Assignment 2
Math 503 - Fall 2006
Prof. J. A. Pelesko
Due September 18, 2006

(1) (5 points) Given the linear space $C(0,1)$, does

$$||f|| = \sqrt{\int_0^1 f^2 dx}$$

define a valid norm?

(2) (5 points) Find the extremals of

$$J[y(x)] = \int_0^{\pi/2} (y'^2 - y^2) dx$$

(3) (15 points) Derive the Euler-Lagrange equation for

$$J[y] = \int_a^b F(x, y, y', y'', y''', y'''') dx$$

(4) (15 points) Formulate the following problem mathematically: Find a curve of length L joining the points A and B in the $x - y$ plane of such shape that its center of gravity is as low as possible. (Note: you do not need to solve this problem, simply formulate.)

(5) (15 points) Consider the set

$$S = \{(u(x), a) : u(x) \text{ continuous on}[0,1], a \text{ is a real number}\}$$

First, show that this is a linear space. Next, construct a norm for this space and show that your norm satisfies the requirements in the definition of a norm.

(6) (15 points) Write down the functional $J[y(x)]$ which gives the arc length of the curve joining the points $(0,0)$ and $(1,0)$. Write down an equation which describes a one-parameter family of parabolas whose zeros correspond to these two points. Call your parameter, a, and denote this family by $y(x;a)$. Insert this into your functional and plot $J[y(x;a)]$ as a function of a. Use a computer to do this! By inspection of the graph, determine the value of a that minimizes your functional. To which parabola does the minimum correspond?

(7) (15 points) In class, we will show that a hanging chain hangs in the shape of a catenary. In this problem, I want you to consider the shape of a hanging *elastic* string. The key difference between the string and the chain is that the string can stretch while the chain's length is fixed. In this problem, assume the ends of the string are at $(0, a)$ and (L, b). Assume the un-stretched length of the string is L.

(a) Derive an energy functional, $E[y]$, for this system by assuming that the total energy is simply the sum of the elastic energy and gravitational energy. Assume that the elastic energy is proportional to the change in length of the string from its un-stretched configuration.

(b) In your energy functional you should have a term $\sqrt{1 + y'^2}$. Linearize your energy functional by Taylor expanding this term about $y'^2 = 0$ and discarding all but the first two terms in the Taylor expansion.

(c) Minimize your linearized functional and find the shape of the hanging string.

(8) (15 points) Find the general solution to the Euler-Lagrange equation for

$$J[y(x)] = \int_a^b f(x)\sqrt{1 + y'^2}dx.$$

Find the solution for the special cases where $f(x) = x^{1/2}$ and $f(x) = x$.