Assignment 3
Math 503 - Fall 2006
Prof. J. A. Pelesko
Due September 25, 2006

(1) (10 points) In class we proved that a necessary condition for a functional to have an extrema is that its first variation, δJ, vanish at the extrema. Here, I ask you to parallel the proof in a simpler setting and prove a similar result for functions. Proceed as follows. First, we define an extrema of a function $f(x)$, by saying that a point x_0 is an extrema iff $f(x)$ has one sign in a neighborhood of that point. Now, assume that $f(x)$ is smooth so that it has a Taylor series expansion wherever you wish. Use the Taylor series expansion to prove that if x_0 is an extrema then $f'(x_0) = 0$. Again, you should parallel the proof in class.

(2) (20 points) A catenoid spans two circular rings. Each ring has radius 2cm. What is the largest distance, L, the rings can be apart and there still be a catenoid between them? Use Maple to make a 3d sketch of this surface.

(3) (30 points) Consider two circular *disks* of radius a. Imagine a catenoid-like surface spans the two disks, but since the disks are solid, imagine that the volume contained between the two disks and the surface can be controlled. That is, the enclosed volume is a constraint in this problem.

(a) Formulate the variational problem for this system and apply the appropriate Euler-Lagrange equation to derive an ODE for the surface.

(b) Show that when the volume constraint is removed, your ODE reduces to the normal catenoid equation.

(c) Show that there is a particular value of the volume for which a cylinder of radius a is a solution to your ODE.

(4) (15 points) Find the shape of the brachistochrone that connects the fixed point $(0, 1)$ to the line $x = 1$.

(5) (25 points) Fermat's principle of least time says that the path that light follows through a medium with varying index of refraction is the one that minimizes its total travel time. Assume that the speed of light in a given medium is a continuous function of one variable, i.e., the speed is given as $u(y)$. If a light ray originates at the point (x_1, y_1) and is observed at (x_2, y_2), write down the functional which when minimized gives the path followed by the ray of light. Now, assume that $u(y) = ay$ where a is a constant and solve for the path of the ray. Plot your result.