Assignment 1
Math 503 - Fall 2006
Prof. J. A. Pelesko
Due September 8, 2005

(1) (5 points) Solve the following first order linear ordinary differential equation

$$t\frac{dy}{dt} + 2y = 4t^2 \tag{1}$$

(2) (5 points) Solve the following initial value problem

$$\frac{dy}{dt} = y(1-y) \tag{2}$$
$$y(0) = a \tag{3}$$

(3) (10 points) A marble is placed on a frictionless track of length L. The track is pinned at the origin and makes an angle θ with the x-axis. At what angle will a marble, released from rest, reach the origin in the least possible time? Justify your answer.

(4) (20 points) In class, we will show that the solution to the brachistochrone problem is given by the solution to the differential equation

$$y(1 + (\frac{dy}{dx})^2) = c \tag{4}$$

where c is a constant. Solve this problem by proceeding as follows:

1. Separate variables in the ODE.
2. Introduce a new variable ϕ by letting $\tan(\phi) = (\frac{y}{c-y})^{1/2}$.
3. Integrate the new ODE with respect to ϕ to obtain x as function of ϕ, you should have two unknown constants, c, and a new constant of integration that we will call c_1.
4. Require that your curve pass through the origin and hence conclude $c_1 = 0$.
5. Let $a = c/2$ and $\theta = 2\phi$ and obtain the standard parametric equations of a cycloid.
6. Plot the cycloid and explain its relationship to a rolling circle.

(5) (15 points) A marble is released from rest at the point $(1, 1)$ in the $x - y$ plane. Gravity points along the negative y-axis. You may place two sections of straight frictionless track in any way you wish such that they connect the origin to the point $(1, 1)$. How should the track be placed so that the marble rolls along the track and reaches the origin as fast as possible?

(6) (5 points) Show that the differential equation

$$(x^2 + y^2)y' = xy$$

has as its solution $2y^2 \log(cy) - x^2 = 0$. (Hint: Remember implicit differentiation?)

(7) (10 points) A bead starting at the origin and sliding down a curved frictionless track ending at the point $(1, -1)$ will make the journey in time T given by

$$T[y(x)] = \int_0^1 \sqrt{\frac{1 + y'^2}{y}} dx$$

If the path is a straight line, how long will the journey take? If the path is the arc of a circle centered at $(1, 0)$, how long will the journey take?

(8) (15 points) Prove that any norm defined on a linear space R is a continuous functional on R.

(9) (15 points) Prove that if $\alpha(x)$ is continuous on $[a, b]$ and if

$$\int_a^b \alpha(x)h''(x)dx = 0 \quad (5)$$

for every function $h(x) \in D_2(a, b)$ such that $h(a) = h(b) = 0$ and $h'(a) = h'(b) = 0$ then $\alpha(x) = c_0 + c_1 x$ for all x in $[a, b]$. (Here c_0 and c_1 are constants.)