(1) Solve $t\dfrac{dy}{dt} + 2y = 4t^2$

$$\Rightarrow \quad \frac{dy}{dt} + \frac{2}{t}y = 4t$$

An integrating factor is $\exp\left(\int \frac{2}{t}\right) = e^{2\log t} = t^2$

$\Rightarrow$

$$t^2 y' + 2ty = 4t^3$$

$$\Rightarrow \quad \frac{d}{dt}\left(t^2 y\right) = 4t^3$$

$$\Rightarrow \quad t^2 y = t^4 + c \quad \Rightarrow \quad y(t) = t^2 + \frac{c}{t^2}$$

2. Solve $\frac{dy}{dt} = y(1-y)$ $\quad y(0) = a$

Separate variables $\Rightarrow$ $\frac{dy}{y(1-y)} = dt$

$$\Rightarrow \frac{dy}{y} + \frac{dy}{1-y} = dt \Rightarrow \frac{dy}{y} - \frac{dy}{y-1} = dt$$

$$\Rightarrow \int \frac{dy}{y} - \int \frac{dy}{y-1} = \int dt$$

$$\Rightarrow \log y - \log(y-1) = t + c$$

$$\Rightarrow \log \frac{y}{y-1} = t + c \Rightarrow \frac{y}{y-1} = e^{t+c}$$

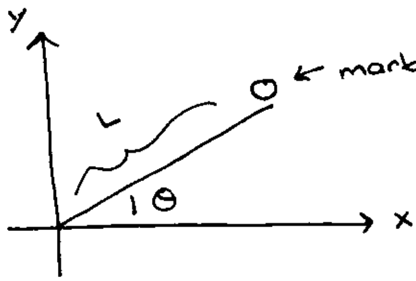
$$\Rightarrow \frac{y}{y-1} = c_1 e^t \Rightarrow y = c_1 e^t y - c_1 e^t$$

$$\Rightarrow y(1 - c_1 e^t) = -c_1 e^t \Rightarrow y(t) = \frac{-c_1 e^t}{1 - c_1 e^t}$$

$$y(0) = a = \frac{-c_1}{1 - c_1} \Rightarrow a(1-c_1) = -c_1 \Rightarrow a - c_1 a = -c_1$$

$$\Rightarrow a = c_1(a-1) \Rightarrow c_1 = \frac{a}{a-1} \Rightarrow \boxed{y(t) = \frac{\frac{a}{1-a} e^t}{1 + \frac{a}{1-a} e^t}}$$

3.



← marble at ~~(L sin θ, L cos θ)~~ $(L\cos\theta, L\sin\theta)$

The y-position satisfies $m\ddot{y} = -mg \Rightarrow \ddot{y} = -g$

with $y(0) = L\sin\theta$ $\Rightarrow$ $\dot{y} = -gt + \cancelto{0}{c}$

$\dot{y}(0) = 0$

$$\Rightarrow \quad y(t) = -\frac{gt^2}{2} + c_1 \quad , \quad c_1 = L\sin\theta$$

$$\Rightarrow \quad y(t) = -\frac{gt^2}{2} + L\sin\theta,$$ reaches the bottom at $t = t_c$

$$\Rightarrow \quad 0 = -\frac{gt_c^2}{2} + L\sin\theta \quad \Rightarrow \quad t_c = \left(\frac{2L}{g}\sin\theta\right)^{1/2}$$

$$\frac{dt_c}{d\theta} = \frac{1}{2}\left(\frac{2L}{g}\sin\theta\right)^{-1/2}\frac{2L}{g}\cos\theta = 0$$

$\Rightarrow$ $\theta = \pi/2$ minimizes travel time.

6. Show that $2y^2 \log(cy) - x^2 = 0$ solves $(x^2+y^2)y' = xy$.

Consider $\frac{d}{dx}\left(2y^2 \log cy - x^2\right) = 0 \quad \Rightarrow$

$$4yy' \log cy + \frac{2y^2}{\cancel{c}y} y'\cancel{c} - 2x = 0$$

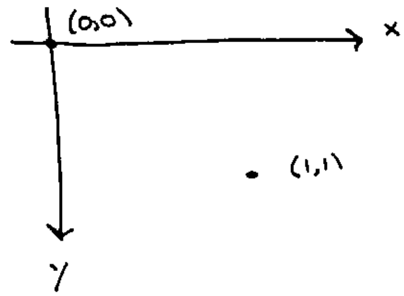
$$\Rightarrow \quad 2y^2y' \log cy + y^2y' - xy = 0$$

$$\Rightarrow \quad (\underbrace{2y^2 \log cy + y^2}_{= x^2+y^2})\,y' = xy$$

$$\Rightarrow \quad (x^2+y^2)\,y' = xy \quad \text{as desired.}$$

9. Proof is in your text.

7.



$$T[y(x)] = \int_0^1 \sqrt{\frac{1+y'^2}{y}}\, dx$$

If the path is a straight line, $y = x$ $\Rightarrow$

$$T[x] = \int_0^1 \frac{\sqrt{2}}{x^{1/2}}\, dx = 2\sqrt{2}\, x^{1/2}\Big|_0^1 = 2\sqrt{2}$$

If the path is the circle $(x-1)^2 + y^2 = 1$

$$\Rightarrow \quad y = \sqrt{1-(x-1)^2} \quad \Rightarrow \quad y' = \frac{-\cancel{2}}{\cancel{2}\sqrt{1-(x-1)^2}} = \frac{-1}{\sqrt{1-x^2+2x-1}}$$

$$= \frac{-1}{\sqrt{2x-x^2}} \quad \Rightarrow \quad y'^2 = \frac{1}{2x-x^2} \quad \Rightarrow$$

$$T = \int_0^1 \sqrt{\frac{1+\frac{1}{2x-x^2}}{\sqrt{2x-x^2}}}\, dx$$, do numerically.