Exam #3
Math 243

This is exam #3 of the Spring 2004 semester. You must show all of your work and be sure to clearly indicate your final answer. Good luck! You may make use of the Maple guide found at: http://www.math.lsa.umich.edu/courses/215/12maple/refcard.pdf

(1) (20 points) Evaluate $\int\int_D (y^2 - x^2) dA$ where D is the region between $y = 2x^2$ and $y = 2 - x^2$.

The answer is found by finding the points of intersection of the two curves and writing as an iterated integral. We have

$$\int\int_D (y^2 - x^2) dA = \int_{-\sqrt{2/3}}^{\sqrt{2/3}} \int_{2x^2}^{2-x^2} (y^2 - x^2) dy dx = \frac{208\sqrt{6}}{189} \tag{1}$$

(2) (a) (15 points) Consider the vector function

$$\vec{F}(x,y) =< x, \frac{1}{y} > .$$

Find $\int_C \vec{F} \cdot d\vec{r}$ where C is the upper half of the circle $(x-5/2)^2+(y-1)^2 = 1/4$ and the circle is traversed counterclockwise.

(b) (5 points) Find $\int_C \vec{F} \cdot d\vec{r}$ where C is the entire circle $(x-5/2)^2+(y-1)^2 = 1/4$ and it is traversed once clockwise.

The vector field is conservative, hence the answer to (b) is zero. The potential function is easy to construct, we find

$$f(x,y) = \frac{x^2}{2} + \log(y) \tag{2}$$

hence part (a) is

$$\int_C \vec{F} \cdot d\vec{r} = f(x,y)|_{(3,1)}^{(2,1)} = 5/2 \tag{3}$$

(3) (20 points) Find the volume of the region under the paraboloid $z = 2x^2 + 5y^2$ and over the rectangle $[0, 2] \times [1, 3]$.

To solve, we use a triple integral

$$V = \int_0^2 \int_1^3 \int_0^{2x^2+5y^2} dzdydz = \frac{292}{3}. \tag{4}$$

(4) (20 points) Consider the vector field $\vec{F} =< z, x, y >$. Suppose a particle moves along the helical path $\vec{r}(t) =< 3\cos(t), t, 3\sin(t) >$. Find the work done by the force field.

The vector $\vec{r}'(t) =< -3\sin(t), 1, 3\cos(t) >$ and the work is given by

$$W = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt = \int_0^{\pi} (-9\sin^2(t) + 3\cos(t) + 3t\sin(t))dt = -\frac{9\pi}{2} - 6 \tag{5}$$

(5) (20 points) Consider the vector field $\vec{F} =< 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 >$. Is this vector field conservative? If so, construct a potential such that $\vec{F} = \nabla f$.

The vector field is conservative as can be seen by constructing the potential

$$f(x, y, z) = x^2y^3z^4. \tag{6}$$