Homework 1
Math 352 - Spring 2003
Prof. John A. Pelesko

This assignment is to be handed in on 3/1. You are encouraged to work together, but the work you hand in must be your own.

1. (Scale changes). Show that if $\mathcal{L}\{f(t)\} = F(s)$, then
   (a) $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$, and
   (b) $\mathcal{L}^{-1}\{F(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right)$.
2. Using the definition of the Laplace transform find
$$\mathcal{L}\left[e^{at}\cos(at)\right].$$
3. From the Laplace transform of $\sin(at)$ and of e^{-bt}, find the Laplace transform of
   (a) $\cos(at)$
   (b) $\cos^2(at)$
4. Compute the inverse Laplace transform of
$$\frac{5}{s^7} + \frac{7}{s-a}.$$
5. Compute the inverse Laplace transform of
$$\frac{2s+5}{s^2-25}.$$
6. Compute the inverse Laplace transform of
$$\frac{2s+14}{(s-2)^2+36}.$$
7. Using the Laplace transform solve
$$y'' + 9y = 12\cos(3t)$$
$$y(0) = 2 \quad y'(0) = 5$$

8. Using the Laplace transform solve

$$y + \int_0^t y(\tau)d\tau = \sin(2t).$$

9. Using the Laplace transform solve

$$\begin{aligned} y'' + 3ty' - 6y &= 1 \\ y(0) = y'(0) &= 0. \end{aligned}$$

10. Using the Laplace transform solve and sketch the solution of

$$\begin{aligned} x' - x = e^{-t}H(t-3) \\ x(0) = 0. \end{aligned}$$

11. Using the Laplace transform solve and sketch the solution of

$$\begin{aligned} x'' + 2x' + x = 10\delta(t-5) \\ x(0) = x'(0) = 0. \end{aligned}$$

12. Let $N(t)$ be the population of a bacterial colony at time t. Let $R(t) = \frac{dN}{dt}$, be the rate of change of the population. In the simplest model of population growth, $R(t) = rN(t)$; so that the production rate is proportional to the total population. In a model which includes age structure in the population, the number of individuals born between times τ and $\tau + \delta\tau$ is $R(\tau)\delta\tau + 0\big((\delta\tau)^2\big)$. At time t, these individuals reproduce at growth rate $\lambda(t-\tau)$ depending on their age $t-\tau$. Hence
$R(t) = N_0\lambda(t) + \int\limits_0^t \lambda(t-\tau)R(\tau)d\tau$, where N_0 = initial population.

(a) Assuming $\tilde{R}(s)$ exists, show

$$\tilde{R} = \frac{N_0\tilde{\lambda}}{1-\tilde{\lambda}}$$

(b) Show that when $\lambda = r$ (constant), $R = N_0 r e^{rt}$ and
$N = N_0 e^{rt}$

(c) Find $R(t)$ and $N(t)$ when $\lambda = e^{-kt}$ (productivity dies exponentially with age).