurlyeq \int_{-T}^{T}\int_D (\mathcal{L}u)^2 + \int_{-T}^{T}\int_{\partial D} u^2 + |\partial u|^2, \qquad \forall\ u \in C^2(\overline{D} \times [-T,T]), \tag{16}$$

$$\int_D |\partial u(.,t)|^2 + u(.,t)^2 \preccurlyeq \int_{-T}^{T}\int_D (\mathcal{L}u)^2 + \int_{-T}^{T}\int_{\partial D} u^2 + |\partial u|^2, \qquad \forall\ u \in C^2(\overline{D} \times [-T,T]), \quad |t| \leq T. \tag{17}$$

Here $\partial u = (u_t, \partial_x u)$ and the constant is independent of u and t.

Remark The condition on T is that the radius of Ω be less than T - this is not guaranteed if the diameter of D is less than $2T$ as seen when D is the interior of an equilateral triangle.

Proof Using $2(\Box u)u_t = (u_t^2 + |\nabla u|^2)_t - 2\nabla \cdot (u_t \nabla u)$, integrating over the region $[s,t] \times D$, and applying Gronwall's inequality, we can show that for all $s,t \in [-T,T]$ and for all $u \in C^2(\overline{D} \times [-T,T])$ we have

$$\int_D |(\partial u)(.,t)|^2 + |u(.,t)|^2 \preccurlyeq \int_D |(\partial u)(.,s)|^2 + |u(.,s)|^2 + \int_{-T}^{T}\int_D |\mathcal{L}u|^2 + \int_{-T}^{T}\int_{\partial D} |u|^2 + |\partial u|^2,$$

with the constant independent of u, s and t. Hence for any $S_1, S_2 \in [-T,T]$, $S_1 < S_2$ and any $t \in [-T,T]$, integrating the above relation with respect to s over S_1, S_2

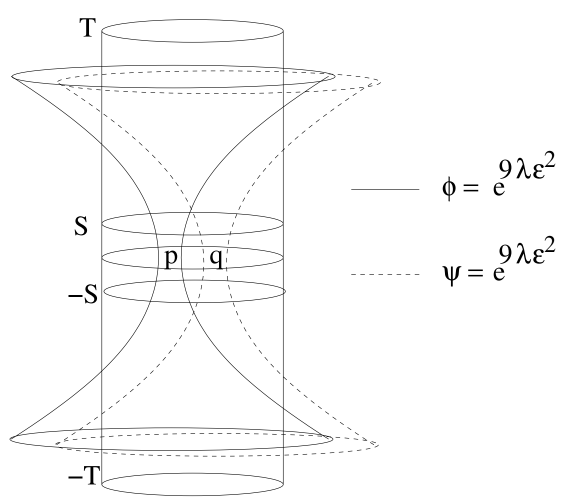
$$\int_D |(\partial u)(.,t)|^2 + |u(.,t)|^2 \preccurlyeq \int_{S_1}^{S_2}\int_D |\partial u|^2 + |u|^2 + \int_{-T}^{T}\int_D |\mathcal{L}u|^2 + \int_{-T}^{T}\int_{\partial D} |u|^2 + |\partial u|^2, \tag{18}$$

for all u in $C^2(\overline{D}\times[-T,T])$, with the constants independent of u, t and its dependence on S_1, S_2 is only with a lower bound on $|S_1 - S_2|$. One consequence of (18) is that (17) follows from (16), so we need to prove only (16).

For any $\beta \in (0,1)$, $a, b \in \mathbb{R}^n$, $a \neq b$, $\lambda > 0$, define

$$\phi(x,t) := e^{\lambda(-\beta^2 t^2 + |x-a|^2)}, \qquad \psi(x,t) := e^{\lambda(-\beta^2 t^2 + |x-b|^2)}.$$

Then, for large enough λ, ϕ has the 'special property' with respect to the operator $\mathcal{L}$ on any closed domain in $\overline{D}\times[0,T]$ which excludes $(a,0)$; there is a problem at $(a,0)$ because $(\nabla\phi, \phi_t)$ is zero at $(a,0)$. Similarly ψ has the special property with respect to the operator $\mathcal{L}$ on any closed domain in $\overline{D}\times[0,T]$ which excludes $(b,0)$.



We claim we can choose $\beta \in (0,1)$, $a, b \in \mathbb{R}^n$, and $\epsilon > 0$ so that

- $\phi(x,t) \leq \phi(a,0) = 1$ and $\psi(x,t) \leq \psi(b,0) = 1$ for all $(x,t) \in \overline{D}\times\{t = \pm T\}$;
- $|a-b| \geq 8\epsilon$.

Let $R = \max_{x\in\overline{D}} |x-a|$ and by hypothesis $R < T$. The first condition is $R < \beta T$ and $\max_{x\in\overline{D}} |x-b| < \beta T$ and these will hold if we choose $\beta \in (0,1)$ with $\beta > R/T$ and we choose $b \neq a$ with $|a-b| < \beta T - R$. So the second condition will hold if we choose any positive ϵ with $\epsilon < |a-b|/8$.

One consequence of our choice of a, b, ϵ is that the the union of the regions $\phi \geq e^{9\lambda\epsilon^2}$ and $\psi \geq e^{9\lambda\epsilon^2}$ contains the vertical cylinder $\overline{D} \times [-2\epsilon, 2\epsilon]$. This is so because for any $x \in \mathbb{R}^n$ we have $\max(|x-a|^2, |x-b|^2) \geq |a-b|^2/4 \geq 16\epsilon^2$ and hence for any $t \in [-2\epsilon, 2\epsilon]$ we have

$$\max(\phi(x,t), \psi(x,t)) \geq \max(e^{\lambda(-4\epsilon^2+|x-a|^2)}, e^{\lambda(-4\epsilon^2+|x-b|^2)}) \geq e^{12\lambda\epsilon^2}.$$

For any $c \in \mathbb{R}^n$, $B_r(c,0)$ will denote the open ball, in $\mathbb{R}^{n+1}$, of radius r and centered at $(c,0)$. Let $\chi(x,t)$ be a smooth function on $\mathbb{R}^{n+1}$ with $0 \leq \chi \leq 1$, $\chi = 1$ in $B_\epsilon(a,0)$ and $\chi = 0$ outside $B_{2\epsilon}(a,0)$. Let $v := (1-\chi)u$.

Define $D_{\epsilon,a,T} := (\overline{D} \times [-T,T]) \setminus B_\epsilon(a,0)$. Since ϕ has the special property with respect to $\mathcal{L}$ on the region $D_{\epsilon,a,T}$, by Theorem 2 (and noting $v = 0$ on the boundary of $B_\epsilon(a,0)$), we have

$$\sigma \int_{D_{\epsilon,a,T}} e^{2\sigma\phi}(|\partial v|^2 + \sigma^2|v|^2) \preccurlyeq \int_{-T}^{T}\int_D e^{2\sigma\phi}|\mathcal{L}v|^2 + \sigma \int_{D,t=\pm T} e^{2\sigma\phi}(|\partial v|^2 + \sigma^2|v|^2)$$
$$+ \sigma \int_{-T}^{T}\int_{\partial D} e^{2\sigma\phi}(|\partial v|^2 + \sigma^2|v|^2). \qquad (19)$$

Let ϕ_ϵ be the subset of $\overline{D} \times [-T,T]$ where $\phi \geq e^{9\lambda\epsilon^2}$ with a similar definition of ψ_ϵ. Since $\phi \leq e^{4\lambda\epsilon^2}$ on the region $B_{2\epsilon}(a,0)$ it is clear that ϕ_ϵ is a subset of $D_{\epsilon,a,T}$ and $v = u$ on ϕ_ϵ. Further $\mathcal{L}v$ is equal to $(1-\chi)\mathcal{L}u$ plus terms involving ∂u and u multiplied with derivatives of $1-\chi$ - so the lower order terms are supported on $B_{2\epsilon}(a,0)$ - where $\phi \leq e^{4\lambda\epsilon^2}$. Also $\phi \leq 1$ on $D \times t = \pm T$. Hence (19) implies

$$\sigma e^{2\sigma e^{9\lambda\epsilon^2}} \int_{\phi_\epsilon} |\partial u|^2 + \sigma^2 u^2 \preccurlyeq \int_{-T}^{T}\int_D e^{2\sigma\phi}|\mathcal{L}u|^2 + \sigma e^{2\sigma e^{4\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + u^2$$
$$+ \sigma e^{2\sigma}\int_{D,t=\pm T} |\partial u|^2 + \sigma^2|u|^2 + \sigma e^{C\sigma}\int_{-T}^{T}\int_{\partial D} |\partial u|^2 + \sigma^2|u|^2. \qquad (20)$$

Similarly we may show that

$$\sigma e^{2\sigma e^{9\lambda\epsilon^2}} \int_{\psi_\epsilon} |\partial u|^2 + \sigma^2 u^2 \preccurlyeq \int_{-T}^{T}\int_D e^{2\sigma\psi}|\mathcal{L}u|^2 + \sigma e^{2\sigma e^{4\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + u^2$$
$$+ \sigma e^{2\sigma}\int_{D,t=\pm T} |\partial u|^2 + \sigma^2|u|^2 + \sigma e^{C\sigma}\int_{-T}^{T}\int_{\partial D} |\partial u|^2 + \sigma^2|u|^2. \qquad (21)$$

Since the union of ψ_ϵ and ϕ_ϵ includes $\overline{D} \times [-2\epsilon, 2\epsilon]$, combining (20) and (21) we conclude that

$$\sigma e^{\sigma e^{9\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + \sigma^2 u^2 \preccurlyeq e^{C\sigma}\int_{-T}^{T}\int_D |\mathcal{L}u|^2 + e^{\sigma e^{4\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D (|\partial u|^2 + \sigma^2 u^2)$$
$$+ \sigma e^{C\sigma}\int_{-T}^{T}\int_{\partial D} |\partial u|^2 + \sigma^2 u^2 + \sigma e^{\sigma}\int_{D,t=\pm T} |\partial u|^2 + \sigma^2 u^2.$$

Hence using (18) with $t = \pm T$ and $S_1 = -2\epsilon$, $S_2 = 2\epsilon$, we conclude that for large σ

$$\sigma e^{\sigma e^{9\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + \sigma^2 u^2 \preccurlyeq e^{C\sigma}\int_{-T}^{T}\int_D |\mathcal{L}u|^2 + + \sigma e^{C\sigma}\int_{-T}^{T}\int_{\partial D} |\partial u|^2 + \sigma^2 u^2$$

which proves (16) in Theorem 3. **QED**

3 Proof of Theorem 1

Let $u = (u_1 - u_2)_t$ and $q = q_1 - q_2$, then $u(.,0) = 0$ and

$$u_t(.,0) = (u_{1tt} - u_{2tt})(.,0) = \Delta(u_1 - u_2)(.,0) + (q_1 u_1 - q_2 u_2)(.,0) = kq.$$

Hence

$$\begin{aligned} \Box u - q_1 u = q u_{2t}, &\qquad \text{on } D \times [0,T];\\ u(.,0) = 0, \quad u_t(.,0) = k(.)q(.), &\qquad \text{on } D;\\ u = 0 &\qquad \text{on } \partial D \times [0,T]. \end{aligned}$$

We take an odd extension of u across $t = 0$; note u will be C^2. Let $r(x,t)$ be the even extension of $u_{2t}(x,t)$; $r(x,t)$ will be C^1. Hence

$$\Box u - q_1 u = qr, \qquad \text{on } D \times [-T,T]; \tag{22}$$

$$u(.,0) = 0, \quad u_t(.,0) = k(.)q(.), \qquad \text{on } D; \tag{23}$$

$$u = 0 \qquad \text{on } \partial D \times [-T,T]. \tag{24}$$

Let $\mathcal{L} := \Box - q_1$, and for any $\beta \in (0,1)$, $a, b \in \mathbb{R}^n$, $\lambda > 0$, define as before

$$\phi(x,t) := e^{\lambda(-\beta^2 t^2 + |x-a|^2)}, \qquad \psi(x,t) := e^{\lambda(-\beta^2 t^2 + |x-b|^2)}.$$

Then we claim we can choose $\beta \in (0,1)$, $a, b \in \mathbb{R}^n$, and $\epsilon > 0$ so that

- $\phi(x,t) \leq \phi(a,0) = 1$ and $\psi(x,t) \leq \psi(b,0) = 1$ for all $(x,t) \in \overline{D} \times \{t = \pm T\}$;
- $|a - b| \geq 8\epsilon$.

Let $R = \max_{x \in \overline{D}} |x - a|$ and by hypothesis $R < T$. The first condition is $R < \beta T$ and $\max_{x \in \overline{D}} |x - b| < \beta T$ and these will hold if we choose $\beta \in (0,1)$ with $\beta > R/T$ and we choose $b \neq a$ with $|a - b| < \beta T - R$. So the second condition will hold if we choose any positive ϵ with $\epsilon < |a - b|/8$.

One consequence of our choice of a, b, ϵ is that the the union of the regions $\phi \geq e^{9\lambda\epsilon^2}$ and $\psi \geq e^{9\lambda\epsilon^2}$ contains the vertical cylinder $\overline{D} \times [-2\epsilon, 2\epsilon]$. This is so because for any $x \in \mathbb{R}^n$ we have $\max(|x-a|^2, |x-b|^2) \geq |a-b|^2/4 \geq 16\epsilon^2$. Hence for any $t \in [-2\epsilon, 2\epsilon]$ we have

$$\max(\phi(x,t), \psi(x,t)) \geq \max(e^{\lambda(-4\epsilon^2 + |x-a|^2)}, e^{\lambda(-4\epsilon^2 + |x-b|^2)}) \geq e^{12\lambda\epsilon^2}.$$

Let $\chi(x,t)$ be a smooth function on $\mathbb{R}^{n+1}$ with $0 \leq \chi \leq 1$, $\chi = 1$ in $B_\epsilon(a,0)$ and $\chi = 0$ outside $B_{2\epsilon}(a,0)$. Let $v := (1 - \chi)u$.

We will need a standard energy estimate, except we need it with weights, so we have to rederive it. Define $w := e^{\sigma\phi} v = e^{\sigma\phi}(1-\chi)u$. We have the identity

$$2(\mathcal{L}w + q_1 w + \sigma^2 w)\, w_t = 2(w_{tt} - \Delta_x w + \sigma^2 w) w_t = (w_t^2 + |\nabla_x w|^2 + \sigma^2 w^2)_t - 2\nabla_x \cdot (w_t \nabla_x w).$$

Integrating this over the region $D \times [0,t]$, we obtain

$$\int_D |(\partial w)(.,0)|^2 + \sigma^2 |w(.,0)|^2 \preccurlyeq \int_D |(\partial w)(.,t)|^2 + \sigma^2 |w(.,t)|^2 + \int_0^t \int_D |\mathcal{L}w|\,|w_t| + \sigma |w_t|^2 + \sigma^3 w^2 + \int_0^t \int_{\partial D} |\partial w|^2 + |w|^2.$$

Integrating this with respect to t over the interval $[-2\epsilon, 2\epsilon]$ we obtain

$$\int_D |(\partial w)(.,0)|^2 + \sigma^2 |w(.,0)|^2 \preccurlyeq \int_{-2\epsilon}^{2\epsilon} \int_D |\mathcal{L}w|\,|w_t| + \sigma |\partial w|^2 + \sigma^3 w^2 + \int_{-T}^{T} \int_{\partial D} |\partial w|^2 + |w|^2. \quad (25)$$

Since $w = e^{\sigma\phi} v$ and $v = e^{-\sigma\phi} w$, one may see that

$$e^{2\sigma\phi}(|\partial v|^2 + \sigma^2 |v|^2) \preccurlyeq |\partial w|^2 + \sigma^2 |w|^2 \preccurlyeq e^{2\sigma\phi}(|\partial v|^2 + \sigma^2 |v|^2),$$
$$|\mathcal{L}w|\,|w_t| \preccurlyeq e^{2\sigma\phi} |(\mathcal{L}v| + \sigma|\partial v| + \sigma^2 |v|)(|\partial v| + \sigma |v|) \preccurlyeq e^{2\sigma\phi}(|\mathcal{L}v|^2 + \sigma |\partial v|^2 + \sigma^3 |v|^2).$$

Hence (25) implies

$$\int_D e^{2\sigma\phi}(|\partial v|^2 + \sigma^2 |v|^2)(.,0) \preccurlyeq \int_{-2\epsilon}^{2\epsilon} \int_D e^{2\sigma\phi} |\mathcal{L}v|^2 + \sigma \int_{-2\epsilon}^{2\epsilon} \int_D e^{2\sigma\phi}(|\partial v|^2 + \sigma^2 |v|^2) + \int_{-T}^{T} \int_{\partial D} e^{2\sigma\phi}(|\partial v|^2 + |v|^2). \quad (26)$$

Now $v = u$ outside $B_{2\epsilon}(a,0)$ and $u(.,0) = 0$, $u_t(.,0) = k(.)q(.)$ with k continuous and $|k| > 0$ on D; further $v = (1-\chi)u$. So (26) implies

$$\int_{D \setminus B_{2\epsilon}(a)} e^{2\sigma\phi} q^2 \preccurlyeq \int_{-T}^{T} \int_D e^{2\sigma\phi} |\mathcal{L}u|^2 + \sigma \int_{-2\epsilon}^{2\epsilon} \int_D e^{2\sigma\phi}(|\partial u|^2 + \sigma^2 |u|^2) + e^{C\sigma} \int_{-T}^{T} \int_{\partial D} |\partial u|^2 + |u|^2. \quad (27)$$

Define $D_{\epsilon,a,T} := (\overline{D} \times [-T,T]) \setminus B_\epsilon(a,0)$. Since ϕ has the special property with respect to $\mathcal{L}$ on the region $D_{\epsilon,a,T}$, by Theorem 2 (and noting $v = 0$ on the boundary of $B_\epsilon(a,0)$), we have

$$\sigma \int_{D_{\epsilon,a,T}} e^{2\sigma\phi}(|\partial v|^2 + \sigma^2 |v|^2) \preccurlyeq \int_{-T}^{T} \int_D e^{2\sigma\phi} |\mathcal{L}v|^2 + \sigma \int_{D, t=\pm T} e^{2\sigma\phi}(|\partial v|^2 + \sigma^2 |v|^2) + \sigma \int_{-T}^{T} \int_{\partial D} e^{2\sigma\phi}(|\partial v|^2 + \sigma^2 |v|^2). \quad (28)$$

Let ϕ_ϵ be the subset of $\overline{D} \times [-T,T]$ where $\phi \geq e^{9\lambda\epsilon^2}$ with a similar definition of ψ_ϵ. Since $\phi \leq e^{4\lambda\epsilon^2}$ on the region $B_{2\epsilon}(a,0)$ it is clear that ϕ_ϵ is a subset of $D_{\epsilon,a,T}$ and $v = u$ on ϕ_ϵ. Further $\mathcal{L}v$ is equal to $(1-\chi)\mathcal{L}u$ plus terms involving ∂u and u multiplied with derivatives of $1-\chi$ - so the lower order terms are supported on $B_{2\epsilon}(a,0)$ - where $\phi \le e^{4\lambda\epsilon^2}$. Also $\phi \le 1$ on $D \times t = \pm T$. Hence (28) implies

$$\sigma \int_{\phi_\epsilon} e^{2\sigma\phi}(|\partial u|^2 + \sigma^2 u^2) \preccurlyeq \int_{-T}^{T}\int_D e^{2\sigma\phi}|\mathcal{L}u|^2 + \sigma e^{2\sigma e^{4\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + u^2$$
$$+ \sigma e^{2\sigma} \int_{D,t=\pm T} |\partial u|^2 + \sigma^2|u|^2 + \sigma e^{C\sigma} \int_{-T}^{T}\int_{\partial D} |\partial u|^2 + \sigma^2 |u|^2. \quad (29)$$

Similarly we may show that

$$\sigma \int_{\psi_\epsilon} e^{2\sigma\psi}(|\partial u|^2 + \sigma^2 u^2) \preccurlyeq \int_{-T}^{T}\int_D e^{2\sigma\psi}|\mathcal{L}u|^2 + \sigma e^{2\sigma e^{4\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + u^2$$
$$+ \sigma e^{2\sigma} \int_{D,t=\pm T} |\partial u|^2 + \sigma^2|u|^2 + \sigma e^{C\sigma} \int_{-T}^{T}\int_{\partial D} |\partial u|^2 + \sigma^2 |u|^2. \quad (30)$$

Now

$$e^{2\sigma e^{4\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + u^2 \le o(\sigma) e^{2\sigma e^{9\lambda\epsilon^2}} \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + u^2$$
$$\le o(\sigma)\left(\int_{\phi_\epsilon} e^{2\sigma\phi}|\partial u|^2 + u^2 + \int_{\phi_\epsilon} e^{2\sigma\phi}|\partial u|^2 + u^2 \right).$$

Hence (29) and (30) give

$$\sigma \int_{\phi_\epsilon} e^{2\sigma\phi}(|\partial u|^2 + \sigma^2 u^2) + \sigma \int_{\psi_\epsilon} e^{2\sigma\psi}(|\partial u|^2 + \sigma^2 u^2) \preccurlyeq \int_{-T}^{T}\int_D (e^{2\sigma\phi} + e^{2\sigma\psi})|\mathcal{L}u|^2$$
$$+ \sigma e^{2\sigma} \int_{D,t=\pm T} |\partial u|^2 + \sigma^2|u|^2 + \sigma e^{C\sigma} \int_{-T}^{T}\int_{\partial D} |\partial u|^2 + \sigma^2 |u|^2. \quad (31)$$

We want to use (31) to estimate the second term of (27). Towards that we note that $\overline{D} \times [-2\epsilon, 2\epsilon]$ is contained in $\phi_\epsilon \cup \psi_\epsilon$. Also, for any (x,t) which is not in ϕ_ϵ and with $|t| \le \epsilon$ we have

$$9\epsilon^2 \ge -\beta^2 t^2 + |x-a|^2 \ge -4\beta^2\epsilon^2 + |x-a|^2.$$

Hence $|x-a|^2 \le 13\epsilon^2$ so $|x-b| \ge |x-a|$ because $|a-b| \ge 8\epsilon$. So for such (x,t) we have $\phi(x,t) \le \psi(x,t)$ and of course $(x,t) \in \psi_\epsilon$. So from (31) used in (27) we conclude that

$$\int_{D \setminus B_{2\epsilon}(a)} e^{2\sigma\phi} q^2 + e^{2\sigma e^{9\lambda\epsilon^2}} \sigma \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + \sigma^2 |u|^2$$
$$\preccurlyeq \int_{-T}^{T}\int_D e^{2\sigma\phi}|\mathcal{L}u|^2 + \sigma e^{2\sigma} \int_{D,t=\pm T} |\partial u|^2 + \sigma^2|u|^2 + e^{C\sigma} \int_{-T}^{T}\int_{\partial D} |\partial u|^2 + |u|^2. \quad (32)$$

We have shown elsewhere using standard energy estimates that

$$\int_{t=\pm T}\int_D |\partial u|^2 + u^2 \preccurlyeq \int_{-2\epsilon}^{2\epsilon}\int_D |\partial u|^2 + \sigma^2|u|^2 + \int_{-T}^{T}\int_{\partial D} |\partial u|^2 + u^2.$$

So using this in (32) we obtain

$$\int_{D\setminus B_{2\epsilon}(a)} e^{2\sigma\phi} q^2 \preccurlyeq \int_{-T}^{T}\int_D e^{2\sigma\phi}|\mathcal{L}u|^2 + e^{C\sigma}\int_{-T}^{T}\int_{\partial D}|\partial u|^2+|u|^2. \tag{33}$$

Define $\rho(x) := e^{\lambda|x|^2}$ then $\phi(x,0)=\rho(x-a)$ and

$$\phi(x,t) = e^{-\lambda\beta^2t^2}e^{\lambda|x-a|^2} = (e^{-\lambda\beta^2t^2}-1)e^{\lambda|x-a|^2}+e^{\lambda|x-a|^2} \leq -(1-e^{-\lambda\beta^2t^2})+\rho(x-a).$$

Now $(\mathcal{L}u)(x,t)=q(x)r(x,t)$ with r bounded on $\overline{D}\times[-T,T]$, hence

$$\begin{aligned}\int_{-T}^{T}\int_D e^{2\sigma\phi}|\mathcal{L}u|^2 &\leq \int_D e^{2\sigma\rho(x-a)}|q|^2\int_{-T}^{T} e^{-2\sigma(1-e^{-\lambda\beta^2t^2})}|r|^2 \preccurlyeq \int_D e^{2\sigma\rho(x-p)}|q|^2\int_{-T}^{T}e^{-2\sigma(e^{1-\lambda\beta^2t^2})}\\ &= o(\sigma)\int_D e^{2\sigma\rho(x-a)}|q|^2\end{aligned}$$

by the Dominated Convergence Theorem. Hence (33) implies

$$\int_{D\setminus B_{2\epsilon}(a)} e^{2\sigma\rho(x-a)}|q|^2 \preccurlyeq o(\sigma)\int_D e^{2\sigma\rho(x-a)}|q|^2 + e^{C\sigma}\int_{-T}^{T}\int_{\partial D}|\partial u|^2+|u|^2. \tag{34}$$

Similarly

$$\int_{D\setminus B_{2\epsilon}(b)} e^{2\sigma\rho(x-b)}|q|^2 \preccurlyeq o(\sigma)\int_D e^{2\sigma\rho(x-b)}|q|^2 + e^{C\sigma}\int_{-T}^{T}\int_{\partial D}|\partial u|^2+|u|^2. \tag{35}$$

Since $|a-b|\geq 8\epsilon$, we have $\rho(x-a)\leq\rho(x-b)$ on $B_{2\epsilon}(a)$ and $\rho(x-b)\leq\rho(x-a)$ on $B_{2\epsilon}(b)$. Further $B_{2\epsilon}(a)\subseteq D\setminus B_{2\epsilon}(b)$ and $B_{2\epsilon}(a)\subseteq D\setminus B_{2\epsilon}(b)$. Hence (34), (35) imply

$$\int_D \max(e^{2\sigma\rho(x-a)},e^{2\sigma\rho(x-b)})|q|^2 \preccurlyeq o(\sigma)\int_D \max(e^{2\sigma\rho(x-a)},e^{2\sigma\rho(x-b)})|q|^2 + e^{C\sigma}\int_{-T}^{T}\int_{\partial D}|\partial u|^2+|u|^2.$$

Since $\rho\geq 0$, taking σ large enough we conclude that

$$\int_D |q|^2 \preccurlyeq \int_{-T}^{T}\int_{\partial D}|\partial u|^2+|u|^2$$

proving the theorem.

QED

4 Proof of Theorem 2

The proof is based on [TaWeb].

Proof Below t will be sometimes written as x_0 and v_j will mean $\partial_j v$. Define $v := e^{\sigma\phi}u$; we will show that there is a smooth function $h(x,t)$ so that for large enough σ

$$e^{2\sigma\phi}|\mathcal{L}u|^2 \succcurlyeq \sigma(|\partial v|^2 + \sigma^2|v|^2) + \sigma\partial\cdot E \tag{36}$$

where the constant is independent of u, σ, x, t and $E = [E_0, \cdots, E_n]$ with

$$E_k = (\epsilon_j\epsilon_k\phi_k v_j^2 - 2\epsilon_j\epsilon_k\phi_j v_j v_k) + \epsilon_k(hvv_k - \frac{1}{2}h_k v^2) - \sigma^2(\epsilon_j\epsilon_k\phi_k\phi_j^2 v^2). \tag{37}$$

Now $|E_i| \preccurlyeq |\partial v|^2 + \sigma^2|v|^2$, so integrating (36) over $\overline{\Omega}$ we get

$$\sigma\int_\Omega |\partial v|^2 + \sigma^2 v^2 \preccurlyeq \int_\Omega e^{2\sigma\phi}|\mathcal{L}u|^2 + \sigma\int_{\partial\Omega}|\partial v|^2 + \sigma^2 v^2. \tag{38}$$

However $v = e^{\sigma\phi}u$ and $u = e^{-\sigma\phi}v$ so $e^{\sigma\phi}\partial u = \partial v - \sigma\partial\phi v$ and $\partial v = e^{\sigma\phi}(\partial u + \sigma\partial\phi u)$. Hence

$$e^{2\sigma\phi}(|\partial u|^2 + \sigma^2 u^2) \preccurlyeq |\partial v|^2 + \sigma^2|v|^2 \preccurlyeq\preccurlyeq e^{2\sigma\phi}(|\partial u|^2 + \sigma^2 u^2)$$

with the constant independent of x, t, σ and u. Applying this to (38) we recover (15). So it remains to prove (36).

Since $u = e^{-\sigma\phi}v$ we have

$$\begin{aligned}
&e^{\sigma\phi}\partial_j u = e^{\sigma\phi}\partial_j(e^{-\sigma\phi}v) = (\partial_j - \sigma\phi_j)v = (\partial_j - \sigma\phi_j)(e^{\sigma\phi}u)\\
&e^{\sigma\phi}\partial_j^2 u = (\partial_j - \sigma\phi_j)(e^{\sigma\phi}\partial_j u) = (\partial_j - \sigma\phi_j)(\partial_j - \sigma\phi_j)v = v_{jj} + \sigma^2\phi_j^2 v^2 - 2\sigma\phi_j v_j - \sigma\phi_{jj}v.
\end{aligned}$$

Hence

$$\begin{aligned}
e^{\sigma\phi}\partial u &= \partial v - \sigma(\partial\phi)v,\\
e^{\sigma\phi}\Box u &= \sum\epsilon_j(v_{jj} + \sigma^2\phi_j^2 - 2\sigma\phi_j v_j - \sigma\phi_{jj}v)\\
&= \Box v + \sigma^2(\sum\epsilon_j\phi_j^2)v - 2\sigma\sum\epsilon_j\phi_j v_j - \sigma(\Box\phi)v.
\end{aligned}$$

Define the operators

$$P_\sigma := \Box + \sigma^2(\sum\epsilon_j\phi_j^2), \qquad Q := -\sum\epsilon_j\phi_j\partial_j, \qquad R := -\Box\phi.$$

then, for any function $h(x,t)$,

$$\begin{aligned}
e^{2\sigma\phi}|\Box u|^2 &= |P_\sigma v + 2\sigma Qv + \sigma Rv|^2 = |(P_\sigma v + 2\sigma Qv + \sigma hv) + \sigma(R-h)v|^2\\
&\succcurlyeq |P_\sigma v + 2\sigma Qv + \sigma hv|^2 - C\sigma^2|v|^2\\
&\geq 2\sigma P_\sigma v(2Qv + hv) + \sigma^2|2Qv + hv|^2 - C\sigma^2|v|^2\\
&\geq 2\sigma P_\sigma v(2Qv + hv) + 2\sigma^2|Qv|^2 - C\sigma^2|v|^2.
\end{aligned}$$

Hence

$$e^{2\sigma\phi}|\mathcal{L}u|^2 \succcurlyeq \sigma P_\sigma v(2Qv + hv) + \sigma^2|Qv|^2 - C\sigma^2|v|^2. \tag{39}$$

The first term on the RHS can be broken into the sum of the product of a second order term and a first order term, the product of a second order term and a zeroth order term, the product of a first order term and a zeroth order terms, and the product of two zeroth order terms. We examine each of these; we use the repeated summation notation below.

$$\begin{aligned}2(P_\sigma v-\sigma^2 \sum \epsilon_j\phi_j^2 v)\, Qv &= -2\epsilon_j v_{jj}\epsilon_k\phi_k v_k = -2\epsilon_j\epsilon_k\left((v_j\phi_k v_k)_j - v_j\phi_{jk}v_k - v_j\phi_k v_{jk}\right)\\ &= -2\epsilon_j\epsilon_k\left((v_j\phi_k v_k)_j - v_j\phi_{jk}v_k - \frac{1}{2}(\phi_k v_j^2)_k + \frac{1}{2}\phi_{kk}v_j^2\right)\\ &= -2(\epsilon_j v_j\epsilon_k\phi_k v_k)_j + 2\epsilon_j\epsilon_k\phi_{jk}v_j v_k + (\epsilon_k\phi_k\epsilon_j v_j^2)_k - \epsilon_k\phi_{kk}\epsilon_j v_j^2\\ &= (\epsilon_k\phi_k\epsilon_j v_j^2 - 2\epsilon_j\epsilon_k\phi_j v_j v_k)_k + 2\epsilon_j\epsilon_k\phi_{jk}v_j v_k - \Box\phi(\epsilon_j v_j^2).\end{aligned}$$

Also

$$\begin{aligned}(P_\sigma v - \sigma^2\epsilon_j\phi_j^2 v)h(x)v = \epsilon_j v_{jj}h(x)v &= \epsilon_j(v_j hv)_j - h\epsilon_j v_j^2 - \epsilon_j h_j v v_j\\ &= \epsilon_j(v_j hv)_j - h\epsilon_j v_j^2 - \frac{1}{2}\epsilon_j h_j(v^2)_j\\ &= \epsilon_j(hvv_j - \frac{1}{2}v^2 h_j)_j - h\epsilon_j v_j^2 + \frac{1}{2}(\Box h)v^2.\end{aligned}$$

Next we consider the term which is a product of a first order term and a zeroth order term.

$$\begin{aligned}2\epsilon_j\phi_j^2 v\, Qv &= -\, 2\epsilon_j\epsilon_k\phi_j^2\phi_k v v_k = -\epsilon_j\epsilon_k\phi_j^2\phi_k(v^2)_k\\ &= -\, \epsilon_j\epsilon_k(\phi_j^2\phi_k v^2)_k + \epsilon_j\epsilon_k(\phi_j^2\phi_k)_k v^2\\ &= -\, \epsilon_j\epsilon_k(\phi_j^2\phi_k v^2)_k + \Box\phi(\epsilon_j\phi_j^2)v^2 + 2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k v^2.\end{aligned}$$

Hence, taking $g = h + \Box\phi$ and $E = [E_1, \cdots, E_n]$ where

$$E_k = (\epsilon_j\epsilon_k\phi_k v_j^2 - 2\epsilon_j\epsilon_k\phi_j v_j v_k) + \epsilon_k(hvv_k - \frac{1}{2}h_k v^2) - \sigma^2(\epsilon_j\epsilon_k\phi_k\phi_j^2 v^2)$$

we have

$$\begin{aligned}P_\sigma v(2Qv + hv) =& \sigma^2\epsilon_j\phi_j^2 hv^2 + \left(2\epsilon_j\epsilon_k\phi_{jk}v_j v_k - \Box\phi\, \epsilon_j v_j^2\right) + \left((\Box h)\frac{v^2}{2} - h\epsilon_j v_j^2\right)\\ &+ \sigma^2 v^2\left(\Box\phi\, \epsilon_j\phi_j^2 + 2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k\right) + \partial\cdot E\\ =& 2\left(\epsilon_j\epsilon_k\phi_{jk}v_j v_k + \sigma^2 v^2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k\right) - g(\epsilon_j v_j^2 - \sigma^2 v^2\epsilon_j\phi_j^2) + \Box h\frac{v^2}{2} + \partial\cdot E. \quad (40)\end{aligned}$$

Hence from (39)

$$\begin{aligned}e^{2\sigma\phi}|\mathcal{L}u|^2 \succcurlyeq& \sigma\left(2\left(\epsilon_j\epsilon_k\phi_{jk}v_j v_k + \sigma^2 v^2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k\right) - g(\epsilon_j v_j^2 - \sigma^2 v^2\epsilon_j\phi_j^2)\right) + \sigma^2|Qv|^2\\ &+ \sigma\partial\cdot E - C\sigma^2|v|^2. \quad (41)\end{aligned}$$

The first, second, and third terms on the RHS of (41) are bilinear in the vector $(\partial v, \sigma v)$. So if we can find a constant $d > 0$ and a smooth function $g(x)$ on $\overline{\Omega}$ so that

$$2\left(\epsilon_j\epsilon_k\phi_{jk}\xi_j\xi_k + \tau^2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k\right) - g(\epsilon_j\xi_j^2 - \tau^2\epsilon_j\phi_j^2) + d(\epsilon_j\phi_j\xi_j)^2 > 0 \tag{42}$$

for all $(x,t) \in \overline{\Omega}$ and $(\xi,\tau) \in R^{n+1}$ with $|\xi|^2 + \tau^2 = 1$, then from (41), for large enough σ,

$$\begin{aligned} e^{2\sigma\phi}|\mathcal{L}u|^2 &\succcurlyeq \sigma(|\partial v|^2 + \sigma^2|v|^2) + \sigma\partial\cdot E - C\sigma^2|v|^2 \\ &\succcurlyeq \sigma(|\partial v|^2 + \sigma^2|v|^2) + \sigma\partial\cdot E, \end{aligned}$$

proving (36). So it remains to prove (42).

Below S will denote the unit sphere $\xi^2 + \tau^2 = 1$. Since ϕ satisfies the special condition we have (see subsection 2.1)

$$2\left(\epsilon_j\epsilon_k\phi_{jk}\xi_j\xi_k + \tau^2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k\right) > 0 \tag{43}$$

when $x \in \overline{\Omega}$, $(\xi,\tau) \in S$, $\epsilon_j\xi_j^2 - \tau^2\epsilon_j\phi_j^2 = 0$ and $\epsilon_j\phi_j\xi_j = 0$. Hence[1] we can find a $d > 0$ so that

$$2\left(\epsilon_j\epsilon_k\phi_{jk}\xi_j\xi_k + \tau^2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k\right) + d(\epsilon_j\phi_j\xi_j)^2 > 0$$

when $x \in \overline{\Omega}$, $(\xi,\tau) \in S$ and $\epsilon_j\xi_j^2 - \tau^2\epsilon_j\phi_j^2 = 0$.

Now fix an $x \in \overline{\Omega}$. Define the quadratic forms (in ξ, τ)

$$\begin{aligned} a(\xi,\tau) &:= 2\left(\epsilon_j\epsilon_k\phi_{jk}\xi_j\xi_k + \tau^2\epsilon_j\epsilon_k\phi_{jk}\phi_j\phi_k\right) + d(\epsilon_j\phi_j\xi_j)^2, \\ q(\xi,\tau) &:= \epsilon_j\xi_j^2 - \tau^2\epsilon_j\phi_j^2, \\ b_\lambda(\xi,\tau) &:= a(\xi,\tau) + \lambda q(\xi,\tau). \end{aligned}$$

Fix an $x \in \overline{\Omega}$; if we can find some constant λ so that $b_\lambda(\xi,\tau) > 0$ for all $(\xi,\tau) \in S$, then the same λ will work in a nbd of this x, hence using a partition of unity argument we could come up with a function g on $\overline{\Omega}$ which satisfies (42).

Fix an $x \in \overline{\Omega}$. Let Z_λ be the zero set of the quadratic form $b_\lambda(\xi,\tau)$ in $\mathbb{R}^{n+1}$ - then Z_λ is a collection of lines in (ξ,τ) space. We claim that Z_λ (or the zero set of any quadratic form) is projectively connected, that is, there is a continuously varying family of lines connecting any two lines in Z_λ. There is no loss of generality in assuming that the quadratic form is generated by a diagonal matrix with m ones, n minus ones, and k zeros - we will prove the result by induction on n. If $m = 0$ then it is trivial so we will assume that $m \geq 1$. If $n = 1$ then the zero set is a cone times $\mathbb{R}^k$ and hence projectively connected. If $n \geq 2$ and $x = (p,q,r)$ is in the zero set with $p \in \mathbb{R}^m, q \in R^n, r \in R^k$ then $|p|^2 = |q|^2$. We can find a $q' \in R^{n-1}$ so that $|q'|^2 = |q|^2$; also we can connect q to $(q', 0)$ by a curve on a ball of radius $|q|$. Hence the zero set is projectively connected to the zero set of a quadratic form with signature $m, n-1, k$ and this zero set is projectively connected by the induction hypothesis.

[1] The LHS of (43) is positive in some neighborhood $|\epsilon_j\phi_j\xi_j|^2 \leq \epsilon$; from compactness choose d large enough so that $d\epsilon$ exceeds the maximum of the LHS of (43) over $\overline{\Omega} \times S$.

Since Z_λ is connected and does not intersect the region $q = 0$ (by hypothesis), it is contained completely either in $q < 0$ or in $q > 0$. Now $a > 0$ on $q = 0$, hence $a > 0$ on a neighborhood of $q = 0$ in S - say the neighborhood $|q(\xi, \tau)| \leq \epsilon$ on S. Then, for $\lambda > \epsilon^{-1} \max a$ we have $b_\lambda > 0$ on the region $q > 0$. Similarly, for $\lambda << 0$, we have $b_\lambda > 0$ on the region $q < 0$. Hence for $\lambda >> 0$, Z_λ is contained in $q < 0$ and for $\lambda << 0$, Z_λ is contained in $q > 0$. We claim that this implies Z_λ is empty for some λ, that is for some λ, b_λ is never zero on S and hence has the same sign at every point on S. But $b_\lambda > 0$ on $q = 0$ so $b_\lambda > 0$ on S which would prove our claim. It remains to show that Z_λ is empty for some λ.

Suppose Z_λ is never empty - we show that the set of λ for which Z_λ is contained in $q > 0$ (similarly $q < 0$) is a closed set. This will contradict the connectedness of $\mathbb{R}$. Suppose $\lambda_n \to \lambda$ and Z_{λ_n} is contained in $q > 0$ for all n. So we have a sequence (ξ_n, τ_n) in S (which may be assumed to be convergent to some (ξ^*, τ^*) because of compactness) with $q(\xi_n, \tau_n) > 0$ and $b_{\lambda_n}(\xi_n, \tau_n) = 0$. Taking the limit we have $b_\lambda(\xi^*, \tau^*) = 0$ and $q(\xi^*, \tau^*) \geq 0$. But Z_λ is in $q > 0$ or $q < 0$ so Z_λ is in $q > 0$ - proving our claim.

QED

References

[BuKl81] Bukhgeim, A. L.; Klibanov, M. V. Uniqueness in the large of a class of multidimensional inverse problems. (Russian) Dokl. Akad. Nauk SSSR 260 (1981), no. 2, 269–272.

[Ho76] Hormander, Lars. Linear Differential Operators, Fourth Printing, Springer-Verlag (1976).

[Kr09] Krantz, Steven. Explorations in Harmonic Analysis, Birkhauser (2009). Also available electronically at `http://www.springerlink.com/content/978-0-8176-4668-4` .

[ImYa01] Imanuvilov, O. Y.; Yamamoto, M. Global uniqueness and stability in determining coefficients of wave equations. Comm Partial Differential Equations 26 (2001), no. 7-8, 1409-1425.

[Ta94] Tataru, D. A priori estimates of Carleman's type in domains with boundary. J. Math. Pures Appl. (9) 73 (1994), no. 4, 355–387.

[TaWeb] Tataru, Daniel. Notes on Carleman estimates, `http://www.math.berkeley.edu/%7Etataru/papers/ucpnotes.ps` .