Electrostatic Deflections of Circular Elastic Membranes

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Abstract

Electrostatically deflected elastic systems are studied in areas as diverse as optical switching and cloud electrification. All such systems exhibit an instability commonly known as the "pull-in" or "snap-down" instability. In this instability, when applied voltages are increased beyond a certain critical voltage, there is no steady-state configuration of the system where mechanical members remain separate. The widespread use of electrostatic actuation in the design of micro and nanoelectromechanical systems (MEMS and NEMS) has rekindled an interest in this instability. Here a mathematical model of a disk shaped electrostatically actuated system is studied. This model represents a configuration typical of many MEMS/NEMS systems. Solutions to the model are examined for the case of a prescribed voltage drop across the system and for the case where the system is embedded in a capacitive control circuit. The curve of solutions, or bifurcation diagram, is shown to bend back upon itself repeatedly implying the existence of multiple steady-state configurations of the system. The effect of the control scheme is studied by examining the effect on the bifurcation diagram.

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1 Introduction

The study of electrostatic actuation began in the late 1960s. In a series of elegant experiments, performed when he was 82 years old, the prolific British scientist G.I. Taylor investigated the coalescence of liquid drops held at differing electric potentials, [1]. Taylor's experimental setup is equivalent to the system sketched in Figure 1; in his system the role of the elastic membrane is played by a soap film. Taylor observed that when the applied potential difference was increased beyond a critical voltage, the two drops would coalesce or in the context of Figure 1, the membrane would collapse onto the bottom plate. Further, for applied voltages just below the critical voltage, the two drops remained a considerable distance apart.

Taylor's experiments were mathematically modeled by Ackerberg, [2]. Ackerberg exploited the small aspect ratio of Taylor's experimental setup to compute an approximate electrostatic potential. Again in the context of Figure 1, Ackerberg assumed the gap between the disks was small compared to disk radius, ignored fringing fields and treated the membrane and plate as if they remained parallel. Assuming cylindrical symmetry these simplifications allowed Ackerberg to reduce the problem to the study of a single nonlinear ordinary differential equation for the deflection of the membrane. Proceeding numerically, Ackerberg discovered the surprising result that for certain values of the applied voltage there were infinitely many steady-state configurations of the system.

At about the same time as Taylor, one of the earliest MEMS researchers, H.C. Nathanson, was investigating electrostatic actuation as a method for designing a resonant gate transistor, [3]. In order to understand the behavior of their device, Nathanson et. al., constructed a mass-spring model of electrostatic actuation. The same instability observed by Taylor was uncovered and it was Nathanson who introduced the name "pull-in" instability. As the field of MEMS has matured and the field of NEMS has arisen the importance of understanding electrostatic actuation and the "pull-in" instability has grown. The design of numerous MEMS/NEMS devices is limited by this instability. Typical is the design of a varactor, [4], where the achievable capacitance range is limited by the instability.

In the MEMS and NEMS communities, many authors, such as Chu et al., [5], have followed Nathanson and relied upon mass-spring models to analyze various features of electrostatically actuated MEMS/NEMS. Other authors, [6, 7, 8, 9], have examined the pull-in instability using direct numerical simulation while more recently a full analysis of a 2-d device has been carried out, [10]. A common feature of these models is the device geometry analyzed and the subsequent implications for solutions to the model equations. In particular, mass-spring models rely upon the assumption that parallel members remain parallel through all deflections. As a result, these models predict that for the range of applied voltages, $[0, V_c]$, two steady-state configurations of the device exist while beyond V_c, no solution exists. Hence, V_c is identified as the pull-in voltage. This theory has been at least partially verified experimentally by Chu et al., [5], for a device that closely approximates the mass-spring scenario. A model of an elastic strip, suspended at two opposite edges and free along the others was investigated by Bernstein et al., [10]. This analysis showed that for the strip geometry where parallel surfaces became non-parallel, the features of the mass-spring theory still persist. Numerical simulations, [6, 7, 8, 9], of strip-like or mass-spring-like devices have also confirmed these results.

Still other researchers have turned their attention towards devising control schemes for the pull-in instability. Chu and Pister, [5], proposed a voltage control algorithm, while Seeger and Crary, [11, 12], and Chan and Dutton, [13], studied capacitive control schemes. In [11], Seeger and Crary analyzed a control scheme which placed a fixed capacitor in series with a MEMS device by studying a mass-spring model. In the course of their analysis, nonlinear terms arising from the electrostatic force and from the additional capacitance were found to cancel. This led to the result that the scheme could stabilize a device over the entire range of motion. In [13], 2-d effects were included in a mass-spring model through the addition of parasitic capacitance's to the circuit. Mathematically, this removed the cancellation of nonlinear terms seen by Seeger and Crary and led to the conjecture that the control scheme is only partially stabilizing. In [14], this conjecture was verified for a strip geometry, similar to that studied in [10]. A common feature of these models is the geometry and the implications for solutions to the model equations. In all of these studies, there was shown to exist either zero, one or two steady-state solutions; nowhere in the MEMS/NEMS literature is the multiplicity result of Ackerberg studied.

In this paper, we analyze a mathematical model of a disk shaped electrostatically actuated system similar to that studied by Ackerberg, [2]. In particular, our device consists of a circular elastic membrane suspended above a rigid plate. The membrane is supported along the entire circumference of the disk. Both the case of a simple applied voltage and of a capacitive control scheme are studied. The governing equations are generalizations of those presented in [10, 14]; the generalization being from a strip to a disk geometry. Instead of using numerical methods, like Ackerberg, we use symmetry methods and rigorously investigate solutions of the mathematical model. Further, our study of the capacitive control schemes of Seeger and Crary, [11, 12], in the context of the disk model require the study of a nonlocal version of Ackerberg's model. Our analysis of the nonlocal system yields insight into how the proposed control scheme partially stabilizes the system.

2 Analysis of the model

We consider an idealized device consisting of a circular elastic membrane suspended above a rigid circular plate. The device is placed in series with a fixed voltage source and a fixed capacitor. The geometry is shown in Figure 1 while the circuit is shown in Figure 2. This circuit is the control circuit proposed and studied by Seeger and Crary in [11, 12] for a mass-spring model and by Pelesko and Triolo in [14] for an elastic strip. Our governing equations in dimensionless form are

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \frac{\beta}{(1+u)^2(1+2\pi\chi\int_0^1 \frac{rdr}{1+u(r)})^2} \tag{1}$$

$$u(1) = 0 \tag{2}$$

$$\frac{du}{dr}(0) = 0. \tag{3}$$

Here, β is a dimensionless parameter proportional to the square of the voltage drop across the device, χ is a ratio of the undeflected capacitance of the device to the capacitance of the fixed capacitor in the circuit while u measures the deflection of the membrane. The radius of the disk and the undeflected gap width between membrane and plate have both been scaled to one. Additionally, an asymptotic solution to the electrostatic potential equation has been used, allowing us to rewrite the forcing term in equation (1) entirely in terms of the membrane's deflection. For a full derivation of these equations, including all scaling and the asymptotics of the potential equation, the reader is referred to [10, 14]. We also note that if we set χ to zero, the integral term in equation (1) drops out. This corresponds to removing the fixed capacitor from the circuit shown in Figure 2.

We observe that our governing equations are both nonlinear and *nonlocal*. We can effectively remove the nonlocal terms by considering the related system

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \frac{\lambda}{(1+u)^2} \tag{4}$$

$$u(1) = 0 \tag{5}$$

$$\frac{du}{dr}(0) = 0 \tag{6}$$

and then noting that any solution of equations (1)-(3) is a solution of equations (4)-(6) for

$$\lambda = \frac{\beta}{(1 + 2\pi\chi \int_0^1 \frac{rdr}{1+u(r)})^2} \tag{7}$$

while any solution of equations (4)-(6) is a solution of equations (1)-(3) for

$$\beta = \lambda(1 + 2\pi\chi \int_0^1 \frac{rdr}{1+u(r)})^2. \tag{8}$$

Hence, it is sufficient to study the local problem, equations (4)-(6). We also note that the local problem is precisely our model for the case $\chi = 0$, i.e. the case where the fixed capacitor is removed from the circuit.

Next, we note that the local problem exhibits a scale invariance that allows us to obtain solutions from a related initial value problem. That is, $u(r)$ is a solution to equations (4)-(6) if and only if

$$u(r) = -1 + \alpha w(\gamma r) \tag{9}$$

where

$$\alpha = \frac{1}{w(\gamma)} \tag{10}$$

$$\frac{\lambda}{\gamma^2\alpha^3} = 1 \tag{11}$$

and w satisfies the initial value problem

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = \frac{1}{w^2} \tag{12}$$

$$w(0) = 1 \tag{13}$$

$$\frac{dw}{dr}(0) = 0. \tag{14}$$

With these identifications, the bifurcation diagram for the local problem, equations (4)-(6), is parameterized in terms of γ and w. That is, to understand solutions, we wish to plot λ versus $||u||_\infty = -u(0)$. But from equations (9)-(11) we have $\lambda = \frac{\gamma^2}{w(\gamma)^3}$ and $||u||_\infty = 1 - \frac{1}{w(\gamma)}$.

It is easy to numerically integrate our initial value problem and use the result to compute the complete bifurcation diagram for our local problem, equations (4)-(6). The result of such a computation is shown in Figure 3. We note that as $r \to \infty$, $w(r)$ and hence λ and u change very little. This makes it difficult to compute the bifurcation diagram as it approaches the barrier $||u||_\infty = 1$. Hence, it is useful to study the equations for $w(r)$ analytically. To proceed, we change variables in equation (12) by setting $\eta = \log(r)$ and $w(r) = r^{2/3}v(\eta)$. This yields the autonomous equation

$$\frac{d^2v}{d\eta^2} + \frac{4}{3}\frac{dv}{d\eta} + \frac{4}{9}v - \frac{1}{v^2} = 0. \tag{15}$$

We may rewrite as a first order system

$$\frac{dv}{d\eta} = h \tag{16}$$

$$\frac{dh}{d\eta} = -\frac{4}{3}h - \frac{4}{9}v + \frac{1}{v^2}.$$

We observe that this system has a single critical point located at $v = (\frac{9}{4})^{1/3}, h = 0$. Linearizing about this critical point we find that the linear system has eigenvalues

$$\mu = -\frac{2}{3} \pm \frac{2\sqrt{2}i}{3} \tag{17}$$

and hence this point is a stable spiral. In fact, using the Lyapanov function

$$V(v,h) = \frac{h^2}{2} + \frac{2}{9}v^2 + \frac{1}{v} - (\frac{4}{9})^{1/3} - \frac{2}{9}(\frac{9}{4})^{2/3} \tag{18}$$

it is easy to see that our critical point is the global attractor for the system. We note that we may restrict our attention to the physical region of phase space, $v > 0$. This is clear since equations (4)-(6) immediately imply that

$w(r) > 1$ for all r and hence $v > 0$. The large η asymptotics for $v(\eta)$ are now simple to obtain, we find

$$v(\eta) \sim (\frac{9}{4})^{1/3} + Ae^{-2\eta/3}\cos(\frac{2\sqrt{2}}{3}\eta + B) + o(e^{-2\eta/3}) \qquad \text{as} \quad \eta \to \infty. \quad (19)$$

This allows us to deduce the large r asymptotics for $w(r)$ and we find

$$w(r) \sim (\frac{9}{4})^{1/3} r^{2/3} + A\cos(\frac{2\sqrt{2}}{3}\log(r) + B) + o(1) \qquad \text{as} \quad r \to \infty. \quad (20)$$

Finally, we may use the behavior we have obtained for $w(r)$ as $r \to \infty$ to deduce the behavior of the bifurcation diagrams for equations (4)-(6) and (1)-(3). First, from

$$||u||_\infty = 1 - \frac{1}{w(\gamma)} \quad (21)$$

and equation (20) we see that $||u||_\infty \to 1$ as we monotonically let $\gamma \to \infty$, while from

$$\lambda = \frac{\gamma^2}{w(\gamma)^3} \quad (22)$$

and (20) we see that $\lambda \to 4/9$ as $\gamma \to \infty$. We also observe that the curve of solutions must oscillate infinitely many times as it heads to the point $\lambda = 4/9$, $||u||_\infty = 1$. A close up view of this part of the bifurcation diagram is plotted in Figure 4. Similarly, we may use the asymptotic result for w together with the mapping, equation (7), to deduce the behavior of the bifurcation diagram for the nonlocal problem. We find that as $\gamma \to \infty$ we have $\beta \to \frac{4}{9}(1 + \frac{3\pi\chi}{2})^2$. To deduce that β oscillates as $\gamma \to \infty$ it is necessary to obtain the first two terms in the asymptotic expansion of the integral appearing in the mapping, equation (7), for $\gamma \to \infty$. This is done in Appendix A. A bifurcation diagram for the controlled problem is sketched in Figure 3.

3 Discussion

In order to understand the implications of our analysis it is useful to compare the bifurcation diagram for the disk with the bifurcation diagram of an infinite strip. In Figure 5 we sketch a family of curves for the uncontrolled and controlled strip. In Figure 5, the uncontrolled case corresponds to the $\chi = 0$ curve, while the remaining curves correspond to control.

First, let us compare the bifurcation diagrams for the uncontrolled strip and disk. That is, consider Figures 3,5. In Figure 5, we see that the bifurcation diagram consists of a single fold and hence for positive values of β less than some critical value β^*, two solutions exist. As β approaches β^* these two solutions merge and disappear at β^*. Hence, the critical value β^* corresponds to the pull-in voltage and the pull-in instability is characterized in terms of this bifurcation diagram. Now, in Figure 3, we see a similar, but also strikingly different behavior. Clearly, there is a critical value, β^*, beyond which no solutions exist. This critical value once again corresponds to the pull-in voltage. However, for $\beta < \beta^*$, the story is quite different. There are values of β for which one, two, three or in fact any number of solutions exist! At $\beta = 4/9$ there are infinitely many solutions as unveiled by our asymptotic results in the previous section. In Figure 6, we sketch three of these solutions for a fixed value of β for which more than two solutions exist. We note the change in concavity of the solutions as we move from branch to branch.

Next, let us compare the bifurcation diagrams for the strip and disk when embedded in the capacitive control scheme. In Figure 5, the diagram is sketched for the strip for various values of χ. We see that as χ is increased, the "nose" of the curve moves up and to the right. This implies that the capacitive scheme is partially stabilizing. That is the achievable maximum displacement has increased. The cost of this is that a higher voltage is now necessary to obtain equivalent displacements. In Figure 3, we also sketch the bifurcation diagram for the disk for various values of χ. We know from our results in the previous section that the limiting value of β as we march along the curve has shifted from $4/9$ to $\frac{4}{9}(1+\frac{3\pi\chi}{2})^2$. That is, the entire curve has shifted to the right. Additionally, we observe that again, the nose of the curve move up and to the right. This implies that the capacitive control scheme is also partially stabilizing for the disk.

Our analysis of the disk shaped device has revealed that the structure and number of solutions differs significantly from the simple 1-d and strip-like structures analyzed previously. These differences may have important implications for the design of current devices or may even lead to new devices based on the unique properties of the disk. For example, we note the pull-in distance, i.e. the maximum achievable displacement, is larger for the disk than for the strip. This may have implications in the design of MEMS based varactors, [4]. If we conjecture that the different branches of the disk bifurcation diagram, Figure 3, have alternating stability, we may perhaps utilize the additional stable steady-states in the construction of new devices.

Finally and most importantly, we note that the fact that the solution set for the disk and strip are so dramatically different suggests that for other device shapes many other behaviors are possible. This would imply that shape is an important design parameter for electrostatic MEMS. The authors are currently working to extend the analysis here to an arbitrarily shaped device.

4 Appendix A

In this appendix, we give the asymptotic expansion of the integral appearing in the mapping between the nonlocal and local problems, equation (7). That is, we examine

$$I(\gamma) = \int_0^1 \frac{rdr}{w(\gamma r)} \tag{23}$$

for $\gamma \gg 1$. First, introducing the change of variables $x = \gamma r$ and breaking the integral into two pieces we obtain

$$I(\gamma) = \frac{1}{\gamma^2}\int_0^a \frac{xdx}{w(x)} + \frac{1}{\gamma^2}\int_a^x \frac{xdx}{w(x)}. \tag{24}$$

Now, noting that w is bounded and bounded away from zero on the interval $[0, a]$ for any a we see that the first term on the right is $O(1/\gamma^2)$. We choose a so large that in the second term on the right w is well approximated by the asymptotic expansion, equation (20), and write

$$I(\gamma) \sim \frac{1}{\gamma^2}\int_a^\gamma \frac{xdx}{(\frac{9}{4})^{1/3}x^{2/3} + A\cos(\frac{2\sqrt{2}}{3}\log(x) + B)} + O(1/\gamma^2). \tag{25}$$

The integrand in the integral in this expression may be expanded and integrated term by term. Retaining terms larger that $O(1/\gamma^2)$ we obtain:

$$\begin{aligned} I(\gamma) \sim \frac{3}{4}\frac{1}{\gamma^{2/3}}(\frac{4}{9})^{1/3} - \frac{A}{4}\frac{1}{\gamma^{2/3}}(2\cos(\frac{2\sqrt{2}}{3}\log(\gamma) + B) \\ +2\sqrt{2}\sin(\frac{2\sqrt{2}}{3}\log(\gamma) + B)) + O(1/\gamma^2). \end{aligned} \tag{26}$$

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