Assignment 3
Math 810 - Spring 2003
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(1) Consider the perturbed eigenvalue problem

$$y'' + \lambda(1+\epsilon x)y = 0$$
$$y(0) = y(\pi) = 0.$$

Assume $\epsilon \ll 1$ and find a leading order approximation to the eigenfunctions and the first *two* terms in the expansion of the eigenvalues. (Hint: You need to expand both y and λ and use the Fredholm alternative theorem.)

(2) Consider the problem of steady-state heat conduction in a rod of length L

$$\kappa \frac{d^2T}{dx'^2} + s(x') = 0$$

where here, κ is the thermal conductivity and $s(x')$ is a source term. Assume that $s(x') = p$ for $0 \le x' \le x_0$ and is identically zero for $x_0 < x' < L$. As boundary conditions, assume that the rod is insulated on the left so that $T'(0) = 0$ and is in an ice bath on the right so that $T(L) = T_b$.

(a) Introduce dimensionless variables and scale your equations.

(b) Assuming a *strong* source argue that a dimensionless parameter from part (a) is small.

(c) Using boundary layer theory find a uniform approximation to the solution.

(3) Consider the nonlinear oscillator equation with a "soft" nonlinearity

$$u'' + u - \epsilon u^3 = 0$$

Impose initial conditions

$$u(0) = 0$$
$$u'(0) = a$$

(a) Using an energy argument, derive a potential for this problem. Sketch the potential well. What conditions must you impose on a in order to guarantee that you have a bounded (and hence periodic) solution?

(b) Assume $\epsilon \ll 1$ and expand the solution to this problem in a regular perturbation series. Compute at least the first two terms. Is your expansion uniformly valid? Why or why not?

(c) Assume $\epsilon \ll 1$ and apply the Poincare-Lindstead method to this problem. Determine at least the first correction to the frequency.

(d) Solve this problem numerically and compare the approximations obtained in parts (b) and (c) with a numerical solution. Discuss your findings.